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FACE MAGIC AND 1-ANTIMAGIC LABELINGS OF ROOTED PRODUCT OF PATH AND CYCLE

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Abstract

In graph theory, the concept of graph labeling is a significant area of research due to its both theoretical and practical implications. It provides valuable insights into the structural properties and labeling constraints of the graphs. This study focuses on the face magic labeling and 1-antimagic labeling of types (1,1,1), (1,1,0), (1,0,0), (1,0,1), (0,1,0) and (0,1,1) of the rooted product of graphs, a construction that combines two graphs, path P_n and cycle C_m, by attaching a copy of C_m to each vertex of P_n. This study establishes the theoretical foundations for face magic and d-antimagic labelings of type (a,b,c) in the rooted product of graphs, providing detailed criteria and methodologies.

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Key Words and Phrases: Face labeling, Graph labeling, Planar graph, Magic labeling, Antimagic labeling

1 Introduction

Graph labeling is a technique used to assign labels to the elements of a graph, such as vertices, edges, or faces, based on specific rules, facilitating the study and application of graph properties. Numerous studies have explored the magic and antimagic labelings of various types of graphs. Magic labelings assign labels to graph elements in such a way that the sums of labels around specified structures are the same, leading to uniformity in the graph's labeling scheme. On the other hand, antimagic labelings ensure that these sums are distinct, introducing a level of uniqueness.

In 1983, Lih introduced a magic-type method for labeling the vertices, edges, and faces of planar graphs [1]. Baca provided consecutive and magic labeling of type (0,1,1), consecutive labeling of type (1,1,1) for planar graphs with hexagonal faces [2] and the definition of (a,d)-face antimagic labeling of a planar graph [3]. Baca and Miller defined the d-antimagic labeling of the type (1,1,1) of a planar graph [4]. Hameed et al. investigated the super d-antimagic labeling of some planar graphs [5]. Liu et al. studied the super (a, 0) edge-antimagic labeling of the rooted product of certain graphs [6]. Face magic and 1-antimagic labelings of specific graph classes are studied [7]. A large number of papers related to graph labeling are compiled in the Gallian survey of graph labeling [8]. Graph labeling has a broad range of applications, including cryptography [9]. These labeling methods can be utilized to design algorithms.

The rooted product of two graphs G and H is constructed by taking $|V(G)|$ copies of H and for every vertex v_i of G, identifying v_i with the root node of the i^{th} copy of H. This product is denoted by $G \circ H$. A magic/antimagic labeling of type (a,b,c) for a planar graph G is called a super magic/antimagic labeling of type (a,b,c) if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. In magic labeling of type (a,b,c), the sum of the labels of specified constraints surrounding the face is

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constant, wherein d-antimagic labeling of type (a,b,c), the sum of the labels of specified constraints surrounding the face constitutes an arithmetic progression with difference d. For detailed definitions of labelings, see the Gallian survey [8].

This paper focusses on the magic labeling and 1-antimagic labeling of all types namely, (1,1,1), (1,1,0), (1,0,0), (1,0,1), (0,1,0) and (0,1,1) of the rooted product of graphs path P_n, and cycle C_m, where m and n are the number of vertices in P_n and C_m, respectively. Additionally, several examples are presented to illustrate these methods and demonstrate the feasibility of the criteria.

2 Face labeling of rooted product of graphs P_n and C_m

Consider the planar graph $G \cong P_n \circ C_m$ with vertex set $V(G)$, edge set $E(G)$ and face set $F(G)$. Let M be the magic constant and W be the sum of the labels in the interior face of the graph (face weight).

The labeling of the graph $P_n \circ C_m$ is analysed for both cases when m is even and when m is odd and the proof covers all possible scenarios. For even m, the analysis includes the cases where m is divisible by 4 and not divisible by 4. For odd m, the proof examines the cases where $m-1$ is divisible by 4 and not divisible by 4. Each scenario is carefully considered to ensure the completeness of the proof and to cover all possible configurations of m.

In Fig. 1, the labelings of a general graph are demonstrated for $m = 4k, k \in N$. The same graph framework applies, but the number of nodes and labelings varies in other instances. The cardinalities of vertices, edges, and faces are $mn, mn+n-1$, and n, respectively and their labelings are given in general as follows:

$$V(G) = \{v_i, v_j^i : 1 \leqslant i \leqslant n; 1 \leqslant j \leqslant m-1\},$$

$$E(G) = \{v_iv_{i+1}, v_iv_q^i, v_q^iv_r^i, v_iv_s^i, v_s^iv_t^i : 1 \leqslant i \leqslant n; 1 \leqslant q, r, s \leqslant \frac{m}{4};$$

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$$1 \leqslant t \leqslant \frac{m}{4} - 1\},$$
$$F(G) = \{F_i : 1 \leqslant i \leqslant n\}.$$

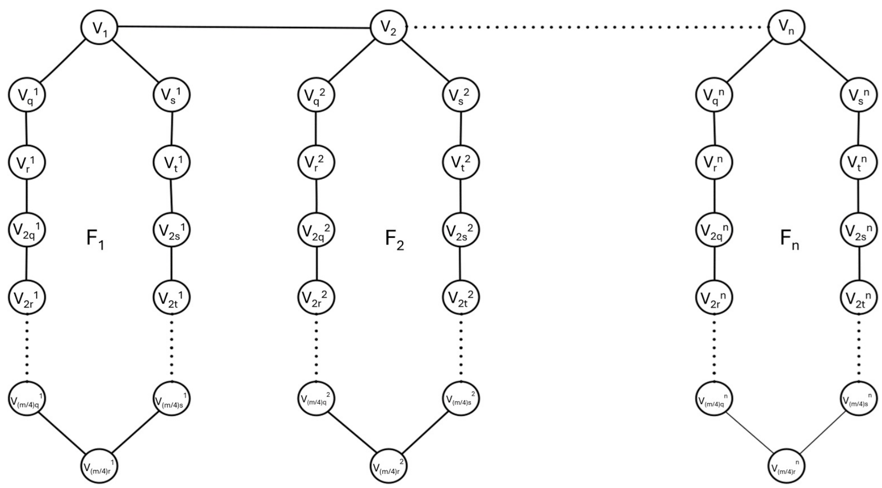

Fig. 1 $P_n \circ C_{4m}$

Theorem 1. *For even m, the graph $P_n \circ C_m$ admits a super magic labeling of type (1,0,0) with magic constant $M = \frac{m}{2}(mn+1)$.*

Proof. Let G be the rooted product of graphs P_n and C_m with vertex set $V(G)$ and $|V(G)| = mn$. The vertices are labeled as described below.

