International Journal of Applied Mathematics

Volume 38 No. 3 2025, 421–435
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v38i3.9

ADJACENCY-DISTANCE AND GENERALIZED ADJACENCY DISTANCE ENERGIES OF NON-COMMUTING GRAPH FOR DIHEDRAL GROUPS

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Received: March 26, 2025

Abstract

The correspondence between matrix and graph is one of the fundamental characteristics of the spectral graph theory. This relation allows us to formulate the characteristic polynomial of the graph and calculate the energy of the graph as the sum of its absolute eigenvalues. This study examines the non-commuting graph for dihedral groups. We establish the spectral radius and energy of this graph using the adjacency-distance and generalized adjacency-distance matrices. It should be noted that the obtained energies are never an odd integer and are always equal to twice its spectral radius.

Math. Subject Classification: 05C25, 15A18

Key Words and Phrases: energy of a graph, non-commuting graph, dihedral group, adjacency-distance matrix, generalized adjacency-distance matrix

1. **Introduction**

Let G be a group and $Z(G)$ be a center of G. The non-commuting graph of G, denoted by Γ_G, has vertex set $G \backslash Z(G)$ and two distinct vertices v_p, v_q in Γ_G are adjacent whenever $v_p v_q \neq v_q v_p$ [1]. Throughout this paper, the vertex set of Γ_G is the non-abelian dihedral group of order $2n$, D_{2n}, where $n \geq 3$. We denote $\Gamma_{D_{2n}}$ as the non-commuting graph for dihedral groups.

Recently, research on the non-commuting graph has been linked with the energy of the graph. Gutman first introduced the concept of graph energy in 1978 [5]. The discussion of the energy of $\Gamma_{D_{2n}}$ can be found in [10] which is associated with the closeness matrix. Other types of graph matrix that involve the degree of a vertex have been discussed by several authors, for instance, the Sombor energy [11], and Wiener-Hosoya energy [12] of $\Gamma_{D_{2n}}$.

Furthermore, in 1989, Schults discussed the adjacency distance matrix of a graph [13], and then this discussion was extended to the generalized adjacency distance matrix in 2023 [7]. In addition, Guo and Zhou established some properties for the adjacency-distance spectral radius of connected graphs [4]. Based on these previous results, this research focuses on the energy of $\Gamma_{D_{2n}}$ corresponding to the adjacency-distance and generalized adjacency-distance matrices.

The details of this research methodology are as follows. First, we need to construct the adjacency and distance matrices of $\Gamma_{D_{2n}}$, then the adjacency-distance and generalized adjacency-distance matrices can be determined. The next method consists of formulating the characteristic polynomial of the respective matrices, finding the eigenvalues and the spectrum of the respective matrix, observing the spectral radius, and analyzing the energies. We also investigate the relationship between the spectral radius and the energies of the respective matrices.

2. **Preliminaries**

In this section, we recall some required basic notations and definitions that play a very important role throughout this manuscript.

Let $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1}\rangle$ [2], $G_1 = \{a^i : 1 \leq i \leq n\} \backslash Z(D_{2n})$ and $G_2 = \{a^i b : 1 \leq i \leq n\}$. Let d_{pq} be the distance between vertex v_p and v_q. The next theorem describes the distance of every pair of vertices in $\Gamma_{D_{2n}}$.

THEOREM 2.1. [10] *If v_p, v_q are two distinct vertices in $\Gamma_{D_{2n}}$, then*

(1) *for odd n,* $d_{pq} = \begin{cases} 2, & \text{if } v_p, v_q \in G_1 \\ 1, & \text{otherwise,} \end{cases}$, *and*

(2) *for even n,* $d_{pq} = \begin{cases} 2, & \text{if } v_p, v_q \in G_1 \\ 2, & v_p \in G_2, v_q \in \left\{a^{\frac{n}{2}+i}b\right\} \\ 1, & \text{otherwise.} \end{cases}$

Moreover, the distance matrix of $\Gamma_{D_{2n}}$ is given by $D(\Gamma_{D_{2n}}) = [d_{pq}]$. Let $A(\Gamma_{D_{2n}})$ be the adjacency matrix of $\Gamma_{D_{2n}}$. The definitions of the adjacency-distance and generalized adjacency-distance matrices are presented below.

