

ON THE PROJECTIVE FLATNESS OF RANDERS CONFORMAL CHANGED FINSLER METRIC

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Abstract

In the present paper, we first find the fundamental metric tensor of the Randers conformally transformed special (α, β) -metric $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ (ϵ and $k \neq 0$ are constants) and then we find the necessary and sufficient condition for this metric to be locally projectively flat.

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1 Introduction

Let $\alpha = \sqrt{a_{ij}y^i y^j}$ be a Riemannian metric and $\beta = b_i y^i$ be a differential 1-form of an n -dimensional differentiable manifold M^n . The Finsler metric $F = \alpha + \beta$ was originated by G. Randers in his unified field theory and named the Randers space by R. S. Ingarden

[8]. The Kropina metric $F = \frac{\alpha^2}{\beta}$, a kind of (α, β) -metric was investigated by C. Shibata [13] and treated by Ingarden in his paper concerned with thermodynamics [9].

In 1976, Hashiguchi [7] introduced the conformal change of Finsler metrics given by $\bar{F} = e^{\sigma(x)}F$. In particular, he studied the special conformal transformation called as C-conformal. This transformation has also been studied by many authors ([10], [14]). In 1984, Shibata [12] extended the notion of β -change to a general case in Finsler geometry, i.e., $\bar{F} = f(F, \beta)$. Under this transformation, he also studied the change of torsion tensor and curvature tensor, and also dealt with some special Finsler spaces corresponding to specific forms of this function. In 2008, Abed ([1], [2]) generalized the conformal, Randers and generalized Randers changes by a new transformation $\bar{F} = e^{\sigma(x)}F + \beta$. In addition to this, he also obtained the correlation between some relevant tensors associated with (M, F) and the corresponding tensors associated with $(M, \bar{F})$. Further, he discussed the invariant and σ -invariant properties and determined an association between the Cartan connection associated with $(M, \bar{F})$ and the transformed Cartan connection associated with $(M, \bar{F})$.

The main purpose of the current paper is to investigate the special (α, β) -metric $\bar{F} = e^{\sigma} \left(\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha} \right) + \beta$, which is obtained by the Randers conformal change of the special (α, β) -metric $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ (ϵ and $k \neq 0$). The paper is organized as follows:

Starting with literature survey in section one, we find fundamental metric tensor $\bar{g}_{ij}$ and its inverse $\bar{g}^{ij}$ in section two for the Randers conformally transformed metric. In section three, we find the necessary and sufficient conditions for the transformed metric to be locally projectively flat (see Theorem 1).

2 Randers conformal change of special (α, β) metric

A Finsler metric [1] where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a differential one form, which was introduced by physicist Randers in 1941 from view point of general theory of relativity. Further, Matsumoto, Ingarden ([5] and [9]) studied this metric as a Finsler metric and investigated its properties.

In 2008, S. Abed introduced a Finsler metric $\bar{F} = e^{\sigma(x)}F + \beta$, where F is a Finsler metric, $\sigma(x)$ is a conformal factor and β is a one form. This metric is called Randers conformal change of Finsler metric.

Here, we consider the Randers conformal change of the special (α, β) -metric $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ given by

$$\bar{F} = e^{\sigma}\alpha + (e^{\sigma}\epsilon + 1)\beta + ke^{\sigma}\frac{\beta^2}{\alpha}. \quad (1)$$

The differentiation of (1) with respect to y^i yields the normalized supporting element $\bar{l}_i$ given by

$$\bar{l}_i = e^{\sigma}l_i + b_i,$$

In view of (2), we have

$$\bar{l}_i = \frac{e^{\sigma}y_i}{\alpha} + (\epsilon e^{\sigma} + 1)b_i + e^{\sigma}k\beta\left(\frac{2b_i}{\alpha} - \frac{\beta y_i}{\alpha^3}\right), \quad (2)$$

