

**SYMMETRY ANALYSIS OF THE GEODESIC
EQUATIONS ON SIX-DIMENSIONAL LIE GROUPS:
THE NON-ABELIAN NILRADICAL AND
NON-ABELIAN COMPLEMENT CASE**

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Abstract

In this article, we continue our investigation and classification of the symmetry Lie algebra of the geodesic equations of the canonical connection on a Lie group. In particular, we consider the list of indecomposable solvable six-dimensional Lie algebras whose nilradical is four-dimensional non-abelian and its complement is two-dimensional non-abelian as well. According to the classification of the six-dimensional Lie algebras given by Turkowski [1], there are thirteen Lie algebras to consider, namely $A_{6,28} - A_{6,40}$. For each of these algebras, we list the system of the geodesic equations, a basis for the symmetry algebra in terms of vector fields and the non-zero brackets of the symmetry Lie algebra. A comprehensive analysis and classification of the symmetry Lie algebra are given.

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1. Introduction

In this paper, we consider the problem of classifying the symmetry Lie algebras of the geodesic equations of the canonical connection on Lie groups for the six-dimensional case where the nilradical is non-abelian and its complement is non-abelian as well. The canonical connection on Lie groups was introduced in 1926 by Cartan and Schouten [2]. Ghanam and Thompson [3] studied the properties of this connection and a summary of its properties will be presented in Section 3. In recent year, there has been a lot of work done on identifying the symmetry Lie algebras of the geodesic equations on Lie groups. Ghanam et al. [4, 5] considered the problem for three and four dimensional Lie groups and they used the list of Lie algebras presented by Winternitz [6]. They also considered the problem for six-dimensional nilpotent Lie algebras [7]. Almusawa et al. [8] considered the problem for the five-dimensional indecomposable Lie algebras with co-dimension one abelian nilradical. Almutiben et al. considered the problem for the six-dimensional Lie algebras where the nilradical is abelian and the complement is non-abelian [9]. In this paper, we will identify the symmetry Lie algebras for thirteen algebras in dimension six: $A_{6,28} - A_{6,40}$, these algebras were classified by Turkowski

[1] and each of them has a non-abelian nilradical and a non-abelian complement. This paper will complete the classification of the symmetry Lie algebras of the geodesic equations of the canonical connection for all indecomposable six-dimensional Lie algebras.

The main motivation for our investigation, is to determine the Lie algebra of the symmetry group in each of the geodesic systems that correspond to one of the low-dimensional Lie algebras occurring in a list, in the case at hand, Algebras A6.28-A40 in [1]. It should be appreciated that the classifications of the low-dimensional Lie algebras are very limited. For nilpotent and solvable Lie algebras such classifications extend only up to dimension seven, [10, 11, 12]. In the present context, we continue the analysis started in [9]. As such, we are able to obtain a complete description of the symmetry Lie algebras of *all* the Algebras in [1]. The determination of such symmetry Lie algebras may be considered to be valuable because they have a very high dimension compared with symmetry Lie algebras of more general geodesic systems and help to shed light on the general structure.

The outline of the paper is as follows: In Section 2, we give the definition of the canonical connection ∇ on a Lie group and a summary of its properties. In Section 3, we review the symmetries of differential equations and the Lie invariance condition. In Section 4, for each Lie algebra $A_{6,28} - A_{6,40}$ in Turkowski's list, we give the geodesic equations, a basis for the symmetry algebra in terms of vector fields, the non-zero brackets for the symmetry algebra, and finally we identify the symmetry Lie algebra in terms of the nilradical and its complement. We will use $\rtimes$ to denote a semi-direct product and $\oplus$ for the direct sum of algebras.

2. The canonical connection on Lie groups

On left-invariant vector fields X and Y , the canonical symmetric connection ∇ on a Lie group G is defined by

$$\nabla_X Y = \frac{1}{2} [X, Y], \quad (1)$$

and then extended to arbitrary vector fields using linearity and the Leibnitz rule. The connection ∇ is left-invariant. One could just as well use right-invariant vector fields to define ∇ , but one must check that ∇ is well-defined. Properties of the canonical connection have been studied in [13], and we will summarize them in the following proposition.