DEFINITION 2.1. [13] The adjacency-distance (AD) matrix of $\Gamma_{D_{2n}}$ is

$$AD(\Gamma_{D_{2n}}) = A(\Gamma_{D_{2n}}) + D(\Gamma_{D_{2n}}).$$

DEFINITION 2.2. [7] The generalized adjacency-distance (GAD) of $\Gamma_{D_{2n}}$ is

$$GAD(\Gamma_{D_{2n}}) = (1-\alpha)A(\Gamma_{D_{2n}}) + \alpha D(\Gamma_{D_{2n}}).$$

We know that the center of D_{2n} is either $Z(D_{2n}) = \{e\}$ for n is odd, or $\left\{e, a^{\frac{n}{2}}\right\}$ for n is even. Then $\Gamma_{D_{2n}}$ has $2n-1$ vertices for odd n, and when n is even, there are $2n-2$ vertices. The characteristic polynomial of $AD(\Gamma_{D_{2n}})$ is denoted by $P_{AD(\Gamma_{D_{2n}})}(\lambda)$. It is defined as $|\lambda I_{2n-1} - AD(\Gamma_{D_{2n}})|$ for odd n and $|\lambda I_{2n-2} - AD(\Gamma_{D_{2n}})|$ for even n. For formulating $P_{AD(\Gamma_{D_{2n}})}(\lambda)$, the following result is very helpful in simplifying the process.

Lemma 2.1. [9] *If a, b, c and d are real numbers, then the determinant of the $(n_1+n_2)\times(n_1+n_2)$ matrix of the form*

$$\begin{vmatrix} (\lambda+a)I_{n_1} - aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\lambda+b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

can be simplified in an expression as

$$(\lambda+a)^{n_1-1}(\lambda+b)^{n_2-1}\left((\lambda-(n_1-1)\,a)\,(\lambda-(n_2-1)\,b) - n_1 n_2 cd\right),$$

where $1 \le n_1, n_2 \le n$ and $n_1+n_2 = n$.

If the above lemma is not suitable for the obtained matrices, then we need to apply row and column operations. We define R_i and C_i as the i-th row and column of the matrix, respectively. Then we have $R_i^{'}$ and $C_i^{'}$ as the new i-th row and column provided from the respective row and column operations.

Furthermore, the roots $P_{AD(\Gamma_{D_{2n}})}(\lambda) = 0$ are the eigenvalues of $AD(\Gamma_{D_{2n}})$ and denoted by $\lambda_1, \lambda_2, \ldots, \lambda_m$. The $AD-$spectral radius of $\Gamma_{D_{2n}}$ [6] is

$$\rho_{AD}(\Gamma_{D_{2n}}) = \max\{|\lambda| : \lambda \in Spec_{AD}(\Gamma_{D_{2n}}). \tag{1}$$

The spectrum of $\Gamma_{D_{2n}}$ is

$$Spec_{AD}(\Gamma_{D_{2n}}) = \left\{(\lambda_1)^{k_1}, (\lambda_2)^{k_2}, \ldots, (\lambda_m)^{k_m}\right\}. \tag{2}$$

Therefore, the $AD-$energy of $\Gamma_{D_{2n}}$ [5] is defined as follows:

$$E_{AD}(\Gamma_{D_{2n}}) = \sum_{i=1}^{m} |\lambda_i|. \tag{3}$$

The above notations also apply to $GAD(\Gamma_{D_{2n}})$.

3. Adjacency-distance energy

This part is devoted to discussing fundamental spectral properties related to the adjacency-distance matrix of $\Gamma_{D_{2n}}$.

THEOREM 3.1. *The characteristic polynomial of $\Gamma_{D_{2n}}$ associated with the adjacency-distance matrix is*

(1) *for n is odd:*

$$P_{AD(\Gamma_{D_{2n}})}(\lambda) = (\lambda+2)^{2n-3}\left(\lambda^2 - 2(2n-3)\lambda - 8(n-1)\right),$$

(2) *for n is even:*

$$P_{AD(\Gamma_{D_{2n}})}(\lambda) = (\lambda+2)^{2(n-2)}\left(\lambda^2 - 4(n-2)\lambda - 4(2n-3)\right).$$