Differentiation of (2) with respect to y^j yields

$$\begin{aligned} \bar{l}_{ij} = & \frac{1}{\alpha}\left(e^{\sigma} - \frac{\beta^2}{\alpha^2}\right)a_{ij} + \frac{1}{\alpha^3}\left(\frac{3\beta^2}{\alpha^2} - e^{\sigma}\right)y_i y_j + \frac{2e^{\sigma}k}{\alpha}b_i b_j \\ & - \frac{2e^{\sigma}k\beta}{\alpha^3}y_j b_i - \frac{2\beta}{\alpha^3}y_i b_j, \end{aligned} \quad (3)$$

From (2) and (3), we obtain

$$\begin{aligned} \bar{h}_{ij} &= \bar{F}\bar{l}_{ij}, \\ &= p_1 a_{ij} + p_2 y_i y_j + p_3 b_i b_j + p_4 (y_j b_i + y_i b_j), \end{aligned} \quad (4)$$

where

$$\begin{aligned} p_1 &= \frac{1}{\alpha} \left(e^\sigma - \frac{\beta^2}{\alpha^2} \right) \left(e^\sigma \alpha + (e^\sigma \epsilon + 1) \beta + \frac{e^\sigma k \beta^2}{\alpha} \right), \\ p_2 &= \frac{1}{\alpha^3} \left(\frac{3\beta^2}{\alpha^2} - e^\sigma \right) \left(e^\sigma \alpha + (e^\sigma \epsilon + 1) \beta + \frac{e^\sigma k \beta^2}{\alpha} \right), \\ p_3 &= 2e^\sigma k \left[e^\sigma + (e^\sigma \epsilon + 1) \frac{\beta}{\alpha} + e^\sigma \frac{k \beta^2}{\alpha^2} \right], \\ p_4 &= -\frac{2e^\sigma k \beta}{\alpha^2} \left(e^\sigma + \frac{(e^\sigma \epsilon + 1) \beta}{\alpha} + \frac{e^\sigma k \beta^2}{\alpha^2} \right). \end{aligned} \quad (5)$$

From (2) and (4), the fundamental metric tensor $\bar{g}_{ij}$ of Finsler space $(M, \bar{F})$ is given by:

$$\begin{aligned} \bar{g}_{ij} &= \bar{h}_{ij} + \bar{l}_i \bar{l}_j, \\ &= \rho a_{ij} + \rho_0 b_i b_j + \rho_1 (b_i \alpha_j + b_j \alpha_i) + \rho_2 (\alpha_i \alpha_j), \end{aligned} \quad (6)$$

where

$$\begin{aligned} \rho &= \frac{1}{\alpha} \left(e^\sigma - \frac{\beta^2}{\alpha^2} \right) \left(e^\sigma \alpha + (e^\sigma \epsilon + 1) \beta + \frac{e^\sigma k \beta^2}{\alpha} \right), \\ \rho_0 &= \frac{2e^{2\sigma} k (3k \beta^2 + \alpha^2) + 4e^\sigma k (\epsilon e^\sigma + 1) \alpha \beta}{\alpha^2}, \\ \rho_1 &= \frac{-4e^{2\sigma} k^2 \beta^3 + e^\sigma (\epsilon e^\sigma + 1) (\alpha^2 - 3k \beta^2) \alpha}{\alpha^5}, \\ \rho_2 &= \frac{e^\sigma (2 - 3e^\sigma k) \beta^2 - e^\sigma (\epsilon e^\sigma + 1) \beta \alpha}{\alpha^6}. \end{aligned} \quad (7)$$