Case 1: $m \equiv 0$ $(mod\ 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \leq i \leq n$ are labeled as follows:
$f(v_i) = i,$
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \leq q \leq \frac{m}{4},$
$f(v_r^i) = (4r-1)n + i, \quad 1 \leq r \leq \frac{m}{4},$
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m}{4},$
$f(v_t^i) = 4tn + i, \quad 1 \leq t \leq \frac{m}{4} - 1.$

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Summing all the vertex labels generated by the labeling function f, then the magic constant is calculated as below.

$$M = i + \sum_{q=1}^{\frac{m}{4}} (4q-1)n + 1 - i + \sum_{r=1}^{\frac{m}{4}} (4r-1)n + i$$

$$+ \sum_{s=1}^{\frac{m}{4}} (4s-2)n + 1 - i + \sum_{t=1}^{\frac{m}{4}-1} 4tn + i$$

Using the formula for the sum of first n natural numbers,

$$\begin{aligned} M &= i + \frac{3.4.n(\frac{m}{4})(\frac{m}{4}+1)}{2} - 4.\frac{m}{4}n + \frac{m}{2} - \frac{m}{4}i \\ &+ \frac{4.n(\frac{m}{4}-1)(\frac{m}{4}-1+1)}{2} + \left(\frac{m}{4}-1\right)i \\ &= \frac{m}{2}(mn+1). \end{aligned}$$

Case 2: $m \equiv 2\ (mod\ 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \leq i \leq n$ are labeled as follows:
$f(v_i) = i$,
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \leq q \leq \frac{m-2}{4}$,
$f(v_r^i) = (4r-1)n + i, \quad 1 \leq r \leq \frac{m-2}{4}$,
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m+2}{4}$,
$f(v_t^i) = 4tn + i, \quad 1 \leq t \leq \frac{m-2}{4}$.
Summing all the vertex labels generated by the labeling function f, then the magic constant is calculated as below.

$$M = i + \sum_{q=1}^{\frac{m-2}{4}} (4q-1)n + 1 - i + (4r-1)n + i + 4tn + i + \sum_{s=1}^{\frac{m+2}{4}} (4s-2)n + 1 - i$$

$$= \frac{3.4.n(\frac{m-2}{4})(\frac{m-2}{4}+1)}{2} - 2\left(\frac{m-2}{4}\right)n + \frac{m-2}{4} + \left(\frac{m-2}{4}\right)i$$

$$+ \frac{4.n(\frac{m+2}{4})(\frac{m+2}{4}+1)}{2} - 2.n\frac{m+2}{4} + \frac{m+2}{4} - \left(\frac{m+2}{4}\right)i + i$$

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$$M = \frac{m}{2}(mn+1).$$

Thus, when m is even, G admits super magic labeling of type (1,0,0) with magic constant $\frac{m}{2}(mn+1)$. □

Theorem 2. *For odd m, the graph $P_n \circ C_m$ admits a super 1-antimagic labeling of type (1,0,0) with the following face weights:*
i) $\frac{nm^2+m-n-1}{2} + i, \; 1 \leq i \leq n$
ii) $\frac{nm^2+n+m+1}{2} - i, \; 1 \leq i \leq n.$

Proof. Let G be the rooted product of graphs P_n and C_m with vertex set $V(G)$ and $|V(G)| = mn$. The vertices are labeled as described below.

Case 1: $m \equiv 1 \; (mod \; 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \leq i \leq n$ are labeled as follows:
$f(v_i) = i$,
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \leq q \leq \frac{m-1}{4}$,
$f(v_r^i) = (4r-1)n + i, \quad 1 \leq r \leq \frac{m-1}{4}$,
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m-1}{4}$,
$f(v_t^i) = 4tn + i, \quad 1 \leq t \leq \frac{m-1}{4}$.
Summing all the vertex labels generated by the vertex labeling function f, then the face weight is calculated as,

$$W = i + \sum_{q=1}^{\frac{m-1}{4}} (4q-1)n+1-i+(4r-1)n+i+(4s-2)n+1-i+4tn+i$$
$$= \frac{nm^2+m-n-1}{2} + i.$$

Thus, when m is odd and $m-1$ is divisible by 4, G admits super 1-antimagic labeling of type (1,0,0) with face weight $\frac{nm^2+m-n-1}{2} + i$.

Case 2: $m \equiv 3 \; (mod \; 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \leq i \leq n$ are labeled as follows:

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$f(v_i) = i,$
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \leq q \leq \frac{m+1}{4},$
$f(v_r^i) = (4r-1)n + i, \quad 1 \leq r \leq \frac{m-3}{4},$
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m+1}{4},$
$f(v_t^i) = 4tn + i, \quad 1 \leq t \leq \frac{m-3}{4}.$
Summing all the vertex labels generated by the vertex labeling function f, then the face weight is calculated as,

$$W = i + \sum_{q=1}^{\frac{m+1}{4}} (4q-1)n + 1 - i + (4s-2)n + 1 - i + \sum_{r=1}^{\frac{m-3}{4}} (4r-1)n + i + 4tn + i$$
$$= \frac{nm^2 + n + m + 1}{2} - i.$$

Thus, when m is odd and $m-1$ is not divisible by 4, G admits super 1-antimagic labeling of type (1,0,0) with face weight $\frac{nm^2+n+m+1}{2} - i$. □

Theorem 3. *For even m, the graph $P_n \circ C_m$ admits a super 1-antimagic labeling of type (1,0,1) with the following face weights:*
i) $\frac{m}{2}(mn + 2n + 1) + n + 1 - i, \quad 1 \leq i \leq n$
ii) $\frac{m}{2}(mn + 2n + 1) + i, \quad 1 \leq i \leq n.$

Proof. Let G be the rooted product of graphs P_n and C_m with vertex set $V(G)$, where $|V(G)| = mn$ and face set $F(G)$. The vertices and faces are labeled as described below.

