

**CREATION OF FRACTAL OF A
GRAPH USING THE GRAPH
POLYNOMIALS RATHER THAN
INDEPENDENCE POLYNOMIAL**

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Abstract

Most things in the world can be modeled into graphs, and each graph into a fractal, which is the independence fractal. For instance, in architecture, the ground plan of a building is converted to a graph by taking rooms as vertices and edges as the direct connection between two rooms. After that, these graphs changed to the corresponding fractal using its independence polynomial, and this forms the very basis of the study of architecture's ground plan. This simple example proves the necessity of the study of independence fractals.

Now the question of consideration is, can other polynomials defined in graphs, such as, chromatic polynomial, dichromatic polynomial, matching polynomial, Tutte polynomial, etc., be used to create fractals instead of independence polynomial.

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1. Introduction

French mathematician Benoit Mandelbrot invented the term fractal from the Latin word fractus in the 1980s, [3]. As the name implies, fractals are self-similar objects, [3]. The concept of independence fractals was established in the 21st century, existing results on the roots of polynomials and their limiting behavior under iteration [1] provide theoretical support for studying the independence fractal. The fractal of a graph is explained using independence polynomials and is called the independence fractals. A graph may have many such polynomials like matching polynomials, chromatic polynomials, dichromatic polynomials, etc. ([7]). Now the question is, is it possible to define the fractal of the graph using these polynomials? In this paper, we intend to discuss the criteria for interpreting the fractal of a graph. That is, to check whether other polynomials of a graph can be used to create a fractal of the graph. To reach at a conclusion on this, we attempt to take general polynomials rather than independence polynomials of a graph and check the same.

DEFINITION 1.1. [6] Let G be a graph. The independence polynomial of G , denoted by $i_G(s)$, is given by $i_G(s) = \sum_{k=0}^{\alpha} i_k s^k$, where i_k be the number of independent sets of order k and α be the independence number of G .

For any graph G $i_0 = 1$ and i_1 is the number of vertices of G , [6].

EXAMPLE 1.1. Let G be the graph given in Figure 1. The independence number of the graph is 3, so need to find i_2 and i_3 . The independent sets of order 2 are $\{V_1, V_4\}$, $\{V_1, V_5\}$, $\{V_2, V_4\}$, $\{V_2, V_5\}$, $\{V_4, V_5\}$ and $i_2 = 5$. The independent set of order 3 are $\{V_1, V_4, V_5\}$, $\{V_2, V_4, V_5\}$ and $i_3 = 2$. So the independence polynomial $i_G(s) = 2s^3 + 5s^2 + 5s + 1$.

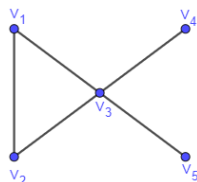


FIGURE 1

DEFINITION 1.2. [6] The lexicographic product or composition $G[H]$ of graphs G and H is a graph such that:

- (1) the vertex set of $G[H]$ is the cartesian product $V(G) \times V(H)$ and
- (2) Any two vertices (u, v) and (x, y) are adjacent in $G[H]$ if and only if either u is adjacent with x in G or $u = x$ and v is adjacent with y in H .

EXAMPLE 1.2. For the graphs $G = P_3$ and $H = P_2$, the lexicographic product $G[H]$ is given in Figure 2.

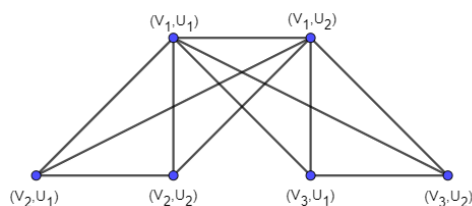


FIGURE 2

THEOREM 1.1. [8] The independence polynomial of $G[H]$, the lexicographic product of G and H , is given by $i_{G[H]}(s) = i_G(I_H(s) - 1)$.

EXAMPLE 1.3. The independence polynomial of lexicographic product of G and H in the above example is $i_{P_3[P_2]}(s) = 1 + 3(2s) + (2s)^2 = 1 + 6s + 4s^2$, since $i_{P_3}(s) = 1 + 3s + s^2$ and $i_{P_2}(s) = 1 + 2s$.