PROPOSITION 2.1. *For the canonical connection defined by (1):*

- (1) *The torsion is zero.*
- (2) *The curvature tensor R is given by $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$.*
- (3) *The curvature tensor R is covariantly constant.*
- (4) *The curvature tensor R is zero if, and only if, the Lie algebra is two-step nilpotent.*
- (5) *The Ricci tensor is symmetric and in fact a multiple of the Killing form.*
- (6) *The Ricci tensor is bi-invariant.*

3. Symmetries of the geodesic equations

In this section, we explain the algorithm for finding the Lie symmetries of the geodesic equations. In local coordinates and in dimension n , the geodesic equations are given by

$$\frac{d^2 x^i}{dt^2} + \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad (2)$$

where Γ_{jk}^i are the connection components or Christoffel symbols, and $i, j, k = 1, \dots, n$. In dimension six, let us take our coordinates to be t, p, q, x, y, z, w , where t is the independent variable and p, q, x, y, z, w are the dependent variables. Define Γ to be

$$\Gamma = T \frac{\partial}{\partial t} + P \frac{\partial}{\partial p} + Q \frac{\partial}{\partial q} + X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + Z \frac{\partial}{\partial z} + W \frac{\partial}{\partial w}, \quad (3)$$

where T, P, Q, X, Y, Z , and W are unknown functions of (t, p, q, x, y, z, w) . The first prolongation Γ^1 and second prolongation Γ^2 of Γ are given by

$$\Gamma^1 = \Gamma + P_t \frac{\partial}{\partial \dot{p}} + Q_t \frac{\partial}{\partial \dot{q}} + X_t \frac{\partial}{\partial \dot{x}} + Y_t \frac{\partial}{\partial \dot{y}} + Z_t \frac{\partial}{\partial \dot{z}} + W_t \frac{\partial}{\partial \dot{w}}, \quad (4)$$

$$\Gamma^2 = \Gamma^1 + P_{tt} \frac{\partial}{\partial \ddot{p}} + Q_{tt} \frac{\partial}{\partial \ddot{q}} + X_{tt} \frac{\partial}{\partial \ddot{x}} + Y_{tt} \frac{\partial}{\partial \ddot{y}} + Z_{tt} \frac{\partial}{\partial \ddot{z}} + W_{tt} \frac{\partial}{\partial \ddot{w}}, \quad (5)$$

where

$$\begin{aligned} P_t &= D_t(P) - \dot{p}D_t(T), & P_{tt} &= D_t(P_t) - \ddot{p}D_t(T), \\ Q_t &= D_t(Q) - \dot{q}D_t(T), & Q_{tt} &= D_t(Q_t) - \ddot{q}D_t(T), \\ X_t &= D_t(X) - \dot{x}D_t(T), & X_{tt} &= D_t(X_t) - \ddot{x}D_t(T), \\ Y_t &= D_t(Y) - \dot{y}D_t(T), & Y_{tt} &= D_t(Y_t) - \ddot{y}D_t(T), \\ Z_t &= D_t(Z) - \dot{z}D_t(T), & Z_{tt} &= D_t(Z_t) - \ddot{z}D_t(T), \\ W_t &= D_t(W) - \dot{w}D_t(T), & W_{tt} &= D_t(W_t) - \ddot{w}D_t(T), \end{aligned} \quad (6)$$

and D_t is given by

$$D_t = \frac{\partial}{\partial t} + \dot{p} \frac{\partial}{\partial p} + \dot{q} \frac{\partial}{\partial q} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} + \dot{w} \frac{\partial}{\partial w} \\ + \ddot{p} \frac{\partial}{\partial \dot{p}} + \ddot{q} \frac{\partial}{\partial \dot{q}} + \ddot{x} \frac{\partial}{\partial \dot{x}} + \ddot{y} \frac{\partial}{\partial \dot{y}} + \ddot{z} \frac{\partial}{\partial \dot{z}} + \ddot{w} \frac{\partial}{\partial \dot{w}}. \quad (7)$$

Finally, Γ is said to be a Lie symmetry of the system of the geodesic equations if

$$\Gamma^2(\Delta_i^{(2)})|_{\Delta_i^{(2)}=0} = 0, \quad (8)$$

where

$$\Delta_i^{(2)} = \frac{d^2 x^i}{dt^2} - f^i(t, x^i), \quad i = 1, 2, \dots, 6. \quad (9)$$

Equation (8) is called the Lie invariance condition. We equate the coefficients of the linearly independent derivation terms to zero, and this yields an over-determined system of partial differential equations (PDEs). In general, the integration of this PDE system is extremely problematic, but in the case of the geodesics of the canonical connection, it always seems to be possible to integrate them. On the other hand, that fact is not so surprising because one knows that each left and right-invariant vector field is a Lie symmetry. For a good reference on symmetries of differential equations, we refer the reader to [14].