Case 1: $m \equiv 0 \ (mod\ 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \leq i \leq n$ are labeled as follows:
$f(v_i) = i,$
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \leq q \leq \frac{m}{4},$
$f(v_r^i) = (4r-1)n + i, \quad 1 \leq r \leq \frac{m}{4},$
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m}{4},$
$f(v_t^i) = 4tn + i, \quad 1 \leq t \leq \frac{m}{4} - 1.$
A face labeling function $g : F(G) \longrightarrow \{mn+1, mn+2, \ldots, mn+n\}$, a bijection which is defined as follows:

$$g(F_i) = mn + n + 1 - i, 1 \leq i \leq n.$$

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Summing all the vertex labels and face labels generated by the labeling functions f and g, then the face weight is calculated as,

$$W = \frac{m}{2}(mn+1) + mn + n + 1 - i$$

$$= \frac{m}{2}(mn+2n+1) + n + 1 - i.$$

Case 2: $m \equiv 2 \ (mod\ 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \leq i \leq n$ are labeled as follows:
$f(v_i) = i$,
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \leq q \leq \frac{m-2}{4}$,
$f(v_r^i) = (4r-1)n + i, \quad 1 \leq r \leq \frac{m-2}{4}$,
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m+2}{4}$,
$f(v_t^i) = 4tn + i, \quad 1 \leq t \leq \frac{m-2}{4}$.
A face labeling function $g : F(G) \longrightarrow \{mn+1, mn+2, \ldots, mn+n\}$, a bijection, is defined as follows:

$$g(F_i) = mn + i, \ \ 1 \leq i \leq n.$$

Summing all the vertex labels and face labels generated by the labeling functions f and g, then the face weight is calculated as,

$$W = \frac{m}{2}(mn+1) + mn + i$$

$$= \frac{m}{2}(mn+2n+1) + i.$$

□

Theorem 4. *For odd m, the graph $P_n \circ C_m$ admits a super magic labeling of type (1,0,1) with magic constant $M = \frac{nm^2+2mn+m+n+1}{2}$.*

Proof. Let G be the rooted product of graphs P_n and C_m with vertex set $V(G)$, where $|V(G)| = mn$ and face set $F(G)$. The vertices and faces are labeled as described below.

Case 1: $m \equiv 1 \ (mod\ 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is

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a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \leq i \leq n$ are labeled as follows:
$f(v_i) = i$,
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \leq q \leq \frac{m-1}{4}$,
$f(v_r^i) = (4r-1)n + i, \qquad 1 \leq r \leq \frac{m-1}{4}$,
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m-1}{4}$,
$f(v_t^i) = 4tn + i, \qquad 1 \leq t \leq \frac{m-1}{4}$.
A face labeling function $g : F(G) \longrightarrow \{mn+1, mn+2, \ldots, mn+n\}$, a bijection, is defined as follows:

$$g(F_i) = mn + n + 1 - i, \;\; 1 \leq i \leq n.$$

Summing all the vertex labels and face labels generated by the labeling function f, then the magic constant is calculated as,

$$M = \frac{nm^2 + m - n - 1}{2} + i + mn + n + 1 - i$$
$$= \frac{nm^2 + 2mn + m + n + 1}{2}.$$

Case 2: $m \equiv 3 \ (mod\ 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \leq i \leq n$ are labeled as follows:
$f(v_i) = i$,
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \leq q \leq \frac{m+1}{4}$,
$f(v_r^i) = (4r-1)n + i, \qquad 1 \leq r \leq \frac{m-3}{4}$,
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m+1}{4}$,
$f(v_t^i) = 4tn + i, \qquad 1 \leq t \leq \frac{m-3}{4}$.
A face labeling function $g : F(G) \longrightarrow \{mn+1, mn+2, \ldots, mn+n\}$, a bijection, is defined as follows:

$$g(F_i) = mn + i, \; 1 \leq i \leq n.$$

Summing all the vertex labels and face labels generated by the labeling functions f and g, then the magic constant is calculated as,

$$M = \frac{nm^2 + n + m + 1}{2} - i + mn + i$$

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$$= \frac{nm^2 + 2mn + m + n + 1}{2}.$$

□

Theorem 5. *For even m, the graph $P_n \circ C_m$ admits a super 1-antimagic labeling of type (1,1,1) with the following face weights:*
i) $m(2mn + 2n + 1) + n + 1 - i, \quad 1 \le i \le n$
ii) $m(2mn + 2n + 1) + i, \quad 1 \le i \le n.$

Proof. Let G be the rooted product of graphs P_n and C_m with vertex set $V(G)$, edge set $E(G)$ and face set $F(G)$. The vertices, edges and faces are labeled as described below.

Case 1: $m \equiv 0 \ (mod\ 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \dots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \le i \le n$ are labeled as follows:
$f(v_i) = i$,
$f(v_q^i) = (4q - 1)n + 1 - i, \quad 1 \le q \le \frac{m}{4}$,
$f(v_r^i) = (4r - 1)n + i, \quad 1 \le r \le \frac{m}{4}$,
$f(v_s^i) = (4s - 2)n + 1 - i, \quad 1 \le s \le \frac{m}{4}$,
$f(v_t^i) = 4tn + i, \quad 1 \le t \le \frac{m}{4} - 1$.
A face labeling function $g : F(G) \longrightarrow \{mn+1, mn+2, \dots, mn+n\}$, a bijection, is defined as follows:

$$g(F_i) = mn + n + 1 - i, \ 1 \le i \le n.$$

A bijective edge labeling function $h : E(G) \longrightarrow \{mn + n + 1, mn + n + 2, \dots, 2mn + 2n - 1\}$ is defined and the edges $v_i v_q^i$, $v_q^i v_r^i$, $v_i v_s^i$, $v_s^i v_t^i$ $(1 \le i \le n)$ and $v_i v_{i+1}$ are labeled as follows:
$h(v_i v_q^i) = mn + (4q - 3)n + i, \quad 1 \le q \le \frac{m}{4}$,
$h(v_q^i v_r^i) = mn + (4r + 1)n + 1 - i, \quad 1 \le r \le \frac{m}{4}$,
$h(v_i v_s^i) = mn + (4s - 1)n + 1 - i, \quad 1 \le s \le \frac{m}{4}$,
$h(v_s^i v_t^i) = mn + (4t - 1)n + i, \quad 1 \le t \le \frac{m}{4}$,
$h(v_i v_{i+1}) = 2mn + n + i, \quad 1 \le i \le n - 1$.
Summing the edge labels in the interior face of G generated by the function h,

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$$\sum_{q,r,s,t=1}^{\frac{m}{4}} mn+(4q-3)n+i+mn+(4r+1)n+1-i$$
$$+mn+(4s-1)n+1-i+mn+(4t-1)n+i$$
$$=\frac{m}{2}(3mn+2n+1).$$

Summing all the vertex labels, face labels and edge labels generated by the labeling functions f, g and h, then the face weight is calculated as,

$$W=\frac{m}{2}(mn+2n+1)+n+1-i+\frac{m}{2}(3mn+2n+1)$$
$$=m(2mn+2n+1)+n+1-i.$$

Case 2: $m \equiv 2\ (mod\ 4)$.