P r o o f. (1) Suppose n is odd, so by Theorem 2.1 (1), we can determine the distance matrix of $\Gamma_{D_{2n}}$. Thus, from Definition 2.1, $AD(\Gamma_{D_{2n}})$ is as the following:

$$\begin{aligned}
AD(\Gamma_{D_{2n}}) &= A(\Gamma_{D_{2n}}) + D(\Gamma_{D_{2n}})\\
&= \begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 1\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & 0 & 1 & \dots & 1\\ 1 & \dots & 1 & 0 & \dots & 1\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 1 & \dots & 1 & 1 & \dots & 0 \end{bmatrix} + \begin{bmatrix} 0 & \dots & 2 & 1 & \dots & 1\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 2 & \dots & 0 & 1 & \dots & 1\\ 1 & \dots & 1 & 0 & \dots & 1\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 1 & \dots & 1 & 1 & \dots & 0 \end{bmatrix}\\
&= \begin{bmatrix} 0 & \dots & 2 & 2 & \dots & 2\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 2 & \dots & 0 & 2 & \dots & 2\\ 2 & \dots & 2 & 0 & \dots & 2\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 2 & \dots & 2 & 2 & \dots & 0 \end{bmatrix}\\
&= \begin{bmatrix} 2(J-I)_{n-1} & 2J_{(n-1)\times n}\\ 2J_{(n-1)\times n} & 2(J-I)_n \end{bmatrix}.
\end{aligned}$$

$P_{AD(\Gamma_{D_{2n}})}(\lambda)$ is given below:

$$\begin{aligned}
P_{AD(\Gamma_{D_{2n}})}(\lambda) &= |\lambda I_{2n-1} - AD(\Gamma_{D_{2n}})|\\
&= \begin{vmatrix} (\lambda+2)I_{n-1} - 2J_{n-1} & -2J_{(n-1)\times n}\\ -2J_{n\times(n-1)} & (\lambda+2)I_n - 2J_n \end{vmatrix}.
\end{aligned}$$

According to Lemma 2.1 with $a = b = c = d = 2$, $n_1 = n-1$, and $n_2 = n$, then we obtain the formula of $P_{AD(\Gamma_{D_{2n}})}(\lambda)$ as follows:

$$P_{AD(\Gamma_{D_{2n}})}(\lambda) = (\lambda + 2)^{2n-3} \left(\lambda^2 - 2(2n-3)\lambda - 8(n-1)\right).$$

(2) Presume that n is even. In that case, we can construct $AD(\Gamma_{D_{2n}})$ by Theorem 2.1 (2) and Definition 2.1. Consequently, we have $(2n-2) \times (2n-2)$ matrix as given below.

$$AD(\Gamma_{D_{2n}}) = A(\Gamma_{D_{2n}}) + D(\Gamma_{D_{2n}})$$

$$= \begin{bmatrix} 0 \dots 0 & 1 \dots 1 & 1 \dots 1 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ 0 \dots 0 & 1 \dots 1 & 1 \dots 1 \\ 1 \dots 1 & 0 \dots 1 & 0 \dots 1 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ 1 \dots 1 & 1 \dots 0 & 1 \dots 0 \\ 1 \dots 1 & 0 \dots 1 & 0 \dots 1 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ 1 \dots 1 & 1 \dots 0 & 1 \dots 0 \end{bmatrix} + \begin{bmatrix} 0 \dots 2 & 1 \dots 1 & 1 \dots 1 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ 2 \dots 0 & 1 \dots 1 & 1 \dots 1 \\ 1 \dots 1 & 0 \dots 1 & 2 \dots 1 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ 1 \dots 1 & 1 \dots 0 & 1 \dots 2 \\ 1 \dots 1 & 2 \dots 1 & 0 \dots 1 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ 1 \dots 1 & 1 \dots 2 & 1 \dots 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & 2 & 2 & \dots & 2 & 2 & \dots & 2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 2 & \dots & 0 & 2 & \dots & 2 & 2 & \dots & 2 \\ 2 & \dots & 2 & 0 & \dots & 2 & 2 & \dots & 2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 2 & \dots & 2 & 2 & \dots & 0 & 2 & \dots & 2 \\ 2 & \dots & 2 & 2 & \dots & 2 & 0 & \dots & 2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 2 & \dots & 2 & 2 & \dots & 2 & 2 & \dots & 0 \end{bmatrix}$$

It consists of four block matrices and can be written as follows:

$$AD(\Gamma_G) = \begin{bmatrix} 2(J-I)_{n-2} & 2J_{(n-2)\times n} \\ 2J_{n\times(n-2)} & 2(J-I)_n \end{bmatrix}.$$

$P_{AD(\Gamma_{D_{2n}})}(\lambda)$ is given below:

$$\begin{aligned} P_{AD(\Gamma_{D_{2n}})}(\lambda) &= |\lambda I_{2n-2} - AD(\Gamma_{D_{2n}})| \\ &= \begin{vmatrix} (\lambda+2)I_{n-2} - 2J_{n-2} & -2J_{(n-2)\times n} \\ -2J_{n\times(n-2)} & (\lambda+2)I_n - 2J_n \end{vmatrix}. \end{aligned}$$