DEFINITION 1.3. [8] The reduced independence polynomial of G is the function $f_G(s) = i_G(s) - 1$.

COROLLARY 1.1. [6] $f_{G[H]}(s) = f_G(f_H(s))$.

DEFINITION 1.4. [8] The independence fractal of a graph G is defined as the set $\mathcal{F}(G) = \lim_{k \rightarrow \infty} \text{Roots}(f_{G^k})$.

EXAMPLE 1.4. For the graph $P_3[P_2]$, the lexicographic product of P_3 and P_2 , the independence fractal is

$$\begin{aligned} \mathcal{F}(P_3[P_2]) = \lim_{k \rightarrow \infty} \text{Roots}(f_{P_3[P_2]^k}) = \{ & -1.5, -1.71510032 + 0.14809861 \\ & i, -1.71510032 - 0.14809861i, -1.61201523 + 0.26286997i, -1.612015 \\ & 23 - 0.26286997i, -1.46912614 + 0.32277849i, -1.46912614 - 0.32277 \\ & 849i, -1.31730480 + 0.32486245i, -1.31730480 - 0.32486245i, -1.181 \\ & 83485 + 0.27861415i, -1.18183485 - 0.27861415i, -1.08033723 + 0.19 \\ & 611604i, -1.08033723 - 0.196116040i, -1.027304930 + 0.092421770i, \\ & -1.02730493 - 0.09242177i, -1.02107599, -0.317638133, -0.31264272 \\ & 4, -0.307666027, -0.300367306, -0.254566930, -0.248416946, -0.237600 \\ & 533, -0.233389867, -0.0548358644, -0.0520452693, -0.0436247372, -0.0 \\ & 399867279, -0.00919568447, -0.00669433069, -0.00153418322, 0, \dots \}. \end{aligned}$$

DEFINITION 1.5. [6, 8] Let f be a polynomial, and z_0 a point which does not lie in any attracting cycle or Siegel disk of f . Then $\lim_{k \rightarrow \infty} f^{-k}(z_0) = J(f)$, where the limit taken with respect to the Hausdorff metric on compact subspaces of $(\mathbb{C}, |\cdot|)$.

DEFINITION 1.6. [8] The forward orbit of a point z_0 with respect to f is the set: $\mathcal{O}^+(z_0) = \{f^k(z_0)\}_{k=0}^{\infty}$.

DEFINITION 1.7. [8] For a polynomial f , its filled Julia set $K(f)$ is the set of all points z whose forward orbit $\mathcal{O}^+(z)$ is bounded in $(\mathbb{C}, |\cdot|)$. The Julia set of f , $J(f)$ is the boundary of $K(f)$. The Fatou set $F(f)$ is the complement of $J(f)$ in $\mathbb{C}$.

2. Fractals from graph polynomials

There are only four non-isomorphic graphs of 3 vertices given in Figure 3. The corresponding independent polynomials are:

$$\begin{aligned} i_{G_1}(s) &= s^3 + 3s^2 + 3s + 1 \\ i_{G_2}(s) &= 2s^2 + 3s + 1 \\ i_{G_3}(s) &= s^2 + 3s + 1 \\ i_{G_4}(s) &= 3s + 1. \end{aligned}$$

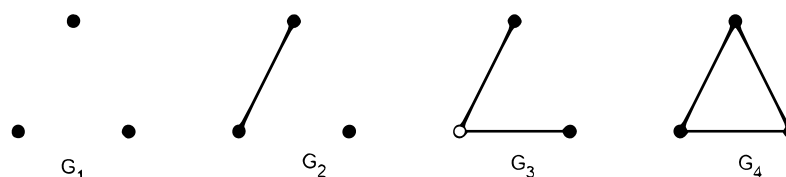


FIGURE 3

It is easy to see that the coefficient of s in each of these polynomials is 3. Since, in the independence polynomial of a graph G , the coefficient of s is the number of vertices of G . But there exists an infinite number of polynomials having three as the coefficient of s . Now our question that:

“Is there a polynomial $p(s)$ having three as the coefficient of s which is not an independence polynomial of a graph of 3 vertices such that:

$$\lim_{k \rightarrow \infty} \text{Roots}\{p^k(s)\} = J(p),”$$

where $J(p)$ is the Julia set of the polynomial $p(s)$. To check whether such a polynomial exists or not, consider some examples.