A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \leq i \leq n$ are labeled as follows:
$f(v_i) = i$,
$f(v_q^i) = (4q-1)n+1-i, \quad 1 \leq q \leq \frac{m-2}{4}$,
$f(v_r^i) = (4r-1)n+i, \quad 1 \leq r \leq \frac{m-2}{4}$,
$f(v_s^i) = (4s-2)n+1-i, \quad 1 \leq s \leq \frac{m+2}{4}$,
$f(v_t^i) = 4tn+i, \quad 1 \leq t \leq \frac{m-2}{4}$.
A face labeling function $g : F(G) \longrightarrow \{mn+1, mn+2, \ldots, mn+n\}$, a bijection, is defined as follows:

$$g(F_i) = mn+i,\ 1 \leq i \leq n.$$

A bijective edge labeling function $h : E(G) \longrightarrow \{mn+n+1, mn+n+2, \ldots, 2mn+2n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \leq i \leq n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = mn+(4q-3)n+i, \quad 1 \leq q \leq \frac{m+2}{4}$,
$h(v_q^iv_r^i) = mn+(4r+1)n+1-i, \quad 1 \leq r \leq \frac{m-2}{4}$,
$h(v_iv_s^i) = mn+(4s-1)n+1-i, \quad 1 \leq s \leq \frac{m+2}{4}$,
$h(v_s^iv_t^i) = mn+(4t-1)n+i, \quad 1 \leq t \leq \frac{m-2}{4}$,
$h(v_iv_{i+1}) = 2mn+n+i, \quad 1 \leq i \leq n-1$.
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$$\sum_{q,s=1}^{\frac{m+2}{4}} mn + (4q-3)n + i + mn + (4s-1)n + 1 - i$$
$$+ \sum_{r,t=1}^{\frac{m-2}{4}} mn + (4r+1)n + 1 - i + mn + (4t-1)n + i$$
$$= \frac{m}{2}(3mn + 2n + 1).$$

Summing all the vertex labels, face labels and edge labels generated by the labeling functions f, g and h, then the face weight is calculated as,

$$W = \frac{m}{2}(mn + 2n + 1) + i + \frac{m}{2}(3mn + 2n + 1)$$
$$= m(2mn + 2n + 1) + i.$$

□

Example 1: The graph $P_3 \circ C_8$ satisfies super 1-antimagic labeling of type (1,1,1). Since m is 8, the face weight of each face is obtained by $m(2mn + 2n + 1) + n + 1 - i$. Thus, the face weights are 443, 442 and 441 which are same as the face weights obtained by summing the vertex, edge and face labels around each face of the graph shown in Fig. 2.

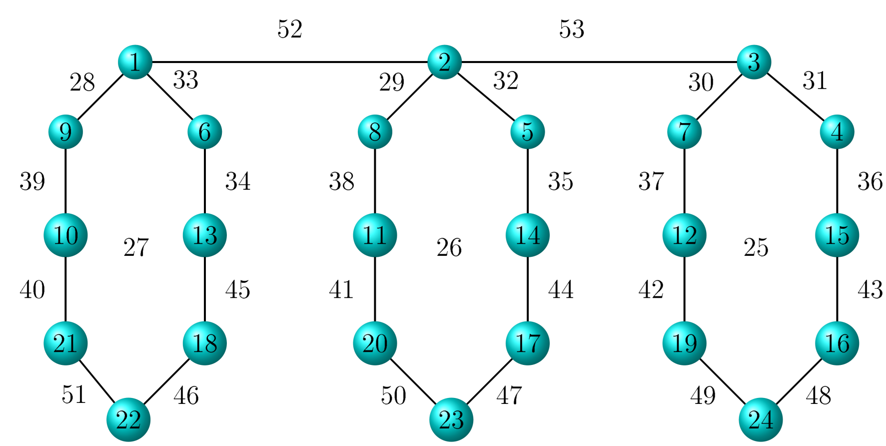

Fig. 2 A super 1-antimagic labeling of type (1,1,1) of $P_3 \circ C_8$

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Theorem 6. *For odd m, the graph $P_n \circ C_m$ admits a super 1-antimagic labeling of type (1,1,1) with face weight $W = m(2mn+2n+1)+i$, $1 \le i \le n$.*

Proof. Let G be the rooted product of graphs P_n and C_m with vertex set $V(G)$, edge set $E(G)$ and face set $F(G)$. The vertices, edges and faces are labeled as described below.

Case 1: $m \equiv 1$ $(mod\ 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \le i \le n$ are labeled as follows:
$f(v_i) = i$,
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \le q \le \frac{m-1}{4}$,
$f(v_r^i) = (4r-1)n + i, \quad 1 \le r \le \frac{m-1}{4}$,
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \le s \le \frac{m-1}{4}$,
$f(v_t^i) = 4tn + i, \quad 1 \le t \le \frac{m-1}{4}$.
A face labeling function $g : F(G) \longrightarrow \{mn+1, mn+2, \ldots, mn+n\}$, a bijection, is defined as follows:

$$g(F_i) = mn + n + 1 - i,\ 1 \le i \le n.$$

A bijective edge labeling function $h : E(G) \longrightarrow \{mn+n+1, mn+n+2, \ldots, 2mn+2n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \le i \le n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = mn + (4q-3)n + i, \quad 1 \le q \le \frac{m+3}{4}$,
$h(v_q^iv_r^i) = mn + (4r+1)n + 1 - i, \quad 1 \le r \le \frac{m-1}{4}$,
$h(v_iv_s^i) = mn + (4s-1)n + 1 - i, \quad 1 \le s \le \frac{m-1}{4}$,
$h(v_s^iv_t^i) = mn + (4t-1)n + i, \quad 1 \le t \le \frac{m-1}{4}$,
$h(v_iv_{i+1}) = 2mn + n + i, \quad 1 \le i \le n-1$.
Summing the edge labels in the interior face of G generated by the function h,

$$\sum_{q=1}^{\frac{m+3}{4}} mn + (4q-3)n + i + \sum_{r,s,t=1}^{\frac{m-1}{4}} mn + (4r+1)n + 1 - i$$
$$+mn + (4s-1)n + 1 - i + mn + (4t-1)n + i$$

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$$= \frac{nm}{2}(3m+2) + \frac{m-n-1}{2} + i.$$

Summing all the vertex labels, face labels and edge labels generated by the labeling functions f, g and h, then the face weight is calculated as,

$$W = \frac{nm^2+2mn+m+n+1}{2} + \frac{nm}{2}(3m+2) + \frac{m-n-1}{2} + i$$
$$= m(2mn+2n+1) + i.$$