By Lemma 2.1 with $a = b = c = d = 2$, $n_1 = n-2$, and $n_2 = n$, we then obtain

$$P_{AD(\Gamma_{D_{2n}})}(\lambda) = (\lambda+2)^{2(n-2)}\left(\lambda^2 - 4(n-2)\lambda - 4(2n-3)\right).$$

□

THEOREM 3.2. *The AD−spectral radius for $\Gamma_{D_{2n}}$ is:*

(1) *for n is odd:* $\rho_{AD}(\Gamma_{D_{2n}}) = 4(n-1)$,
(2) *for n is even:* $\rho_{AD}(\Gamma_{D_{2n}}) = 2(2n-3)$.

P r o o f. (1) Let n is odd. By the result of Theorem 3.1 (1), we can provide the eigenvalues of $\Gamma_{D_{2n}}$. They are $\lambda_1 = -2$ of multiplicity $2(n-1)$ and $\lambda_2 = 4(n-1)$ of multiplicity 1. Consequently, we have the spectrum of Γ_G and so

$$Spec_{AD}(\Gamma_{D_{2n}}) = \left\{(4(n-1))^1, (-2)^{2(n-1)}\right\}.$$

The spectral radius of $\Gamma_{D_{2n}}$ is

$$\rho_{AD}(\Gamma_{D_{2n}}) = \max\{|4(n-1)|, |-2|\} = 4(n-1).$$

(2) In the same way for the even n case, since Theorem 3.1 (2) provides the characteristic polynomial of $\Gamma_{D_{2n}}$, we have two eigenvalues, which are $\lambda_1 = -2$ of multiplicity $2n-3$, and $\lambda_2 = 2(2n-3)$ of multiplicity 1. Hence, the spectrum of $\Gamma_{D_{2n}}$ as the following:

$$Spec_{AD}(\Gamma_{D_{2n}}) = \left\{(2(2n-3))^1, (-2)^{2n-3}\right\},$$

and so

$$\rho_{AD}(\Gamma_{D_{2n}}) = \max\{|2(2n-3)|, |-2|\} = 2(2n-3).$$

□

THEOREM 3.3. *The AD−energy for $\Gamma_{D_{2n}}$ is:*

(1) *for n is odd:* $E_{AD}(\Gamma_{D_{2n}}) = 8(n-1)$,
(2) *for n is even:* $E_{AD}(\Gamma_{D_{2n}}) = 4(2n-3)$.

P r o o f. (1) We have observed the spectrum of $\Gamma_{D_{2n}}$ for odd n in Theorem 3.2 (1). Thus, the $AD-$energy of $\Gamma_{D_{2n}}$ follows from Equation (3),

$$E_{AD}(\Gamma_{D_{2n}}) = 2(n-1)\,|-2| + (1)\,|4(n-1)| = 8(n-1).$$

(2) Since n is even, from Theorem 3.2 (2), we see from Equation (3) that the $AD-$energy of $\Gamma_{D_{2n}}$ is given by

$$E_{AD}(\Gamma_{D_{2n}}) = (1)|2(2n-3)| + (2n-3)\,|-2| = 4(2n-3).$$

□

4. Generalized adjacency-distance energy

Several spectral properties of $\Gamma_{D_{2n}}$ have been presented in this section including the characteristic polynomial, spectral radius, spectrum, and energy of $\Gamma_{D_{2n}}$ corresponding to the generalized adjacency-distance matrix.

THEOREM 4.1. *The characteristic polynomial of $\Gamma_{D_{2n}}$ associated with the generalized adjacency-distance matrix is:*

(1) *for n is odd:*

$$P_{GAD(\Gamma_{D_{2n}})}(\lambda) = (\lambda+2\alpha)^{n-2}(\lambda+1)^{n-1}$$
$$\left(\lambda^2 - (n-1+2\alpha(n-2))\lambda - (n-1)(n-2\alpha(n-2)\right),$$

(2) *for n is even:*

$$P_{GAD(\Gamma_{D_{2n}})}(\lambda) = (\lambda+2\alpha)^{\frac{3n-6}{2}}(\lambda-2\alpha+2)^{\frac{n}{2}-1}$$
$$(\lambda^2 - (n-2)(2\alpha+1)\lambda + 2\alpha(n-3)(2\alpha+n-2) - n(n-2)).$$