EXAMPLE 2.1. Let $p(s) = s^3 + s^2 + 3s$, $p(s)$ is not an independence polynomial of any graph of 3 vertices. Since $p(0) = 0$, 0 is fixed point of $p(s)$, and $p'(s) = 3s^2 + 2s + 3$, so $p'(0) = 3$. This shows that 0 is a repelling fixed point of $p(s)$. It follows that 0 lies in the $J(p)$. Also since 0 is a repelling fixed point, which does not lie on any attracting cycle and therefore $\lim_{k \rightarrow \infty} p^{-k}(s) = J(p)$. That is, $\lim_{k \rightarrow \infty} \text{Roots}\{p^k(s)\} = J(p)$.

EXAMPLE 2.2. Let $h(s) = s^4 + 2s^3 + 4s^2 + 3s$, $h(s)$ is not an independence polynomial of any graph of 3 vertices. Since $h(0) = 0$, 0 is fixed point of $h(s)$, and $h'(s) = 4s^3 + 6s^2 + 8s + 3$, so $h'(0) = 3$. This shows that 0 is a repelling fixed point of $h(s)$. It follows that 0 lies in the $J(h)$. Also since 0 is a repelling fixed point, which does not lie on any attracting cycle and therefore $\lim_{k \rightarrow \infty} h^{-k}(s) = J(h)$. That is $\lim_{k \rightarrow \infty} \text{Roots}\{h^k(s)\} = J(h)$.

Now check whether every polynomial with integer coefficients satisfies this property. If not, what are the conditions to satisfy this property?

EXAMPLE 2.3. Let $g(s) = 6s^3 + 4s^2 + s$, $g(s)$ is not an independence polynomial of any graph. Since $g(0) = 0$, 0 is fixed point of $g(s)$, and $g'(s) = 18s^2 + 8s + 1$ so $g'(0) = 1$. This shows that 0 is a

rationally indifferent fixed point of $g(s)$. It follows that 0 lies in the $J(g)$. Also since 0 is a rationally indifferent fixed point, which does not lie on any attracting cycle and therefore $\lim g^{-k}(s) = J(g(s))$. That is, $\lim_{k \rightarrow \infty} \text{Roots}\{g^k(s)\} = J(g)$.

EXAMPLE 2.4. Let $f(s) = s^2 + 1$, $f(s)$ is not an independence polynomial of any graph. Here $f(0) = 1$ and therefore 0 is not a fixed point of $f(s)$. Now $f^2(s) = (s^2 + 1)^2 + 1$, $f^3(s) = f(f^2(s)) = ((s^2 + 1)^2 + 1)^2 + 1, \dots$ and $f^2(0) = 2, f^3(0) = 5, f^4(0) = 26, \dots$, this means $f^k(0) \neq 0$ for $k = 1, 2, 3, \dots$ that is 0 is not a periodic point of $f(s)$. To check whether $\lim_{k \rightarrow \infty} \text{Roots}\{f^k(s)\} = J(f)$, consider the zeros of $f(s)$, that is i and $-i$, and the forward orbit of i and $-i$ are given below:

$$\mathcal{O}^+(i) = \{i, 0, 1, 2, 5, 26, \dots\}; \quad \mathcal{O}^+(-i) = \{-i, 0, 1, 2, 5, 26, \dots\}.$$

This gives the forward orbit of i and $-i$ are unbounded, so $i, -i \notin J(f)$ but both are lies in $\lim_{k \rightarrow \infty} \text{Roots}\{f^k(s)\}$. Hence $\lim_{k \rightarrow \infty} \text{Roots}\{f^k(s)\} \neq J(f)$.

