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GENERALIZED COLOR COMPLEMENTS IN GRAPHS

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Abstract

Let $G_c = (V, E)$ be a color graph, and $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of order $k \geq 1$. The k and $k(i)$-color complement of G_c is defined as follows: For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j and add the edges which are not in G_c such that end vertices have different colors. For each subset V_r in the partition P, remove the edges of G_c that exist within V_r and add the edges of $\overline{G_c}$ joining the vertices of V_r. The resulting graph $(G_c)^P_{k(i)}$ is known as $k(i)$-color complement of G_c with respect to the partition P of V.

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This paper establishes connectivity conditions for the k-color complement and $k(i)$-color complement of a connected graph based on specific vertex partitioning and color assignments. Additionally, the relationship between clique numbers and independence numbers in the generalized color complements is explored with respect to same color class partitions, and the number of edges is determined for certain graph families.

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Key Words and Phrases: generalized color complement, connectivity, clique, independent set, partition

1. **Introduction**

In this work, all graphs are finite, undirected, no loops and multiple edges. Let $n = |V|$ and $m = |E|$ denote the number of vertices and edges in a graph G, respectively. The complement of graph G, denoted by $\overline{G}$, shares the same vertex set as G, with two vertices being adjacent in G if and only if they are non-adjacent in $\overline{G}$. A graph G is called self-complementary if it is isomorphic to its complement. For any two vertices u and v in G, adjacency is denoted as $u \sim v$, and non-adjacency is denoted as $u \nsim v$. The degree of a vertex v in a graph, denoted by $d(v)$, is the number of edges incident to v. For notation and graph theory terminology we generally follow [2]. Graph partitioning is a key problem with applications in computing, engineering, and network science, including clustering, route optimization, and biological networks. In 1998, Sampathkumar et al. [4] introduced a complement definition based on vertex partitions. Later in 2020, Swati Nayak et al. [3] defined generalized color complements of graphs. Coloring all the vertices of a graph such that no two adjacent vertices have the same color is called the proper coloring of a graph. A graph in which every vertex has been assigned a color according to a proper coloring is called a properly colored graph. The proper coloring which is of interest to us is one that requires the minimum number of colors. A graph G that requires k different colors for its proper coloring, and no less, is called a k-chromatic graph, and the number k is called the chromatic number of G and is denoted by $\chi(G) = k$. In this article, we denote $G_c = (V, E)$ as a colored graph. Then the complement of graph G_c, denoted by $\overline{G_c}$ has the same coloring of G_c with the following properties [3]:

- u and v are adjacent in $\overline{G_c}$, if u and v are non-adjacent in G_c with $c(u) \neq c(v)$.
- u and v are non-adjacent in $\overline{G_c}$, if u and v are non-adjacent in G_c with $c(u) = c(v)$.
- u and v are non-adjacent in $\overline{G_c}$, if u and v are adjacent in G_c.

DEFINITION 1.1. [1] The tadpole graph, $T_{p,q}$ is a special type of graph consisting of a cycle graph on p (at least 3) vertices and a path graph on q vertices, connected with a bridge.

DEFINITION 1.2. A set M of vertices in a graph G is independent if no two vertices of M are adjacent. The number of vertices in a maximum independent set of G is denoted by $\beta(G)$. Opposite to an independent set of vertices in a graph is a clique. A clique in a graph G is a complete subgraph of G. The clique number of a graph G, denoted $\omega(G)$, is the order of the largest clique in G.

2. **Main results**

THEOREM 2.1. *The k-color complement of a connected graph G_c remains connected if the vertex set is partitioned as $P = \{V_1, V_2, \ldots, V_k\}$, under the following conditions:*

i. *There exists a path between every pair of vertices within each partite set V_i.*
ii. *For each V_i, there is at least one vertex that is non-adjacent to at least one vertex in a different partite set $V_j, i \neq j$, and these non-adjacent vertices must have distinct colors.*