The contravariant metric tensor $\bar{g}^{ij}$ of Finsler space $(M, \bar{F})$ is given by

$$\begin{aligned} \bar{g}^{ij} &= \rho^{-1} \{ a^{ij} - \tau b^i b^j - \eta Y^i Y^j \} \\ &= \rho^{-1} \left\{ a^{ij} - \left(\frac{\delta}{1 + \delta b^2} \right) b^i b^j - \frac{\mu \alpha}{1 + \{ \alpha + (\lambda + \xi) \beta + \lambda \xi b^2 \alpha \}} \mu \right. \\ &\quad \left. \left(\frac{y^i}{\alpha} + \lambda b^i \right) \left(\frac{y^j}{\alpha} + \lambda b^j \right) \right\}, \end{aligned} \quad (8)$$

where

$$\begin{aligned}
 \rho &= \frac{1}{\alpha^3} \left(e^\sigma - \frac{\beta^2}{\alpha^2} \right) \left(e^\sigma \alpha + (e^\sigma \epsilon + 1) \beta + \frac{e^\sigma k \beta^2}{\alpha} \right), \\
 \delta &= \frac{l_1 \alpha^6 + l_2 \alpha^5 \beta + l_3 \alpha^4 \beta^2 + l_4 \alpha^3 \beta^3 + l_5 \alpha^2 \beta^4 + l_6 \alpha \beta^5 - 16 e^{4\sigma} k^4 \beta^6}{l_7 \alpha^{11} \beta + l_8 \alpha^{10} \beta^2 + l_9 \alpha^9 \beta^3 + l_{10} \alpha^8 \beta^4 + l_{11} \alpha^7 \beta^5 + l_{12} \alpha^6 \beta^6}, \\
 \mu &= \frac{e^\sigma (2 - 3e^\sigma k) \beta^2 - e^\sigma (\epsilon e^\sigma + 1) \alpha \beta}{\alpha^2 \left(e^\sigma \alpha^2 - \beta^2 \right) \left(e^\sigma \alpha^2 + (e^\sigma \epsilon + 1) \alpha \beta + e^\sigma k \beta^2 \right)}, \\
 \lambda &= \frac{\alpha \rho_1 \rho - (\rho_2 \rho_0 - \rho_1^2) \beta}{\alpha \rho \rho_2 + (\rho_2 \rho_0 - \rho_1^2) b^2 \alpha}, \\
 &= \left[\alpha^9 q_1 + \alpha^8 \beta q_2 + \alpha^7 \beta^2 q_3 + \alpha^6 \beta^3 q_4 + \alpha^5 \beta^4 q_5 + \alpha^4 \beta^5 q_6 + \alpha^3 \beta^6 q_7 + \right. \\
 &\quad \left. \alpha^2 \beta^7 q_8 - \alpha^6 \beta q_9 - \alpha^5 \beta^2 q_{10} - \alpha^4 \beta^3 q_{11} - \alpha^3 \beta^4 q_{12} - \alpha^2 \beta^5 q_{13} - \alpha \beta^6 q_{14} \right. \\
 &\quad \left. - q_{15} \right] \left[\alpha^{11} \beta^2 q'_1 + \alpha^{10} \beta^3 q'_2 + \alpha^9 \beta^4 q'_3 + \alpha^8 \beta^5 q'_4 + \alpha^7 \beta^6 q'_5 + b^2 (\alpha^7 q_9 + \right. \\
 &\quad \left. \alpha^6 \beta q_{10} + \alpha^5 \beta^2 q_{11} + \alpha^4 \beta^3 q_{12} + \alpha^3 \beta^4 q_{13} + \alpha^2 \beta^5 q_{14} + \alpha \beta^5 q_{15}) \right], \\
 \xi &= \frac{-4e^\sigma k^2 \alpha \beta^3 + (\epsilon e^\sigma + 1) \alpha^2 (\alpha^2 - 3k \beta^2)}{-(\epsilon e^\sigma + 1) \alpha \beta + (2 - 3e^\sigma k) \beta^2},
 \end{aligned} \tag{9}$$