THEOREM 2.1. For any polynomial $p(s) = a_0 + a_1s + a_2s^2 + \dots + a_ns^n$ where $a_i \in \mathbb{Z}$ for $i = 0, 1, 2, \dots, n$. If $a_0 = 0$ and $a_1 \neq 0$ or 0 is a periodic point of order m , $m > 1$, and $a_m \neq 0$ then $\lim_{k \rightarrow \infty} \text{Roots}\{p^k(s)\} = J(p)$, where $J(p)$ is the Julia set of the polynomial $p(s)$. Moreover the result holds if $p(s)$ is a complex polynomial, 0 is a periodic point of $p(s)$ of order m and $|a_m| \geq 1$.

P r o o f. Let $p(s) = a_0 + a_1s + a_2s^2 + \dots + a_ns^n$, where $a_i \in \mathbb{Z}$ for $i = 0, 1, 2, \dots, n$.

Case 1: $a_0 = 0$ and $a_1 \neq 0$. Since $a_0 = 0$ we have $p(s) = a_1s + a_2s^2 + \dots + a_ns^n$ and $p(0) = 0$. That is zero is a fixed point of p . Now $p'(s) = a_1 + 2a_2s + 3a_3s^2 + \dots + na_ns^{n-1}$, then $p'(0) = a_1$. Given that $a_1 \neq 0$ so $|a_1| \geq 1$, thus 0 is not in the attracting cycle of p . Therefore, $\lim_{k \rightarrow \infty} \text{Roots}\{p^k(s)\} = \lim_{k \rightarrow \infty} p^{-k}(0) = J(p)$.

Case 2: 0 is a periodic point of order m and $a_m \neq 0$.

Subcase 1: If $m > n$, then $a_m = 0$, nothing to prove.

Subcase 2: if $m \leq n$, given 0 is a periodic point and $a_m \neq 0$.

$$\begin{aligned} p(s) &= a_0 + a_1s + a_2s^2 + \dots + a_ns^n \\ p'(s) &= a_1 + 2a_2s + 3a_3s^2 + \dots + na_ns^{n-1} \\ p''(s) &= 2a_2 + 6a_3s + 12a_4s^2 + \dots + n(n-1)a_ns^{n-2}. \end{aligned}$$

Continuing like this we get the m^{th} derivative of $p(s)$ as $p^m(s) = m!a_m + (m+1)!a_{m+1}s + (m+2)(m+1)m...3a_{m+2}s^2 + ... + n(n-1)(n-2)...(n-m+1)a_ns^n$. Thus $|p^m(0)| = |m!a_m| \geq 1$, since $a_m \neq 0$. This shows that 0 is not in the attracting cycle. Thus we have

$$\lim_{k \rightarrow \infty} Roots\{p^k(s)\} = \lim_{k \rightarrow \infty} p^{-k}(0) = J(p).$$

Let $p(s)$ be a complex polynomial and 0 is a periodic point of order m and also $|a_m| \geq 1$. Thus $|p^m(0)| = |m!a_m| \geq 1$ and 0 is not in the attracting cycle. Hence $\lim_{k \rightarrow \infty} Roots\{p^k(s)\} = \lim_{k \rightarrow \infty} p^{-k}(0) = J(p)$. $\square$

EXAMPLE 2.5. Let $t(s) = s^2 - 1$. The zeros of $t(s)$ are 1 and -1 , it shows that 0 is not a fixed point of $t(s)$. Now $t(t(s)) = t(s^2 - 1) = (s^2 - 1)^2 - 1 = s^4 - 2s^2 + 1 - 1 = s^4 - 2s^2$, and $t^2(0) = 0$. Therefore, 0 is a periodic point of $t(s)$ with periodicity 2. Also $(t^2)'(s) = 4s^3 - 4s$ and $(t^2)'(0) = 0$, hence 0 is in the super attracting cycle and therefore lies in $F(t)$. 0 is a zero of t^2 and lies in $F(t)$, it shows that $\lim_{k \rightarrow \infty} Roots\{t^k(s)\} \neq J(t)$. To prove that the forward orbit of each zeros of t^n is bounded, find a bound for each. Let z_{n_k} be the zeros of $t^n(s)$, $k = 1, 2, 3, \dots, d$, where d is the degree of $t^n(s)$. Since $z_{1_k} = 1, -1$, it follows that $|z_{1_k}| = 1$. Assume that $z_{(n-1)_k}$ are the zeros of $t^{n-1}(s)$ and which satisfies the condition $|z_{(n-1)_k}| \leq 1.6$. Let z_{n_k} be the zeros of $t^n(s)$. $t(z_{n_k}) = 0 \Rightarrow t^{k-1}(t(z_{n_k})) = 0 \Rightarrow t(z_{n_k})$ is a zero of $t^{k-1}(s)$. Thus,