Case 2: $m \equiv 3\ (mod\ 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \leq i \leq n$ are labeled as follows:
$f(v_i) = i$,
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \leq q \leq \frac{m+1}{4}$,
$f(v_r^i) = (4r-1)n + i, \qquad 1 \leq r \leq \frac{m-3}{4}$,
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m+1}{4}$,
$f(v_t^i) = 4tn + i, \qquad 1 \leq t \leq \frac{m-3}{4}$.
A face labeling function $g : F(G) \longrightarrow \{mn+1, mn+2, \ldots, mn+n\}$, a bijection, is defined as follows:

$$g(F_i) = mn + i,\ 1 \leq i \leq n.$$

A bijective edge labeling function $h : E(G) \longrightarrow \{mn+n+1, mn+n+2, \ldots, 2mn+2n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \leq i \leq n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = mn + (4q-3)n + i, \qquad 1 \leq q \leq \frac{m+1}{4}$,
$h(v_q^iv_r^i) = mn + (4r+1)n + 1 - i, \qquad 1 \leq r \leq \frac{m-3}{4}$,
$h(v_iv_s^i) = mn + (4s-1)n + 1 - i, \qquad 1 \leq s \leq \frac{m+1}{4}$,
$h(v_s^iv_t^i) = mn + (4t-1)n + i, \qquad 1 \leq t \leq \frac{m+1}{4}$,
$h(v_iv_{i+1}) = 2mn + n + i, \qquad 1 \leq i \leq n-1$.
Summing the edge labels in the interior face of G generated by the function h,

$$\sum_{q,s,t=1}^{\frac{m+1}{4}} mn + (4q-3)n + i + mn + (4s-1)n + 1 - i$$

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$$+mn+(4t-1)n+i+\sum_{r=1}^{\frac{m-3}{4}} mn+(4r+1)n+1-i$$
$$=\frac{mn}{2}(3m+2)+\frac{m-n-1}{2}+i.$$

Summing all the vertex labels, face labels and edge labels generated by the labeling functions f, g and h, then the face weight is calculated as,

$$W=\frac{nm^2+2mn+m+n+1}{2}+\frac{mn}{2}(3m+2)+\frac{m-n-1}{2}+i$$
$$=m(2mn+2n+1)+i.$$

□

Example 2: The graph $P_4 \circ C_5$ satisfies super 1-antimagic labeling of type (1,1,1). Since m is 5, the face weight of each face is obtained by $m(2mn+2n+1)+i$. Thus, the face weights are 246, 247, 248 and 249 which are same as the face weights obtained by summing the vertex, edge and face labels around each face of the graph shown in Fig. 3.

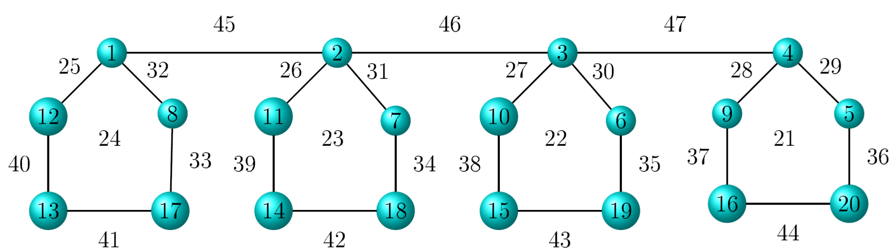

Fig. 3 A super 1-antimagic labeling of type (1,1,1) of $P_4 \circ C_5$

Theorem 7. *For even m, the graph $P_n \circ C_m$ admits a super magic labeling of type (1,1,0) with magic constant $M = m(2mn+1)$.*

Proof. Let G be the rooted product of graphs P_n and C_m with vertex set $V(G)$ and edge set $E(G)$. The vertices and edges are

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labeled as described below.

Case 1: $m \equiv 0\ (mod\ 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \leq i \leq n$ are labeled as follows:
$f(v_i) = i$,
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \leq q \leq \frac{m}{4}$,
$f(v_r^i) = (4r-1)n + i, \quad 1 \leq r \leq \frac{m}{4}$,
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m}{4}$,
$f(v_t^i) = 4tn + i, \quad 1 \leq t \leq \frac{m}{4} - 1$.
A bijective edge labeling function $h : E(G) \longrightarrow \{mn+1, mn+2, \ldots, 2mn+n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \leq i \leq n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = mn + (4q-4)n + i, \quad 1 \leq q \leq \frac{m}{4}$,
$h(v_q^iv_r^i) = mn + 4rn + 1 - i, \quad 1 \leq r \leq \frac{m}{4}$,
$h(v_iv_s^i) = mn + (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m}{4}$,
$h(v_s^iv_t^i) = mn + (4t-2)n + i, \quad 1 \leq t \leq \frac{m}{4}$,
$h(v_iv_{i+1}) = 2mn + i, \quad 1 \leq i \leq n-1$.
Summing the edge labels in the interior face of G generated by the function h,

$$\sum_{q,r,s,t=1}^{\frac{m}{4}} mn + (4q-4)n + i + mn + 4rn + 1 - i$$
$$+mn + (4s-2)n + 1 - i + mn + (4t-2)n + i$$
$$= \frac{m}{2}(3mn+1).$$

Summing all the vertex labels and edge labels generated by the labeling functions f and h, then the magic constant is calculated as,

$$M = \frac{m}{2}(mn+1) + \frac{m}{2}(3mn+1)$$
$$= m(2mn+1).$$

Case 2: $m \equiv 2\ (mod\ 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is

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a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \leq i \leq n$ are labeled as follows:
$f(v_i) = i$,
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \leq q \leq \frac{m-2}{4}$,
$f(v_r^i) = (4r-1)n + i, \qquad 1 \leq r \leq \frac{m-2}{4}$,
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m+2}{4}$,
$f(v_t^i) = 4tn + i, \qquad\qquad 1 \leq t \leq \frac{m-2}{4}$.
A bijective edge labeling function $h : E(G) \longrightarrow \{mn+1, mn+2, \ldots, 2mn+n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \leq i \leq n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = mn + (4q-4)n + i, \qquad 1 \leq q \leq \frac{m+2}{4}$,
$h(v_q^iv_r^i) = mn + 4rn + 1 - i, \qquad 1 \leq r \leq \frac{m-2}{4}$,
$h(v_iv_s^i) = mn + (4s-2)n + 1 - i, \qquad 1 \leq s \leq \frac{m+2}{4}$,
$h(v_s^iv_t^i) = mn + (4t-2)n + i, \qquad 1 \leq t \leq \frac{m-2}{4}$,
$h(v_iv_{i+1}) = 2mn + i, \qquad 1 \leq i \leq n-1$.
Summing the edge labels generated by the function h,

$$\sum_{q,s=1}^{\frac{m+2}{4}} mn + (4q-4)n + i + mn + (4s-2)n + 1 - i$$
$$+ \sum_{r,t=1}^{\frac{m-2}{4}} mn + 4rn + 1 - i + mn + (4t-2)n + i$$
$$= \frac{m}{2}(3mn+1).$$