4. Lie algebras: $A_{6,28} - A_{6,40}$

6.28: $[e_4, e_6] = -e_3$, $[e_5, e_6] = -e_4$, $[e_1, e_3] = e_3$, $[e_1, e_5] = -e_5$, $[e_1, e_6] = e_6$, $[e_2, e_4] = e_4$, $[e_2, e_5] = 2e_3$, $[e_2, e_6] = -e_6$.

The corresponding system of geodesic equations is given by:

$$\begin{aligned} \ddot{w} &= 0 \\ \ddot{z} &= 0 \\ \ddot{y} &= -\dot{q}(\dot{x} + x\dot{z}) + \dot{x}\dot{w}(q + y) + \dot{y}(\dot{z} + x\dot{w}) \\ \ddot{p} &= \dot{p}\dot{w} + \dot{x}(\dot{y} - y\dot{z} - x\dot{q}) + x^2(\dot{y}\dot{w} - \dot{q}\dot{z}) + x\dot{x}\dot{w}(y + q) \\ \ddot{q} &= \dot{q}(2\dot{z} - \dot{w}), \\ \ddot{x} &= -\dot{x}(\dot{z} + \dot{w}). \end{aligned} \quad (10)$$

The symmetry Lie algebra is spanned by the following basis:

$$\begin{aligned} e_1 &= D_t & e_2 &= D_z & e_3 &= zD_t \\ e_4 &= D_p & e_5 &= wD_t & e_6 &= e^w D_p \\ e_7 &= tD_t & e_8 &= D_w & e_9 &= pD_p + qD_q + yD_y. \end{aligned} \quad (11)$$

The nonzero brackets for the symmetry Lie algebra are given by

$$\begin{aligned} [e_1, e_7] &= e_1, [e_2, e_3] = e_1, [e_3, e_7] = e_3, [e_4, e_9] = e_4, [e_5, e_7] = e_5, \\ [e_5, e_8] &= -e_1, [e_6, e_8] = -e_6, [e_6, e_8] = -e_6. \end{aligned} \quad (12)$$

PROPOSITION 4.1. *The symmetry Lie algebra is a nine-dimensional indecomposable solvable Lie algebra. The nilradical is a six-dimensional decomposable Lie algebra spanned by $e_1, e_2, e_3, e_4, e_5, e_6$ and it is a direct sum of $A_{3,1}$ and $\mathbb{R}^3$. The complement is a three-dimensional abelian Lie algebra spanned by e_7, e_8, e_9 . Hence the symmetry algebra is $(A_{3,1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^3$.*

6.29 ($ab : a^2 + b^2 \neq 0$): $[e_4, e_5] = -e_3, [e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_1, e_6] = ae_6, [e_2, e_3] = e_3, [e_2, e_5] = e_5, [e_2, e_6] = be_6$.

The corresponding system of geodesic equations is given by:

$$\begin{aligned} \ddot{w} &= 0 \\ \ddot{z} &= 0 \\ \ddot{p} &= \dot{p}(\dot{z} + \dot{w}) + \dot{q}(\dot{y} - y\dot{w}) \\ \ddot{y} &= \dot{y}\dot{w} \\ \ddot{q} &= \dot{q}\dot{z} \\ \ddot{x} &= \dot{x}(b\dot{z} + a\dot{w}). \end{aligned} \quad (13)$$

The symmetry Lie algebra is spanned by the following basis:

$$\begin{aligned} e_1 &= D_p & e_2 &= D_q & e_3 &= qD_p + D_y & e_4 &= e^w D_y \\ e_5 &= e^{z+w} D_p & e_6 &= ye^z D_p + e^z D_q & e_7 &= D_t & e_8 &= D_x \\ e_9 &= wD_t & e_{10} &= zD_t & e_{11} &= e^{aw+bz} D_x & e_{12} &= D_z \\ e_{13} &= D_w & e_{14} &= tD_t & e_{15} &= xD_x \\ e_{16} &= pD_p + yD_y & e_{17} &= pD_p + qD_q, \end{aligned} \quad (14)$$