$$\begin{aligned} |t(z_{n_k})| &\leq 1.6 \\ -1.6 \leq t(z_{n_k}) &\leq 1.6 \\ -1.6 \leq (z_{n_k})^2 - 1 &\leq 1.6 \\ 0 \leq (z_{n_k})^2 &\leq 2.6 \\ -1.6 \leq z_{n_k} &\leq 1.6 \\ |z_{n_k}| &\leq 1.6. \end{aligned}$$

This shows that each zero of t^n , for all n , lies in the interval $[-1.6, 1.6]$ and $t([-1.6, 1.6]) \subseteq [-1.6, 1.6]$. Therefore the forward orbit of each of the zero of t^n is bounded. Thus all the zeros of t^n lies in the $K(t)$, the filled Julia set of t .

THEOREM 2.2. For any polynomial $p(s) = a_0 + a_1s + a_2s^2 + \dots + a_ns^n$, where $a_i \in \mathbb{Z}$ for $i = 0, 1, 2, \dots, n$. If 0 is not a fixed point and not a periodic point of p and $a_i \geq 0$ for $i = 0, 1, 2, \dots, n$, $a_i \leq 0$ for $i =$

0, 1, 2, ..., n or $a_i \neq 0$ and the sequence $(p^n(0))$ converges to $\pm\infty$. Then $\lim_{k \rightarrow \infty} \text{Roots}\{p^k(s)\} \neq J(p)$.

P r o o f. Let $p(s) = a_0 + a_1s + a_2s^2 + \dots + a_ns^n$ be a non-constant polynomial, where $a_i \in \mathbb{Z}$. Suppose that 0 is not a fixed point and not a periodic point of p , that is $p^k(0) \neq 0$ for all $k \in \mathbb{N}$. Since $p(s)$ is non-constant $a_i \neq 0$ for some i . Let k be the largest integer in $\{0, 1, 2, \dots, n\}$ such that $a_k \neq 0$.

Case 1: $a_i \geq 0$ for $i = 0, 1, 2, \dots, n$.

To prove the sequence $(p^n(0))$ is strictly monotonic increasing sequence. Since 0 is not a fixed point of p , $p(0) = a_0 > 0$ and $p^2(0) = p(p(0)) = p(a_0) = a_0 + a_1a_0 + a_2a_0^2 + \dots + a_ka_0^k + \dots + a_na_0^n$, $a_ka_0^k > a_0$ since $a_k \neq 0$. Hence $p(0) < p^2(0)$. In general $p^{n+1}(0) = p(p^n(0)) = a_0 + a_1p^n(0) + a_2p^n(0)^2 + \dots + a_kp^n(0)^k + \dots + a_np^n(0)^n$, here $a_kp^n(0)^k > p^n(0)$ for all $n \in \mathbb{N}$. This shows that the sequence $(p^n(0))$ is strictly monotonic increasing sequence. Therefore the sequence $(p^n(0))$ converges to ∞ , since $a_i \geq 0$.

Case 2: $a_i \leq 0$ for $i = 0, 1, 2, \dots, n$.

In this case the sequence $(p^n(0))$ is strictly monotonic decreasing (as in Case 1) and therefore, converges to $-\infty$.

Case 3: $a_i \neq 0$ for $i = 0, 1, 2, \dots, n$. Given that the sequence $(p^n(0))$ converges to $\pm\infty$ in this case.