Summing all the vertex labels and edge labels generated by the labeling functions f and h, then the magic constant is calculated as,

$$M = \frac{m}{2}(mn+1) + \frac{m}{2}(3mn+1)$$
$$= m(2mn+1).$$

□

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Theorem 8. *For odd m, the graph $P_n \circ C_m$ admits a super magic labeling of type (1,1,0) with magic constant $M = m(2mn+1)$.*

Proof. Let G be the rooted product of graphs P_n and C_m with vertex set $V(G)$, where $|V(G)| = mn$ and edge set $E(G)$. The vertices and edges are labeled as described below.

Case 1: $m \equiv 3$ $(mod\ 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \leq i \leq n$ are labeled as follows:
$f(v_i) = i,$
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \leq q \leq \frac{m+1}{4},$
$f(v_r^i) = (4r-1)n + i, \quad 1 \leq r \leq \frac{m-3}{4},$
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m+1}{4},$
$f(v_t^i) = 4tn + i, \quad 1 \leq t \leq \frac{m-3}{4}.$
A bijective edge labeling function $h : E(G) \longrightarrow \{mn+1, mn+2, \ldots, 2mn+n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \leq i \leq n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = mn + (4q-4)n + i, \quad 1 \leq q \leq \frac{m+1}{4},$
$h(v_q^iv_r^i) = mn + 4rn + 1 - i, \quad 1 \leq r \leq \frac{m-3}{4},$
$h(v_iv_s^i) = mn + (4s-2)n + 1 - i, \quad 1 \leq s \leq \frac{m+1}{4},$
$h(v_s^iv_t^i) = mn + (4t-2)n + i, \quad 1 \leq t \leq \frac{m+1}{4},$
$h(v_iv_{i+1}) = 2mn + i, \quad 1 \leq i \leq n-1.$
Summing the edge labels in the interior face of G generated by the function h,

$$\frac{mn}{2}(3m+2) + \frac{m-n-1}{2} + i - mn$$
$$= \frac{m}{2}(3mn+1) - \left(\frac{n+1}{2}\right) + i.$$

Summing all the vertex labels and edge labels generated by the labeling functions f and h, then the magic constant is calculated as,

$$M = \frac{nm^2+m+n+1}{2} - i + \frac{m}{2}(3mn+1) - \left(\frac{n+1}{2}\right) + i$$

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$$= m(2mn + 1).$$

Case 2: $m \equiv 1 \ (mod\ 4)$.
A vertex labeling function $f : V(G) \longrightarrow \{1, 2, \dots, mn\}$, which is a bijection, is defined and the vertices v_i, v_q^i, v_r^i, v_s^i and v_t^i, where $1 \le i \le n$ are labeled as follows:
$f(v_i) = i$,
$f(v_q^i) = (4q-1)n + 1 - i, \quad 1 \le q \le \frac{m-1}{4}$,
$f(v_r^i) = (4r-1)n + i, \quad 1 \le r \le \frac{m-1}{4}$,
$f(v_s^i) = (4s-2)n + 1 - i, \quad 1 \le s \le \frac{m-1}{4}$,
$f(v_t^i) = 4tn + i, \quad 1 \le t \le \frac{m-1}{4}$.
A bijective edge labeling function $h : E(G) \longrightarrow \{mn+1, mn+2, \dots, 2mn+n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \le i \le n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = mn + (4q-3)n + 1 - i, \quad 1 \le q \le \frac{m+3}{4}$,
$h(v_q^iv_r^i) = mn + (4r-1)n + i, \quad 1 \le r \le \frac{m-1}{4}$,
$h(v_iv_s^i) = mn + (4s-3)n + i, \quad 1 \le s \le \frac{m-1}{4}$,
$h(v_s^iv_t^i) = mn + (4t-1)n + 1 - i, \quad 1 \le t \le \frac{m-1}{4}$,
$h(v_iv_{i+1}) = 2mn + i, \quad 1 \le i \le n-1$.
Summing the edge labels generated by the function h,

$$\sum_{q=1}^{\frac{m+3}{4}} mn + (4q-3)n + 1 - i + \sum_{r=1}^{\frac{m-1}{4}} mn + (4r-1)n + i$$
$$+ mn + (4s-3)n + i + mn + (4t-1)n + 1 - i$$
$$= \frac{n}{2}(3m^2 + 1) + \frac{m+1}{2} - i.$$

Summing all the vertex labels and edge labels generated by the labeling functions f and h, then the magic constant is,

$$M = \frac{nm^2 + m - n - 1}{2} + i + \frac{n}{2}(3m^2+1) + \frac{m+1}{2} - i$$
$$= m(2mn + 1).$$

$\square$

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Theorem 9. *For even m, the graph $P_n \circ C_m$ admits a super magic labeling of type (0,1,0) with magic constant $M = \frac{m}{2}(mn+1)$.*

Proof. Let G be the rooted product of graphs P_n and C_m with edge set $E(G)$. The edges are labeled as described below.

Case 1: $m \equiv 0 \ (mod\ 4)$.
A bijective edge labeling function $h : E(G) \longrightarrow \{1, 2, \ldots, mn+n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \leq i \leq n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = (4q-4)n + i, \qquad 1 \leq q \leq \frac{m}{4},$
$h(v_q^iv_r^i) = 4rn + 1 - i, \qquad 1 \leq r \leq \frac{m}{4},$
$h(v_iv_s^i) = (4s-2)n + 1 - i, \qquad 1 \leq s \leq \frac{m}{4},$
$h(v_s^iv_t^i) = (4t-2)n + i, \qquad 1 \leq t \leq \frac{m}{4},$
$h(v_iv_{i+1}) = mn + i, \qquad 1 \leq i \leq n-1.$
Summing the edge labels in the interior face of G generated by the function h, then the magic constant M is calculated as follows:

$$M = \sum_{q,r,s,t=1}^{\frac{m}{4}} (4q-4)n+i+4rn+1-i+(4s-2)n+1-i+(4t-2)n+i$$
$$= \frac{m}{2}(mn+1).$$