P r o o f. (1) Assume that n is odd. The distance matrix of $\Gamma_{D_{2n}}$ follows by Theorem 2.1 (1). We observe from Definition 2.2 that $GAD(\Gamma_{D_{2n}})$ is as the following:

$$
\begin{aligned}
GAD(\Gamma_{D_{2n}}) &= (1-\alpha)A(\Gamma_{D_{2n}}) + \alpha D(\Gamma_{D_{2n}}) \\
&= (1-\alpha)\begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 1 & \dots & 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 & \dots & 2 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 2 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 1 & \dots & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \dots & 2\alpha & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 2\alpha & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 1 & \dots & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2\alpha(J-I)_{n-1} & J_{(n-1)\times n} \\ J_{(n-1)\times n} & (J-I)_n \end{bmatrix}.
\end{aligned}
$$

Hence, we can write $P_{GAD(\Gamma_{D_{2n}})}(\lambda)$ as given below:

$$
\begin{aligned}
P_{GAD(\Gamma_{D_{2n}})}(\lambda) &= |\lambda I_{2n-1} - GAD(\Gamma_{D_{2n}})| \\
&= \begin{vmatrix} (\lambda+2\alpha)I_{n-1} - 2\alpha J_{n-1} & -J_{(n-1)\times n} \\ -J_{n\times(n-1)} & (\lambda+1)I_n - J_n \end{vmatrix}.
\end{aligned}
$$

According to Lemma 2.1 with $a = 2\alpha$, $b = c = d = 1$, $n_1 = n-1$, and $n_2 = n$, then the formula of $P_{GAD(\Gamma_{D_{2n}})}(\lambda)$ is

$$
P_{GAD(\Gamma_{D_{2n}})}(\lambda) = (\lambda+2\alpha)^{n-2}(\lambda+1)^{n-1}
\left(\lambda^2 - (n-1+2\alpha(n-2))\lambda - (n-1)(n-2\alpha(n-2)\right).
$$

(2) Now for the even n case, the combination of Theorem 2.1 and Definition 2.2 gives us the $GAD(\Gamma_{D_{2n}})$ with the size $(2n-2)\times(2n-2)$,

$$
GAD(\Gamma_{D_{2n}}) = (1-\alpha)A(\Gamma_{D_{2n}}) + \alpha D(\Gamma_{D_{2n}})
$$

$$= (1-\alpha)\begin{bmatrix} 0 \dots 0 & 1 \dots 1 & 1 \dots 1 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ 0 \dots 0 & 1 \dots 1 & 1 \dots 1 \\ 1 \dots 1 & 0 \dots 1 & 0 \dots 1 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ 1 \dots 1 & 1 \dots 0 & 1 \dots 0 \\ 1 \dots 1 & 0 \dots 1 & 0 \dots 1 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ 1 \dots 1 & 1 \dots 0 & 1 \dots 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 \dots 2 & 1 \dots 1 & 1 \dots 1 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ 2 \dots 0 & 1 \dots 1 & 1 \dots 1 \\ 1 \dots 1 & 0 \dots 1 & 2 \dots 1 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ 1 \dots 1 & 1 \dots 0 & 1 \dots 2 \\ 1 \dots 1 & 2 \dots 1 & 0 \dots 1 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ 1 \dots 1 & 1 \dots 2 & 1 \dots 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & 2\alpha & 1 & \dots & 1 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 2\alpha & \dots & 0 & 1 & \dots & 1 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 1 & 2\alpha & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 1 & \dots & 0 & 1 & \dots & 2\alpha \\ 1 & \dots & 1 & 2\alpha & \dots & 1 & 0 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 1 & \dots & 2\alpha & 1 & \dots & 0 \end{bmatrix}.$$

The above matrix can be written as the partition of nine block matrices as presented below:

$$GAD(\Gamma_{D_{2n}}) = \begin{bmatrix} 2\alpha(J-I)_{n-2} & J_{(n-2)\times\frac{n}{2}} & J_{(n-2)\times\frac{n}{2}} \\ J_{\frac{n}{2}\times(n-2)} & (J-I)_{\frac{n}{2}} & (2\alpha-1)I_{\frac{n}{2}} + J_{\frac{n}{2}} \\ J_{\frac{n}{2}\times(n-2)} & (2\alpha-1)I_{\frac{n}{2}} + J_{\frac{n}{2}} & (J-I)_{\frac{n}{2}} \end{bmatrix}.$$