in which,

$$\begin{aligned}
l_1 &= e^{2\sigma}(\epsilon e^\sigma + 1)^2, \\
l_2 &= -2e^{3\sigma}k(\epsilon e^\sigma + 1), \\
l_3 &= 3e^{3\sigma}k^2(4 - 6\epsilon) - 9k^2(1 + 2k)e^{4\sigma} - 9k^2e^{2\sigma}, \\
l_4 &= (\epsilon e^\sigma + 1)(2e^{3\sigma}k(2k - 3) + 8e^{2\sigma}k), \\
l_5 &= 6k^2e^{3\sigma}(2 - 3e^\sigma k) - 9k^2e^{2\sigma}(\epsilon e^\sigma + 1)^2, \\
l_6 &= 24k^3e^{3\sigma}(\epsilon e^\sigma + 1), \\
l_7 &= -e^{3\sigma}(\epsilon e^\sigma + 1), \\
l_8 &= 2e^{3\sigma}(1 - \epsilon) - e^{4\sigma}(3k + \epsilon^2) - e^{2\sigma}, \\
l_9 &= (e^\sigma \epsilon + 1)(3e^{2\sigma} - 4e^{3\sigma}k), \\
l_{10} &= (5k + \epsilon^2)e^{3\sigma} + 2(\epsilon - 1)e^{2\sigma} - 3e^{4\sigma}k^2 + e^\sigma, \\
l_{11} &= (\epsilon e^\sigma + 1)(4e^{2\sigma}k - 2e^\sigma), \\
l_{12} &= e^{2\sigma}k(2 - 3e^\sigma k), \\
q_1 &= e^{3\sigma}(\epsilon e^\sigma + 1), \\
q_2 &= e^{2\sigma}(\epsilon e^\sigma + 1)^2, \\
q_3 &= -e^{2\sigma}(\epsilon e^\sigma + 1)(1 + 2e^\sigma k), \\
q_4 &= -e^{4\sigma}(4k + 2\epsilon^2)k + e^{3\sigma}(\epsilon^2 - 6\epsilon k) + e^{2\sigma}(2\epsilon - 3k) + e^\sigma, \\
q_5 &= (2 - 7e^\sigma k)(\epsilon e^\sigma + 1)e^{2\sigma}k, \\
q_6 &= -4e^{4\sigma}k + e^{3\sigma}(3\epsilon^2 k + 4) + 6\epsilon k e^{2\sigma}, \\
q_7 &= e^{2\sigma}(\epsilon e^\sigma + 1)(4 + 3k^2), \\
q_8 &= 4e^{3\sigma}k^3, \\
q_9 &= -2ke^{4\sigma}(k - 5\epsilon^2) + 4e^{3\sigma}k(1 - 5\epsilon) - 6ke^{2\sigma} - 4e^\sigma k, \\
q_{10} &= -(\epsilon e^\sigma + 1)\{(2k - 3)2e^{3\sigma}k + 8e^{2\sigma}k\}, \\
q_{11} &= e^{2\sigma}(\epsilon e^\sigma + 1)^2, \\
q_{12} &= -2e^{3\sigma}k(\epsilon e^\sigma + 1), \\
q_{13} &= -(\epsilon^2 + 2k)9e^{4\sigma}k^2 + (2 - 3\epsilon)6k^2e^{3\sigma} + 9k^2e^{2\sigma}, \\
q_{14} &= 24e^{3\sigma}(\epsilon e^\sigma + 1)k^3, \\
q_{15} &= -16e^{4\sigma}k^4\beta^7. \\
q'_1 &= 2e^{3\sigma}(1 - \epsilon) - e^{4\sigma}(3k + \epsilon^2) - e^{2\sigma}, \\
q'_2 &= (\epsilon e^\sigma + 1)(3e^{2\sigma} - 4ke^{3\sigma}), \\
q'_3 &= -e^{3\sigma}(\epsilon e^\sigma + 1), \\
q'_4 &= e^{2\sigma}(ke^\sigma - 1)(2 - 3e^\sigma k) + e^\sigma(\epsilon e^\sigma + 1)k, \\
q'_5 &= e^{2\sigma}k(2 - 3e^\sigma k).
\end{aligned}$$