Now we need to prove the result, so let $x_1, x_2, x_3, \dots, x_n$ be zeros of $p(s)$, thus it belongs to $\lim_{k \rightarrow \infty} \text{Roots}\{p^k(s)\}$. Let us consider the forward orbit of each of these zeros. $\mathcal{O}^+(x_t) = \{p^k(x_t)\}_{k=0}^\infty = \{x_t, 0, p(0), p^2(0), \dots\}$, $t = 1, 2, \dots, n$, which is unbounded since in each case $(p^n(0))$ is converges to $\pm\infty$. It follows that $x_t \notin J(p)$, $t = 1, 2, \dots, n$. Hence,

$$\lim_{k \rightarrow \infty} \text{Roots}\{p^k(s)\} \neq J(p).$$

□

This theorem has been already illustrated in Example 2.4. As a conclusion, polynomials of graph that satisfy the conditions in Theorem 2.2 can be used to create a fractal of the graph.

3. Special case

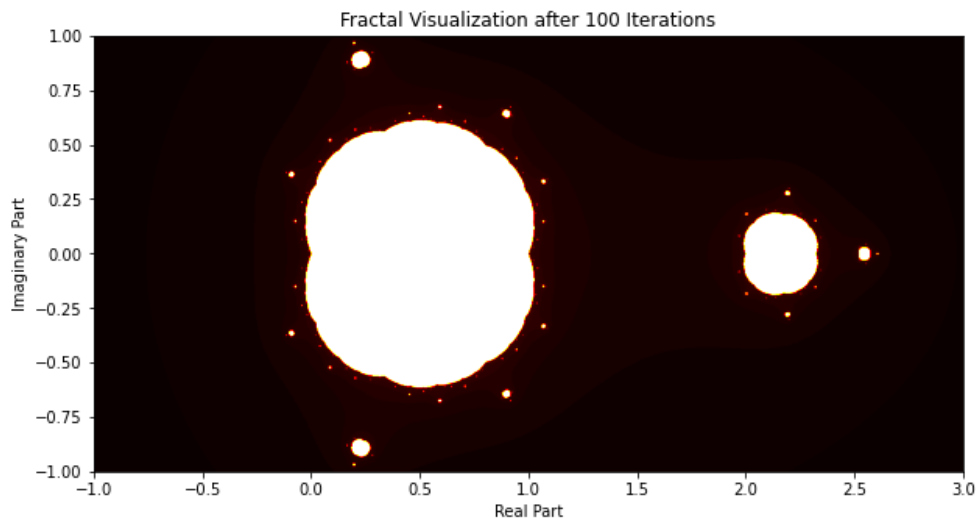
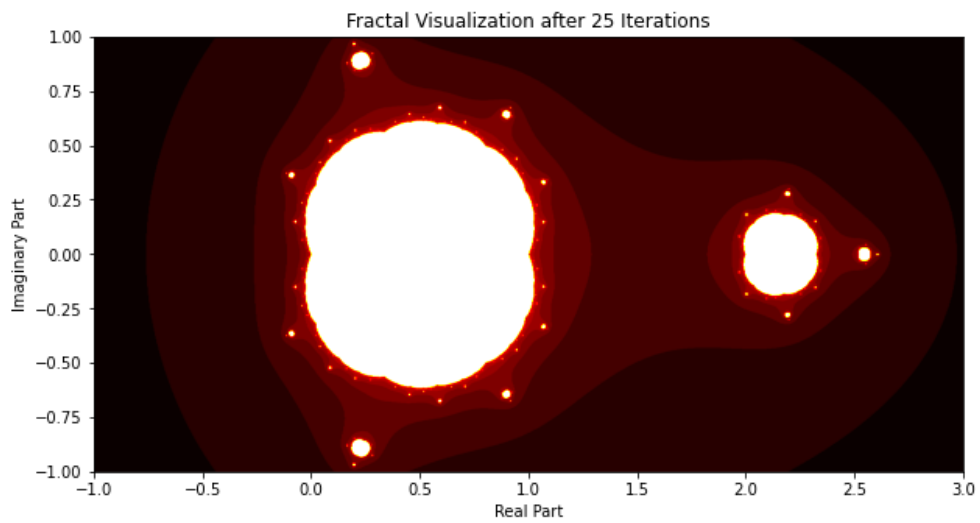
The number of proper coloring of a graph G using s colors is the chromatic polynomial and is denoted as $\chi_G(s)$, [11]. For a graph G ,