Consequently, $P_{GAD(\Gamma_{D_{2n}})}(\lambda)$ is given below:

$$\begin{vmatrix} (\lambda+2\alpha)I_{n-2} - 2\alpha J_{n-2} & -J_{(n-2)\times\frac{n}{2}} & -J_{(n-2)\times\frac{n}{2}} \\ -J_{\frac{n}{2}\times(n-2)} & (\lambda+1)\, I_{\frac{n}{2}} - J_{\frac{n}{2}} & (1-2\alpha)I_{\frac{n}{2}} - J_{\frac{n}{2}} \\ -J_{\frac{n}{2}\times(n-2)} & -(1-2\alpha)I_{\frac{n}{2}} - J_{\frac{n}{2}} & (\lambda+1)\, I_{\frac{n}{2}} - J_{\frac{n}{2}} \end{vmatrix}.$$

To solve this determinant, we need to apply the row and column operations. First step is replacing $R_{n-2+\frac{n}{2}+i}$ by $R'_{n-2+\frac{n}{2}+i}$ $= R_{n-2+\frac{n}{2}+i} - R_{n-2+i}$, for every $1 \le i \le \frac{n}{2}$. Then we see $P_{GAD(\Gamma_{D_{2n}})}(\lambda)$ as the following:

$$\begin{vmatrix} (\lambda+2\alpha)I_{n-2}-2\alpha J_{n-2} & -J_{(n-2)\times\frac{n}{2}} & -J_{(n-2)\times\frac{n}{2}} \\ -J_{\frac{n}{2}\times(n-2)} & (\lambda+1)\, I_{\frac{n}{2}}-J_{\frac{n}{2}} & (1-2\alpha)I_{\frac{n}{2}}-J_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & -(\lambda+2\alpha)I_{\frac{n}{2}} & (\lambda+2\alpha)\, I_{\frac{n}{2}} \end{vmatrix}.$$

The next step is replacing C_{n-2+i} by $C'_{n-2+i} = C_{n-2+i} + C_{n-2+\frac{n}{2}+i}$, for every $1 \le i \le \frac{n}{2}$, then $P_{GAD(\Gamma_{D_{2n}})}(\lambda)$ is

$$\begin{vmatrix} (\lambda+2\alpha)I_{n-2}-2\alpha J_{n-2} & -2J_{(n-2)\times\frac{n}{2}} & -J_{(n-2)\times\frac{n}{2}} \\ -J_{\frac{n}{2}\times(n-2)} & (\lambda-2\alpha+2)\, I_{\frac{n}{2}}-2J_{\frac{n}{2}} & (1-2\alpha)I_{\frac{n}{2}}-J_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & 0_{\frac{n}{2}} & (\lambda+2\alpha)\, I_{\frac{n}{2}} \end{vmatrix}.$$

Now, we consider the following steps:

(a) Replace R_{n_1+1+i} by $R'_{n-1+i} = R_{n-1+i} - R_{n-1}$, for every $1 \le i \le \frac{n}{2} - 1$,
(b) Replace C_{n-1} by $C'_{n-1} = C_{n-1} + C_n + C_{n+1} + \ldots + C_{n-2+\frac{n}{2}}$,
(c) Replace C_i by $C'_i = C_i - C_{n-2}$, for all $1 \le i \le n-3$,
(d) Replace R_{n-2} by $R'_{n-2} = R_{n-2} + R_1 + R_2 + \ldots + R_{n-3}$,

then we see $P_{GAD(\Gamma_{D_{2n}})}(\lambda)$ as follows:

$$\begin{vmatrix} (\lambda+2\alpha)I_{n-3} & -2\alpha J_{(n-3)\times1} & -nJ_{(n-3)\times1} & -2I_{(n-3)\times(\frac{n}{2}-1)} & -J_{(n-3)\times1} & -J_{(n-3)\times(\frac{n}{2}-1)} \\ 0_{1\times(n-3)} & \lambda-2\alpha(n-3) & -n & -2J_{1\times(\frac{n}{2}-1)} & -1 & -J_{1\times(\frac{n}{2}-1)} \\ 0_{1\times(n-3)} & -1 & \lambda-2\alpha-n+2 & -2J_{1\times(\frac{n}{2}-1)} & -2\alpha & -J_{1\times(\frac{n}{2}-1)} \\ 0_{(\frac{n}{2}-1)\times(n-3)} & 0_{(\frac{n}{2}-1)\times1} & 0_{(\frac{n}{2}-1)\times1} & (\lambda-2\alpha+2)I_{\frac{n}{2}-1} & (2\alpha-1)J_{(\frac{n}{2}-1)\times1} & (1-2\alpha)I_{\frac{n}{2}-1} \\ 0_{1\times(n-3)} & 0 & 0 & 0_{1\times(\frac{n}{2}-1)} & \lambda+2\alpha & 0_{1\times(\frac{n}{2}-1)} \\ 0_{(\frac{n}{2}-1)\times(n-3)} & 0_{(\frac{n}{2}-1)\times1} & 0_{(\frac{n}{2}-1)\times1} & 0_{\frac{n}{2}-1} & 0_{(\frac{n}{2}-1)\times1} & (\lambda+2\alpha)I_{\frac{n}{2}-1} \end{vmatrix}.$$