3 Projectively flat (α, β) -metric

A Finsler metric $F = F(x, y)$ on an open subset $U \subset R^n$ is projectively flat [6] if and only if

$$F_{x^k y^l} y^k - F_{x^l} = 0 \quad (10)$$

By (10), we have the following lemma [11]: An (α, β) - metric $F = \alpha\phi(s)$, where $s = \frac{\beta}{\alpha}$, is projectively flat on an open subset $U \subset R^n$ if and only if

$$(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + \alpha^3 Q_{s l 0} + \psi\alpha(-2\alpha Q_{s 0} + r_{00})(b_l\alpha - s y_l) = 0. \quad (11)$$

In this section, we consider the metric $\bar{F} = e^\sigma \left(\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha} \right) + \beta$ which is obtained by the Randers conformal change of the special (α, β) -metric.

$$\bar{F} = \alpha\phi(s), \quad \phi(s) = e^\sigma + (\epsilon e^\sigma + 1)s + e^\sigma k s^2 \quad (12)$$

Let $b_0 > 0$ be the largest number such that

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \leq b < b_0) \quad (13)$$

that is,

$$1 + 2kb^2 - 3ks^2 > 0, \quad (|s| \leq b < b_0) \quad (14)$$

$\bar{F} = e^\sigma \left(\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha} \right) + \beta$ is a Finsler metric iff $\|\beta\|_\alpha < 1$

Proof. If $\bar{F} = e^\sigma \left(\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha} \right) + \beta$ is a Finsler metric, then

$$1 + 2kb^2 - 3ks^2 > 0, \quad (|s| \leq b < b_0) \quad (15)$$

Let $s = b$, then we get $b < \frac{1}{\sqrt{k^2}}$, $\forall b < b_0$. Let $b \rightarrow b_0$, then $b_0 < \frac{1}{\sqrt{k^2}}$. So $\|\beta\|_\alpha < 1$. Now, if

$$|s| \leq b < \frac{1}{\sqrt{k^2}} \quad (16)$$

then

$$1 + 2kb^2 - 3ks^2 > 0, \quad (|s| \leq b < b_0) \quad (17)$$

Thus $\bar{F} = e^\sigma \left(\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha} \right) + \beta$ is a Finsler metric. $\square$

To express formulae for the spray coefficients $\bar{G}^i$ of $\bar{F}$ in terms of α and β , we need to introduce some notations. Let $b_{i;j}$ be a covariant derivative of b_i with respect to y^j . Denote

$$\begin{aligned} r_{ij} &= \frac{1}{2}(b_{i|j} + b_{j|i}), & s_{ij} &= \frac{1}{2}(b_{i|j} - b_{j|i}), \\ s_j^i &= a^{ih}s_{hj}, & s_j &= b_i s_j^i = s_{ij}b^i, & r_j &= r_{ij}b^i, \\ r_0 &= r_j y^j, & s_0 &= s_j y^j, & r_{00} &= r_{ij}y^i y^j. \end{aligned}$$

The spray coefficients $\bar{G}^i$ of the Randers conformally transformed special (α, β) - metric are related to G_α^i by

$$\bar{G}^i = G_\alpha^i + \alpha Q s_0^i + \Theta(-2\alpha Q s_0 + r_{00}) \frac{y^i}{\alpha} + \psi(-2\alpha Q s_0 + r_{00}) b^i, \quad (18)$$