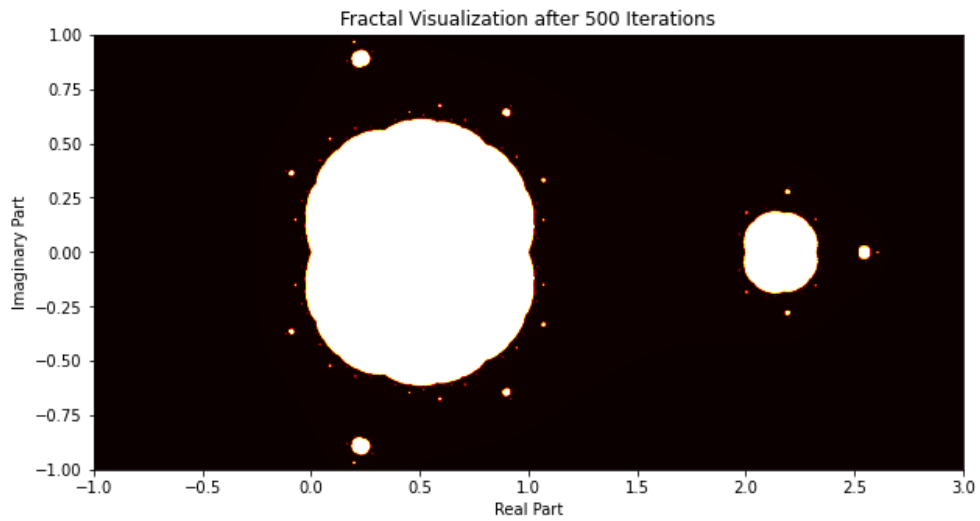
the constant term of $\chi_G(s)$ is always zero and hence 0 is a fixed point of $\chi_G(s)$, [11]. For a connected graph G the coefficient of s in $\chi_G(s)$ is always nonzero, [11]. Therefore using Theorem 2.2, define chromatic fractals of connected graphs as follows.

DEFINITION 3.1. The chromatic fractal of a connected graph G is defined as $\mathcal{F}_\chi(G) = \lim_{k \rightarrow \infty} \text{Roots}(\chi_G^k)$, where $\chi_G(s)$ is the chromatic polynomial of G .

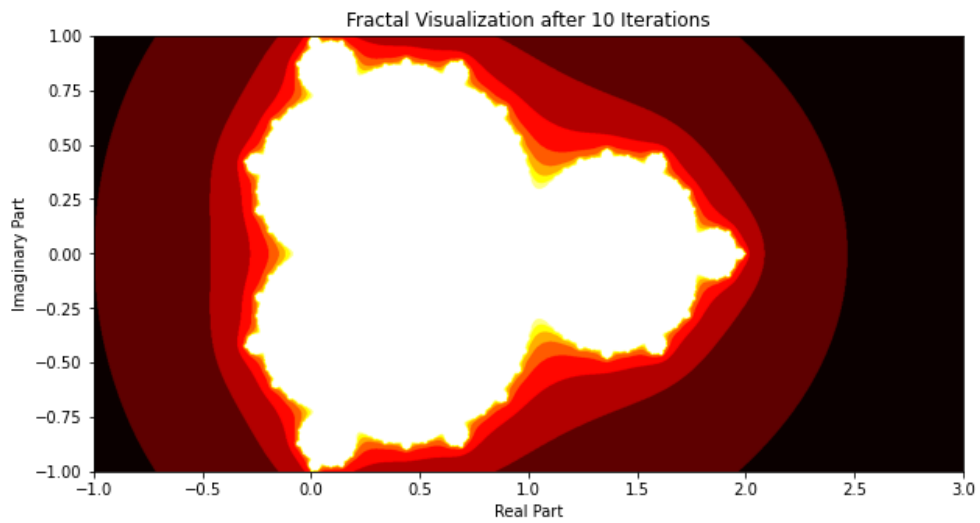
EXAMPLE 3.1. Consider the graph $G = K_3$, the complete graph on 3 vertices. The chromatic polynomial of K_3 is $s(s-1)(s-2)$ or $s^3 - 3s^2 + 2s$. Here the chromatic fractal of K_3 .