$P_{GAD(\Gamma_{D_{2n}})}(\lambda)$ can be written as:

$$P_{GAD(\Gamma_{D_{2n}})}(\lambda) = (\lambda+2\alpha)^{\frac{3n-6}{2}}(\lambda-2\alpha+2)^{\frac{n}{2}-1}$$
$$(\lambda^2-(n-2)(2\alpha+1)\lambda+2\alpha(n-3)(2\alpha+n-2)-n(n-2)).$$

□

THEOREM 4.2. *The $GAD-$spectral radius for $\Gamma_{D_{2n}}$ is:*

(1) *for n is odd:*
$$\rho_{GAD}(\Gamma_{D_{2n}}) = \alpha(n-2) + \tfrac{1}{2}\left(n-1+\sqrt{(n-1-2\alpha(n-2))^2+4n(n-1)}\right),$$
(2) *for n is even:*
$$\rho_{GAD}(\Gamma_{D_{2n}}) = \frac{(n-2)(2\alpha+1)+\sqrt{(n-2)^2(2\alpha+1)^2-8\alpha(n-3)(2\alpha-n-2)+4n(n-2)}}{2}.$$

P r o o f. (1) We consider the odd n case. The eigenvalues of $GAD(\Gamma_{D_{2n}})$ follows by Theorem 4.1 (1). Thus, we have $\lambda_1 = -2\alpha$ of multiplicity $n-2$, $\lambda_2 = -1$ of multiplicity $n-1$, and $\lambda_{3,4} = \alpha(n-2) + \frac{n-1}{2} \pm \frac{\sqrt{(n-1-2\alpha(n-2))^2+4n(n-1)}}{2}$ each of multiplicity 1. Hence, the spectrum of $\Gamma_{D_{2n}}$ as the following:

$$Spec_{GAD}(\Gamma_{D_{2n}}) = \left\{\left(\alpha(n-2) + \frac{1}{2}\left(n-1+\sqrt{(n-1-2\alpha(n-2))^2+4n(n-1)}\right)\right)^1, \left(\alpha(n-2) + \frac{1}{2}\left(n-1-\sqrt{(n-1-2\alpha(n-2))^2+4n(n-1)}\right)\right)^1, (-1)^{n-1}, (-2\alpha)^{n-2}\right\}.$$

From Equation (1), the spectral radius of $\Gamma_{D_{2n}}$ is

$$\rho_{GAD}(\Gamma_{D_{2n}}) = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|, |\lambda_4|\} = \alpha(n-2) + \frac{1}{2}\left(n-1+\sqrt{(n-1-2\alpha(n-2))^2+4n(n-1)}\right).$$

(2) When n is even, we see that $\Gamma_{D_{2n}}$ has four eigenvalues which by Theorem 4.1 (2). Consequently, we get $\lambda_1 = -2\alpha$ of multiplicity $\frac{3n-6}{2}$, $\lambda_2 = 2(\alpha-1)$ of multiplicity $\frac{n}{2}-1$, and

$$\lambda_{3,4} = \frac{(n-2)(2\alpha+1) \pm \sqrt{(n-2)^2(2\alpha+1)^2 - 8\alpha(n-3)(2\alpha-n-2) + 4n(n-2)}}{2},$$

each of multiplicity 1. Hence, by Equation (2), the spectrum of $\Gamma_{D_{2n}}$ as the following:

$$Spec_{GAD}(\Gamma_{D_{2n}}) = \left\{\left(\frac{(n-2)(2\alpha+1) + \sqrt{(n-2)^2(2\alpha+1)^2 - 8\alpha(n-3)(2\alpha-n-2) + 4n(n-2)}}{2}\right)^1, \left(\frac{(n-2)(2\alpha+1) - \sqrt{(n-2)^2(2\alpha+1)^2 - 8\alpha(n-3)(2\alpha-n-2) + 4n(n-2)}}{2}\right)^1, (2(\alpha-1))^{\frac{n}{2}-1}, (-2\alpha)^{\frac{3n-6}{2}}\right\}.$$