where

$$\begin{aligned} Q &= \frac{(e^\sigma + 1)\alpha^2 + 2ke^\sigma \alpha \beta}{e^\sigma \alpha^2 - ke^\sigma \beta^2}, \\ \Theta &= \left[2e^{2\sigma} k \alpha^4 - (e^{2\sigma} \epsilon^2 + 1)\alpha^3 \beta + (2e^{2\sigma} k^2 - 6\epsilon k e^{2\sigma} - 6e^\sigma k)\alpha^2 \beta^2 - \right. \\ &\quad \left. 6k^2 e^{2\sigma} \beta^4 \right] / \left[-\{4k^2 e^\sigma (1 + e^\sigma)\} \beta^4 - [2k(e^\sigma \epsilon + 1)(e^\sigma + 1)] \alpha \beta^3 - \right. \\ &\quad \left. [4e^{2\sigma} k(1 - b^2 k)] \alpha^2 \beta^2 + [2e^\sigma (\epsilon e^\sigma + 1)] \alpha^3 \beta + 2e^\sigma (1 + 2b^2 k) \alpha^4 \right], \\ \psi &= \frac{e^\sigma k \alpha^2}{(1 + 3kb^2)\alpha^2 - 3k\beta^2}. \end{aligned}$$

Equation (11) is reduced to the following form:

$$\begin{aligned} (a_{ml}\alpha^2 - y_m y_l) G_\alpha^m + \alpha^3 \left[\frac{(e^\sigma + 1)\alpha^2 + 2ke^\sigma \alpha \beta}{e^\sigma \alpha^2 - ke^\sigma \beta^2} \right] s_{l0} + \left[\frac{e^\sigma k \alpha^3}{(1 + 3kb^2)\alpha^2 - 3k\beta^2} \right] \\ \left[-2\alpha \left[\frac{(e^\sigma + 1)\alpha^2 + 2ke^\sigma \alpha \beta}{e^\sigma \alpha^2 - ke^\sigma \beta^2} \right] s_0 + r_{00} \right] \left(b_l \alpha - \frac{\beta}{\alpha} y_l \right) = 0. \end{aligned} \quad (19)$$

If $(a_{ml}\alpha^2 - y_m y_l) G_\alpha^m = 0$, then α is projectively flat.

Proof. If $(a_{ml}\alpha^2 - y_m y_l) G_\alpha^m = 0$, then

$$a_{ml}\alpha^2 = y_m y_l G_\alpha^m,$$

then there is a $\eta = \eta(x, y)$ such that $y_m G_\alpha^m = \alpha^2 \eta$, we get

$$a_{ml} G_\alpha^m = \eta y_l.$$

Contracting with a^{il} yields $G_\alpha^i = \eta y^i$, and thus α is projectively flat. $\square$

Theorem 1. *The (α, β) -metric $\bar{F} = e^\sigma \left(\alpha + \epsilon \beta + k \frac{\beta^2}{\alpha} \right) + \beta$ is locally projectively flat iff*

(i) β is parallel with respect to α .

(ii) α is locally projectively flat, i. e., of constant curvature.

Proof. Suppose that $\bar{F}$ is locally projectively flat. First, we rewrite (19) as a polynomial in y^i and α . This gives,

$$\begin{aligned} & (a_{ml} \alpha^2 - y_m y_l) G_\alpha^m \left[\{ (1 + 3kb^2) \alpha^2 - 3k\beta^2 \} \{ e^\sigma \alpha^2 - k e^\sigma \beta^2 \} \right] + \\ & \alpha^3 \{ (e^\sigma + 1) \alpha^2 + 2k e^\sigma \alpha \beta \} \{ e^\sigma \alpha^2 - k e^\sigma \beta^2 \} s_{l0} - \{ 4k^2 e^{2\sigma} \alpha^4 \beta s_0 - \\ & e^{2\sigma} k \gamma_{00} \alpha^4 \} (b_l \alpha^2 - \beta y_l) + \alpha \left\{ -2k e^\sigma (e^\sigma + 1) \alpha^4 s_0 - e^{2\sigma} k^2 \alpha r_{00} e^\sigma \beta^2 \right\} \\ & (b_l \alpha^2 - \beta y_l) = 0. \end{aligned}$$

or

$$U + \alpha V = 0, \quad (20)$$

where

$$\begin{aligned} U &= (a_{ml} \alpha^2 - y_m y_l) G_\alpha^m \left[\{ (1 + 3kb^2) \alpha^2 - 3k\beta^2 \} \{ \alpha^2 - k\beta^2 \} \right] + \\ & 2k e^\sigma \alpha^4 \beta [\alpha^2 - k\beta^2] s_{l0}, \end{aligned}$$

and

$$V = \left[-2k \alpha^2 \{ (e^\sigma + 1) \alpha^2 + 2k e^\sigma \alpha \beta \} s_0 + k \alpha r_{00} \{ e^\sigma \alpha^2 - k e^\sigma \beta^2 \} \right] (b_l \alpha^2 - \beta y_l).$$