P r o o f. Let G_c be a connected graph with a vertex partition $P = \{V_1, V_2, \ldots, V_k\}$. Since each partite set is internally connected, all vertices within V_i have a path between them. Additionally, at least one vertex in V_i is non-adjacent to at least one vertex in V_j for $j \neq i$. If non-adjacent vertices shared the same color, no edge would be added between them in the k-color complement, potentially causing disconnection. However, when non-adjacent vertices receive distinct colors, the necessary edges are introduced to preserve connectivity. These edges create links between different partite sets, ensuring that every vertex remains reachable. Therefore, the k-color complement of G_c remains connected. $\square$

THEOREM 2.2. *The $k(i)$-color complement of a connected graph G_c remains connected if the vertex set is partitioned as $P = \{V_1, V_2, \ldots, V_k\}$ such that no two vertices in V_i are adjacent, and at least one vertex in V_i has a color different from the rest of the vertices.*

Proof. Let G_c be a connected graph with a partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G_c)$, where no two vertices within V_i are adjacent, ensuring that each V_i consists of an independent set of vertices. Additionally, at least one vertex in V_i possesses a unique color compared to the others. In the $k(i)$-color complement, edges are introduced within each partition V_i between non-adjacent vertices that have distinct colors. This ensures that no vertex remains isolated within its own partition, effectively preventing the formation of disconnected components in $(G_c)^P_{k(i)}$. Furthermore, as G_c was originally connected, and connectivity is preserved within each partition through these new edges, the connectivity of the graph $(G_c)^P_{k(i)}$ is preserved. Thus, $(G_c)^P_{k(i)}$ is a connected graph. $\square$

2.1. **Generalized color complements of graphs with respect to partition of same color class.**

REMARK 2.1.

i. Let G_c be a (n, m) graph. Then, $\chi((G_c)^P_k) = 1$ and $\chi((G_c)^P_{k(i)}) = n$ if and only if G_c is isomorphic to K_n or $\overline{K_n}$ or $K_{r_1,r_2,\ldots,r_n}$.

ii. The graph G_c is k-co-self color complementary and $k(i)$-self color complementary with respect to the partition of same color class.

PROPOSITION 2.1. *Let G_c be a non-trivial (n, m) graph. Then,*

i. $\beta((G_c)^P_{k(i)}) = \beta(G_c) \leq \beta((G_c)^P_k)$,

ii. $\omega((G_c)^P_k) \leq \omega(G_c) = \omega((G_c)^P_{k(i)})$.

Proof. Let G_c be a graph of order n.

i. Since $G_c \cong (G_c)^P_{k(i)}$, $\beta((G_c)^P_{k(i)}) = \beta(G_c)$. However, for $(G_c)^P_k$, partitioning may increase the independence number, ensuring $\beta(G_c) \leq \beta((G_c)^P_k)$.

ii. As the graph G_c is $k(i)$-self color complementary, $\omega(G_c) = \omega((G_c)^P_{k(i)})$. Furthermore, in $(G_c)^P_k$, partitioning might break some edges within the largest clique, leading to $\omega((G_c)^P_k) \leq \omega(G_c)$. Hence, the result follows.

□

THEOREM 2.3. *Let G_c be (n, m) graph and m' be the number of pairs of non-adjacent vertices in G_c that receive the same color. The number of edges in $(G_c)_k^P$ is $m((G_c)_k^P) = \binom{n}{2} - m(G_c) - m'$ for the partition of same color class.*

P r o o f. We begin by noting that the graph G_c is k-co-self color complementary and $k(i)$-self color complementary under the partition of the same color class, as stated in Remark 2.1.

This implies that, $(G_c)_k^P \cong \overline{G_c} \implies m((G_c)_k^P) = m(\overline{G_c})$.

Since $m(G_c) + m(\overline{G_c}) + m' = m(K_n)$, where m' denotes the number of pairs of non-adjacent vertices that receive the same color,

$$\implies m(\overline{G_c}) = m(K_n) - m(G_c) - m',$$
$$\implies m((G_c)_k^P) = \binom{n}{2} - m(G_c) - m'.$$

□

The table below presents the number of edges in $(G_c)_k^P$ for specific families of graphs G_c.