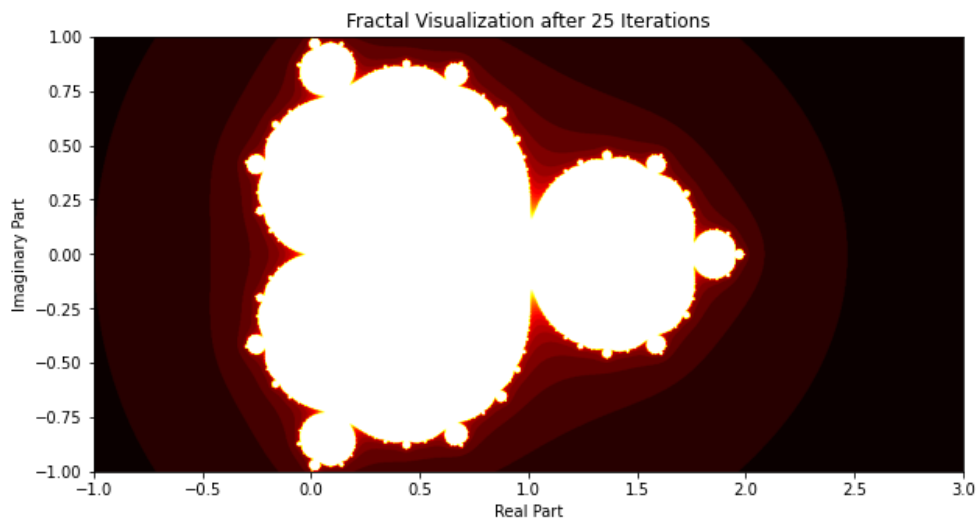


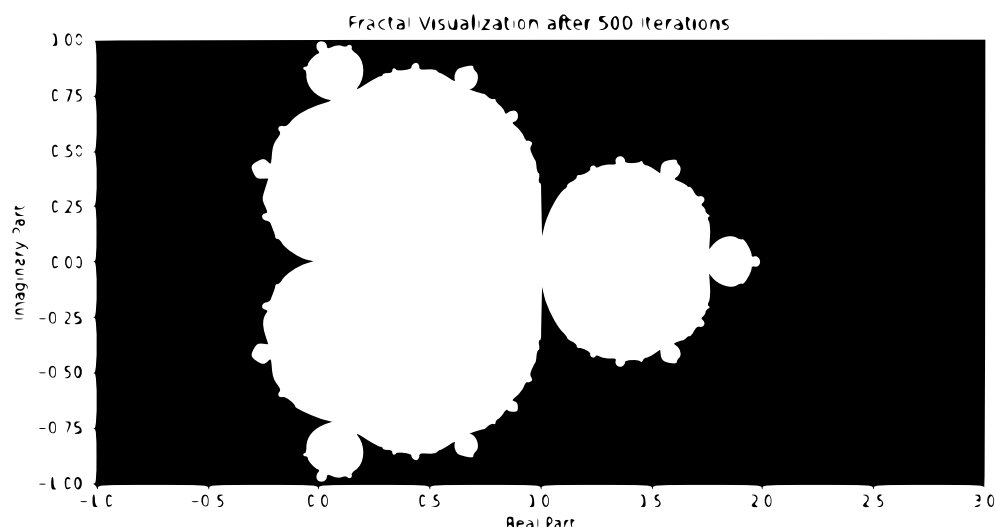




EXAMPLE 3.2. Consider the graph $G = P_3$, the path on 3 vertices. The chromatic polynomial of P_3 is $s(s-1)^2$ or $s^3 - 2s^2 + s$. Here the chromatic fractal of P_3 .







4. Conclusion

Independence fractals have many applications in our daily lives. So the study of fractal of a graph is essential to understand its impact on our day-to-day activities. In this paper, we intended to discuss the fractal of a graph and also the criteria for defining the fractal of graphs using graph polynomials. In conclusion, we reached the inference that it is possible to create the fractal of a graph using any polynomials be it matching polynomials, chromatic polynomials, di-chromatic polynomials, or independence polynomials.

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