Now, (20) is a polynomial in (y^i) , such that U and V are rational in y^i and α is irrational. Therefore, we must have

$$U = 0 \text{ and } V = 0, \quad (21)$$

which implies that

$$(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m \left[\{(1 + 3kb^2)\alpha^2 - 3k\beta^2\} \{\alpha^2 - k\beta^2\} \right] + 2k\alpha^4\beta[e^\sigma\alpha^2 - ke^\sigma\beta^2]_{s_{l0}} = 0, \quad (22)$$

and

$$\left[-2k\alpha^2 \{ (e^\sigma + 1)\alpha^2 + 2ke^\sigma\alpha\beta \} s_0 + e^\sigma k\alpha r_{00} \{ \alpha^2 - k\beta^2 \} \right] (b_l\alpha^2 - \beta y_l) = 0.$$

From (20), considering only terms which do not contain β . There exists a homogenous polynomial V_7 of degree seven in y^i such that

$$\left\{ (1 + 3kb^2)(e^\sigma\epsilon + 1)s_{l0} - 2e^\sigma k(e^\sigma\epsilon + 1)b_l \right\} \alpha^7 = \beta V_7. \quad (23)$$

Since $\alpha^2 o(mod\beta)$, we must have a function $u^l = u^l(x)$ satisfying

$$(1 + 3kb^2)(e^\sigma\epsilon + 1)s_{l0} - 2e^\sigma k(e^\sigma\epsilon + 1)b_l = u^l\beta. \quad (24)$$

Transvecting (24) by b_l , we have

$$(\epsilon e^\sigma + 1)(1 + 3kb^2 - 2e^\sigma kb^2)s_0 = u^l\beta b^l \quad (25)$$

That is,

$$(\epsilon e^\sigma + 1)(1 + 3kb^2 - 2e^\sigma kb^2)s_j = u^l b_l b_j. \quad (26)$$

Further transvecting by b^j , we have $u^i b_i b^2 = 0$, which implies $u^l b_l = 0$. Substituting this equation into (25), we get $s_0 = 0$. Now, from (22), by contracting with b^l , we get

$$(b_m\alpha^2 - y_m\beta)G_\alpha^m \left[\{(1 + 3kb^2)\alpha^2 - 3k\beta^2\}(\alpha^2 - k\beta^2) \right] + 2k\alpha^4\beta[e^\sigma\alpha^2 - ke^\sigma\beta^2]_{s_0} = 0, \quad (27)$$

$$\left[-2k\alpha^2 \left((e^\sigma + 1)\alpha^2 + 2ke^\sigma\alpha\beta \right) s_0 + k\alpha r_{00} \left(e^\sigma\alpha^2 - ke^\sigma\beta^2 \right) \right] (b^2\alpha^2 - \beta^2) = 0.$$

(28)

Since $s_0 = 0$, from (27) and (28), we get

$$(b_m \alpha^2 - y_m \beta) G_\alpha^m = 0, \quad (29)$$

$$\text{and} \quad r_{00} = 0. \quad (30)$$

From (29) and (22), we get

$$s_{l0} = 0. \quad (31)$$

Then by (29) and Lemma (3), α is projectively flat. From (29)-(31), $b_{i;j} = 0$, i. e., β is parallel to α .

Conversely, if β is parallel with respect to α and α is locally projectively flat, then by Lemma (3), we can easily see that F is locally projectively flat. $\square$

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