PROPOSITION 2.2. *Let G_c be a Tadpole graph $T_{p,q}$ with $n = p + q$ vertices and $p+q$ edges. Then, the number of edges in k-color complement of G_c is given by:*

$$m((G_c)_k^P) = \begin{cases} \frac{n^2-2n-4}{4}, & \text{if both } p \text{ and } q \text{ are odd,} \\ \frac{n^2-2n-3}{4}, & \text{if } p \text{ is odd and } q \text{ is even,} \\ \frac{n^2-4n}{4}, & \text{if both } p \text{ and } q \text{ are even,} \\ \frac{n^2-4n-1}{4}, & \text{if } p \text{ is even and } q \text{ is odd.} \end{cases}$$

Graphs	m'	$m((G_c)_k^P)$
C_n	$\begin{cases} \frac{n(n-2)}{4}, & \text{if } n \text{ is even} \\ \frac{n^2-4n+3}{4}, & \text{if } n \text{ is odd} \end{cases}$	$\begin{cases} \frac{n^2-2n-3}{4}, & \text{if } n \text{ is even} \\ \frac{n(n-4)}{4}, & \text{if } n \text{ is odd} \end{cases}$
P_n	$\begin{cases} \frac{n(n-2)}{4}, & \text{if } n \text{ is even} \\ (\frac{n-1}{2})^2, & \text{if } n \text{ is odd} \end{cases}$	$\begin{cases} (\frac{n}{2}-1)^2, & \text{if } n \text{ is even} \\ \frac{n^2-4n+3}{4}, & \text{if } n \text{ is odd} \end{cases}$
W_n, $n \geq 4$	$\begin{cases} \frac{n^2-6n+8}{4}, & \text{if } n \text{ is even} \\ \frac{n^2-4n+3}{4}, & \text{if } n \text{ is odd} \end{cases}$	$\begin{cases} \frac{n(2-n)}{4}, & \text{if } n \text{ is even} \\ \frac{5-n^2}{4}, & \text{if } n \text{ is odd} \end{cases}$
K_{r_1,r_2}	$\frac{r_1(r_1-1)+r_2(r_2-1)}{2}$	0
$K_{p\times 2}$	p	0
F_p	$p(p-1)$	$p(p-1)$
L_p	$p(p-1)$	p^2-3p+2
S_p^0	$p(p-1)$	p
$S(p,q)$	$\frac{p(p-1)+q(q-1)}{2}$	$(p-1)(q-1)$
BF_p	$\begin{cases} p(p-1), & \text{if } p \text{ is even} \\ p^2-p+1, & \text{if } p \text{ is odd} \end{cases}$	$\begin{cases} (p-1)^2+1, & \text{if } p \text{ is even} \\ (p-1)^2, & \text{if } p \text{ is odd} \end{cases}$
H_p	$\begin{cases} \frac{3p^2-8p+8}{4}, & \text{if } p \text{ is even} \\ \frac{3p^2-6p+3}{4}, & \text{if } p \text{ is odd} \end{cases}$	$\begin{cases} \frac{5p^2-16p+8}{4}, & \text{if } p \text{ is even} \\ \frac{5p^2-18p+13}{4}, & \text{if } p \text{ is odd} \end{cases}$
f_p	$\begin{cases} \frac{p(p-2)}{4}, & \text{if } p \text{ is even} \\ (\frac{p-1}{2})^2, & \text{if } p \text{ is odd} \end{cases}$	$\begin{cases} (\frac{p}{2}-1)^2, & \text{if } p \text{ is even} \\ \frac{p^2-4p+3}{4}, & \text{if } p \text{ is odd} \end{cases}$
G_p	p^2	p^2-2p

Note: We denote Cycle, Path, Wheel, Complete Bipartite, Cocktail Party, Friendship, Ladder, Crown, Double Star, Butterfly, Helm, Fan and Gear graphs by C_n, P_n, W_n, K_{r_1,r_2}, $K_{p\times 2}$, F_p, L_p, S_p^0, $S(p,q)$, BF_p, H_p, f_p and G_p respectively.

3. **Conclusion**

We analyzed the conditions under which the k-color and $k(i)$-color complements remain connected and explored their structural properties. Additionally, we investigated the relationship between clique and independence number in generalized color complements and determined the edge counts for specific graph families with respect to the partition of same color class. These findings provide valuable insights into graph modifications and their combinatorial implications.

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