Case 2: $m \equiv 2 \ (mod\ 4)$.
A bijective edge labeling function $h : E(G) \longrightarrow \{1, 2, \ldots, mn+n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \leq i \leq n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = (4q-4)n + i, \qquad 1 \leq q \leq \frac{m+2}{4},$
$h(v_q^iv_r^i) = 4rn + 1 - i, \qquad 1 \leq r \leq \frac{m-2}{4},$
$h(v_iv_s^i) = (4s-2)n + 1 - i, \qquad 1 \leq s \leq \frac{m+2}{4},$
$h(v_s^iv_t^i) = (4t-2)n + i, \qquad 1 \leq t \leq \frac{m-2}{4},$
$h(v_iv_{i+1}) = mn + i, \qquad 1 \leq i \leq n-1.$
Summing the edge labels generated by the function h,

$$M = \sum_{q,s=1}^{\frac{m+2}{4}} (4q-4)n+i+(4s-2)n+1-i+\sum_{r,t=1}^{\frac{m-2}{4}} 4rn+1-i+(4t-2)n+i$$

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$$= \frac{m}{2}(mn+1).$$

□

Theorem 10. *For odd m, the graph $P_n \circ C_m$ admits a super 1-antimagic labeling of type (0,1,0) with face weight $W = \frac{nm^2-n+m-1}{2} + i$, $1 \leq i \leq n$.*

Proof. Let G be the rooted product of graphs P_n and C_m with edge set $E(G)$. The edges are labeled as described below.

Case 1: $m \equiv 1$ $(mod\ 4)$.
A bijective edge labeling function $h : E(G) \longrightarrow \{1, 2, \ldots, mn+n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \leq i \leq n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = (4q-4)n + i$, $\quad 1 \leq q \leq \frac{m+3}{4}$,
$h(v_q^iv_r^i) = 4rn + 1 - i$, $\quad 1 \leq r \leq \frac{m-1}{4}$,
$h(v_iv_s^i) = (4s-2)n + 1 - i$, $\quad 1 \leq s \leq \frac{m-1}{4}$,
$h(v_s^iv_t^i) = (4t-2)n + i$, $\quad 1 \leq t \leq \frac{m-1}{4}$,
$h(v_iv_{i+1}) = mn + i$, $\quad 1 \leq i \leq n-1$.
Summing the edge labels generated by the function h, then the face weight is calculated as,

$$W = \sum_{r,s,t=1}^{\frac{m-1}{4}} 4rn+1-i+(4s-2)n+1-i+(4t-2)n+i+\sum_{q=1}^{\frac{m+3}{4}}(4q-4)n+i$$
$$= \frac{nm^2-n+m-1}{2} + i.$$

Case 2: $m \equiv 3$ $(mod\ 4)$.
A bijective edge labeling function $h : E(G) \longrightarrow \{1, 2, \ldots, mn+n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \leq i \leq n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = (4q-4)n + i$, $\quad 1 \leq q \leq \frac{m+1}{4}$,
$h(v_q^iv_r^i-) = 4rn + 1 - i$, $\quad 1 \leq r \leq \frac{m-3}{4}$,
$h(v_iv_s^i) = (4s-2)n + 1 - i$, $\quad 1 \leq s \leq \frac{m+1}{4}$,
$h(v_s^iv_t^i) = (4t-2)n + i$, $\quad 1 \leq t \leq \frac{m+1}{4}$,
$h(v_iv_{i+1}) = mn + i$, $\quad 1 \leq i \leq n-1$.

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Summing the edge labels in the interior face of G generated by the function h, then the face weight is calculated as,

$$W = \sum_{q,s,t=1}^{\frac{m+1}{4}} (4q-4)n+i+(4s-2)n+1-i+(4t-2)n+i+\sum_{r=1}^{\frac{m-3}{4}} 4rn+1-i$$

$$= \frac{nm^2-n+m-1}{2}+i.$$

□

Theorem 11. *For even m, the graph $P_n \circ C_m$ admits a super 1-antimagic labeling of type (0,1,1) with face weight $W = \frac{m}{2}(mn+1)+n(m+1)+2-i$, $1 \le i \le n$.*

Proof. Let G be the rooted product of graphs P_n and C_m with edge set $E(G)$ and face set $F(G)$. The edges are labeled as described below.

Case 1: $m \equiv 0$ $(mod\ 4)$.
A bijective edge labeling function $h : E(G) \longrightarrow \{1, 2, \ldots, mn+n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \le i \le n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = (4q-4)n+i$, $\quad 1 \le q \le \frac{m}{4}$,
$h(v_q^iv_r^i) = 4rn+1-i$, $\quad 1 \le r \le \frac{m}{4}$,
$h(v_iv_s^i) = (4s-2)n+1-i$, $\quad 1 \le s \le \frac{m}{4}$,
$h(v_s^iv_t^i) = (4t-2)n+i$, $\quad 1 \le t \le \frac{m}{4}$,
$h(v_iv_{i+1}) = mn+i$, $\quad 1 \le i \le n-1$.
The sum of the edge labeles generated by the function h is $\frac{m}{2}(mn+1)$.
A face labeling function $g : F(G) \longrightarrow \{mn+n, mn+n+1, \ldots, mn+2n-1\}$, a bijection, is defined as follows:

$$g(F_i) = mn+n+2-i,\ 1 \le i \le n.$$

Summing the edge labels and face labels in the interior face of G generated by the function h and g, then the face weight W is calculated as follows:

$$W = \frac{m}{2}(mn+1)+n(m+1)+2-i.$$

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Case 2: $m \equiv 2\ (mod\ 4)$.
A bijective edge labeling function $h : E(G) \longrightarrow \{1, 2, \ldots, mn+n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \leq i \leq n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = (4q-4)n + i, \qquad 1 \leq q \leq \frac{m+2}{4},$
$h(v_q^iv_r^i) = 4rn + 1 - i, \qquad 1 \leq r \leq \frac{m-2}{4},$
$h(v_iv_s^i) = (4s-2)n + 1 - i, \qquad 1 \leq s \leq \frac{m+2}{4},$
$h(v_s^iv_t^i) = (4t-2)n + i, \qquad 1 \leq t \leq \frac{m-2}{4},$
$h(v_iv_{i+1}) = mn + i, \qquad 1 \leq i \leq n-1.$
The sum of the edge labeles generated by the function h is $\frac{m}{2}(mn+1)$.
A face labeling function $g : F(G) \longrightarrow \{mn+n, mn+n+1, \ldots, mn+2n-1\}$, a bijection, is defined as follows:

$$g(F_i) = mn + n + 2 - i,\ 1 \leq i \leq n.$$

Summing the edge labels and face labels in the interior face of G generated by the function h and g, then the face weight W is calculated as follows:

$$W = \frac{m}{2}(mn+1) + n(m+1) + 2 - i.$$

□

Theorem 12. *For odd m, the graph $P_n \circ C_m$ admits a super magic labeling of type (0,1,1) with magic constant $M = \frac{nm^2-n+m-1}{2} + n(m+1) + 2$.*

Proof. Let G be the rooted product of graphs P_n and C_m with edge set $E(G)$ and face set $F(G)$. The edges are labeled as described below.