By the same argument following Equation (1), the spectral radius of $\Gamma_{D_{2n}}$ is:

$$\rho_{GAD}(\Gamma_{D_{2n}}) = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|, |\lambda_4|\} = \frac{(n-2)(2\alpha+1)+\sqrt{(n-2)^2(2\alpha+1)^2-8\alpha(n-3)(2\alpha-n-2)+4n(n-2)}}{2}.$$

□

THEOREM 4.3. *The $GAD-$energy for $\Gamma_{D_{2n}}$ is:*

(1) *for n is odd:*

$$E_{GAD}(\Gamma_{D_{2n}}) = 2\alpha(n-2) + n - 1 + \sqrt{(n-1-2\alpha(n-2))^2 + 4n(n-1)},$$

(2) *for n is even:*

$$E_{GAD}(\Gamma_{D_{2n}}) = (n-2)(2\alpha+1) + \sqrt{(n-2)^2(2\alpha+1)^2 - 8\alpha(n-3)(2\alpha-n-2) + 4n(n-2)}.$$

Proof. (1) We consider the first case when n is odd. The $AD-$energy of $\Gamma_{D_{2n}}$ follows by Theorem 4.2 (1) and Equation (3) as follows:

$$E_{GAD}(\Gamma_{D_{2n}}) = (n-1)|-1| + (n-2)|-2\alpha| + \left|\alpha(n-2) + \frac{n-1}{2} \pm \frac{\sqrt{(n-1-2\alpha(n-2))^2 + 4n(n-1)}}{2}\right|$$
$$= 2\alpha(n-2) + n - 1 + \sqrt{(n-1-2\alpha(n-2))^2 + 4n(n-1)}.$$

(2) For even n, by the same argument, by Theorem 4.2 (2) and Equation (3), then the $GAD-$energy of $\Gamma_{D_{2n}}$ is

$$E_{GAD}(\Gamma_{D_{2n}}) = \left(\frac{n}{2} - 1\right)|2(\alpha-1)| + \left(\frac{3n-6}{2}\right)|-2\alpha| + \left|\frac{(n-2)(2\alpha+1)\pm\sqrt{(n-2)^2(2\alpha+1)^2-8\alpha(n-3)(2\alpha-n-2)+4n(n-2)}}{2}\right|$$
$$= (n-2)(2\alpha+1) + \sqrt{(n-2)^2(2\alpha+1)^2 - 8\alpha(n-3)(2\alpha-n-2) + 4n(n-2)}.$$

□

5. Discussion

In this part, we consider the discussion based on the results of Theorems 3.2, 3.3, 4.2, and 4.3.

COROLLARY 5.1. *In $\Gamma_{D_{2n}}$,*

$E_{AD}(\Gamma_{D_{2n}})$ is always an even integer.

Corollary 5.2. *In $\Gamma_{D_{2n}}$,*

$$E_{GAD}(\Gamma_{D_{2n}}) \text{ is never an odd integer.}$$

The above corollaries restate the well-known fact from [3] and [8]. Moreover, we also highlight the interesting relationship between energy and spectral radius.

Corollary 5.3. *In $\Gamma_{D_{2n}}$,*

$$\begin{aligned} E_{AD}(\Gamma_{D_{2n}}) =& 2 \cdot \rho_{AD}(\Gamma_{D_{2n}}), \\ E_{GAD}(\Gamma_{D_{2n}}) =& 2 \cdot \rho_{GAD}(\Gamma_{D_{2n}}). \end{aligned}$$

We recall the obtained energies in Theorems 3.3 and 4.3, and we obtain the link between the adjacency-distance and generalized adjacency-distance energies of $\Gamma_{D_{2n}}$ as given below:

Corollary 5.4. *For $\alpha \in [0, 1]$,*

$$E_{AD}(\Gamma_{D_{2n}}) > E_{GAD}(\Gamma_{D_{2n}}).$$

Acknowledgements

The authors thank the referees for their helpful comments and recommendations on this article. We also thank the University of Mataram, Indonesia, for funding assistance through the Overseas Collaborative Research Scheme PNBP No.2490/UN18.L1/PP/2025.

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