Case 1: $m \equiv 1\ (mod\ 4)$.
A bijective edge labeling function $h : E(G) \longrightarrow \{1, 2, \ldots, mn+n-1\}$ is defined and the edges $v_iv_q^i$, $v_q^iv_r^i$, $v_iv_s^i$, $v_s^iv_t^i$ $(1 \leq i \leq n)$ and v_iv_{i+1} are labeled as follows:
$h(v_iv_q^i) = (4q-4)n + i, \qquad 1 \leq q \leq \frac{m+3}{4},$
$h(v_q^iv_r^i) = 4rn + 1 - i, \qquad 1 \leq r \leq \frac{m-1}{4},$

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$h(v_i v_s^i) = (4s-2)n + 1 - i, \qquad 1 \leq s \leq \frac{m-1}{4},$
$h(v_s^i v_t^i) = (4t-2)n + i, \qquad 1 \leq t \leq \frac{m-1}{4},$
$h(v_i v_{i+1}) = mn + i, \qquad 1 \leq i \leq n-1.$
The sum of the edge labels generated by the function h is $\frac{nm^2-n+m-1}{2}+i$.
A face labeling function $g : F(G) \longrightarrow \{mn+n, mn+n+1, \ldots, mn+2n-1\}$, a bijection, is defined as follows:

$$g(F_i) = mn + n + 2 - i,\ 1 \leq i \leq n.$$

Summing the edge labels and face labels in the interior face of G generated by the function h and g, then the magic constant M is calculated as follows:

$$M = \frac{nm^2 + 2nm + m + n + 3}{2}.$$

Case 2: $m \equiv 3\ (mod\ 4)$.
A bijective edge labeling function $h : E(G) \longrightarrow \{1, 2, \ldots, mn+n-1\}$ is defined and the edges $v_i v_q^i$, $v_q^i v_r^i$, $v_i v_s^i$, $v_s^i v_t^i$ $(1 \leq i \leq n)$ and $v_i v_{i+1}$ are labeled as follows:
$h(v_i v_q^i) = (4q-4)n + i, \qquad 1 \leq q \leq \frac{m+1}{4},$
$h(v_q^i v_r^i -) = 4rn + 1 - i, \qquad 1 \leq r \leq \frac{m-3}{4},$
$h(v_i v_s^i) = (4s-2)n + 1 - i, \qquad 1 \leq s \leq \frac{m+1}{4},$
$h(v_s^i v_t^i) = (4t-2)n + i, \qquad 1 \leq t \leq \frac{m+1}{4},$
$h(v_i v_{i+1}) = mn + i, \qquad 1 \leq i \leq n-1.$
Sum of the edge labels generated by the function h is $\frac{nm^2-n+m-1}{2}+i$.
A face labeling function $g : F(G) \longrightarrow \{mn+n, mn+n+1, \ldots, mn+2n-1\}$, a bijection, is defined as follows:

$$g(F_i) = mn + n + 2 - i,\ 1 \leq i \leq n.$$

Summing the edge labels and face labels in the interior face of G generated by the function h and g, then the magic constant M is,

$$M = \frac{nm^2 + 2nm + m + n + 3}{2}.$$

□

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3 Summary of face magic and antimagic labelings of $P_n \circ C_m$

Table 1 shows super magic labeling of $P_n \circ C_m$ with magic constants for different configurations of m and type (a,b,c).

Table 1: Summary of super magic labeling of type (a,b,c) of $P_n \circ C_m$

m	**Type**	**Magic constant**
Even	(1,0,0)	$\frac{m}{2}(mn+1)$
Odd	(1,0,1)	$\frac{nm^2+2mn+m+n+1}{2}$
Even	(1,1,0)	$m(2mn+1)$
Odd	(1,1,0)	$m(2mn+1)$
Even	(0,1,0)	$\frac{m}{2}(mn+1)$
Odd	(0,1,1)	$\frac{nm^2+2nm+m+n+3}{2}$

Table 2 shows super 1-antimagic labeling of $P_n \circ C_m$ with face weights for different configurations of m and type (a,b,c).

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Table 2: Summary of super 1-antimagic labeling of type (a,b,c) of $P_n \circ C_m$

m	**Type**	**Face weights** $(1 \leq i \leq n)$
Odd	(1,0,0)	$\dfrac{nm^2+m-n-1}{2}+i, when\, m \equiv 1(mod4)$ $\dfrac{nm^2+n+m+1}{2}-i,\ when\, m \equiv 3(mod4)$
Even	(1,0,1)	$\dfrac{m}{2}(mn+2n+1)+n+1-i,\ when\, m \equiv 0(mod4)$ $\dfrac{m}{2}(mn+2n+1)+i,\ when\, m \equiv 2(mod4)$
Even	(1,1,1)	$m(2mn+2n+1)+n+1-i,\ when\, m \equiv 0(mod4)$ $m(2mn+2n+1)+i,\ when\, m \equiv 2(mod4)$
Odd	(1,1,1)	$m(2mn+2n+1)+i$
Odd	(0,1,0)	$\dfrac{nm^2-n+m-1}{2}+i$
Even	(0,1,1)	$\dfrac{m}{2}(mn+1)+n(m+1)+2-i$

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4 Conclusion

This study has investigated the face magic and 1-antimagic labelings of all types such as, (1,1,1), (1,1,0), (1,0,0), (1,0,1), (0,1,0) and (0,1,1) of the rooted product of graphs P_n and C_m, aiding in the discovery of various relationships between vertices, edges and faces. Exploration of these labeling techniques has highlighted their significance in understanding the structural properties and constraints of graphs.

The continuous development of graph labeling theories, including magic and antimagic labelings, not only advances theoretical knowledge but also drives innovation in practical applications, demonstrating the profound impact of graph theory on diverse scientific and engineering disciplines. It has potential applications in network theory, coding theory and cryptography. By assigning unique labels to vertices, edges, or faces, graph labeling can create robust cryptographic keys and improve encryption techniques. By enhancing the understanding of labeling properties in rooted product graphs, this paper contributes to the ongoing development of graph labeling theory and its applications.

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