$$\begin{aligned} [e_1, e_{16}] &= e_1, & [e_1, e_{17}] &= e_1, & [e_2, e_3] &= e_1, & [e_2, e_{17}] &= e_2, \\ [e_3, e_{16}] &= e_3, & [e_4, e_6] &= e_5, & [e_4, e_{13}] &= -e_4, & [e_4, e_{16}] &= e_4, \\ [e_5, e_{12}] &= -e_5, & [e_5, e_{13}] &= -e_5, & [e_5, e_{16}] &= e_5, & [e_5, e_1] &= e_5, \\ [e_6, e_{12}] &= -e_6, & [e_6, e_{17}] &= e_6, & [e_7, e_{14}] &= e_7, & [e_8, e_{15}] &= e_8, \\ [e_9, e_{13}] &= -e_7, & [e_9, e_{14}] &= e_9, & [e_{10}, e_{12}] &= -e_7, & [e_{10}, e_{14}] &= e_{10}, \\ [e_{11}, e_{12}] &= -be_{11}, & [e_{11}, e_{13}] &= -ae_{11}, & [e_{11}, e_{15}] &= e_{11}. \end{aligned} \quad (15)$$

PROPOSITION 4.2. *The symmetry Lie algebra is a seventeen-dimensional indecomposable solvable Lie algebra. The nilradical is a eleven-dimensional decomposable Lie algebra spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}$ and it is a direct sum of two copies $A_{3,1}$ and $\mathbb{R}^5$. The complement is a six-dimensional abelian Lie algebra spanned by $e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}$. Hence the symmetry algebra is $(A_{3,1} \oplus A_{3,1} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^6$.*

6.30 (a) : $[e_4, e_5] = -e_3, [e_1, e_3] = 2e_3, [e_1, e_4] = e_4, [e_1, e_5] = e_5, [e_1, e_6] = e_6, [e_2, e_4] = e_5, [e_2, e_6] = e_6$.

The corresponding system of geodesic equations is given by:

$$\begin{aligned} \ddot{w} = 0, \quad \ddot{z} = 0, \quad \ddot{p} = \dot{w}(2\dot{p} + q\dot{y} + y\dot{q}) + y\dot{z}(\dot{y} - y\dot{w}), \quad \ddot{y} = \dot{y}\dot{w}, \\ \ddot{q} = -\dot{q}\dot{w} - \dot{z}(\dot{y} + y\dot{w}), \quad \ddot{x} = \dot{x}(\dot{z} + a\dot{w}). \end{aligned}$$

The symmetry Lie algebra is spanned by the following basis:

$$\begin{aligned} e_1 = D_p \quad e_2 = D_x \quad e_3 = D_t \\ e_4 = wD_t \quad e_5 = zD_t \quad e_6 = -yD_p + D_q \\ e_7 = e^{2w}D_p \quad e_8 = e^{aw+z}D_x \quad e_9 = \frac{-y}{3}e^{-w}D_p + e^{-w}D_q \\ e_{10} = D_w \quad e_{11} = D_z \quad e_{12} = tD_t \\ e_{13} = xD_x \quad e_{14} = 2pD_p + qD_q + yD_y. \end{aligned} \tag{16}$$

The non-zero brackets of the symmetry Lie algebra are given by:

$$\begin{aligned} [e_1, e_{14}] = 2e_1, \quad [e_2, e_{13}] = e_2, \quad [e_3, e_{12}] = e_3, \quad [e_4, e_{10}] = -e_3, \\ [e_4, e_{12}] = e_4, \quad [e_5, e_{11}] = -e_3, \quad [e_5, e_{12}] = e_5, \quad [e_6, e_{14}] = e_6, \\ [e_7, e_{10}] = -2e_7, \quad [e_7, e_{14}] = 2e_7, \quad [e_8, e_{10}] = -ae_8, \quad [e_8, e_{11}] = -e_8, \\ [e_8, e_{13}] = e_8, \quad [e_9, e_{10}] = e_9, \quad [e_9, e_{14}] = e_9. \end{aligned} \tag{17}$$

PROPOSITION 4.3. *The symmetry Lie algebra is a fourteen-dimensional indecomposable solvable Lie algebra. The nilradical is a nine-dimensional abelian Lie algebra spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$. The complement is a five-dimensional abelian Lie algebra spanned by $e_{10}, e_{11}, e_{12}, e_{13}, e_{14}$. Hence the symmetry algebra is $\mathbb{R}^9 \rtimes \mathbb{R}^5$.*

6.31: $[e_4, e_5] = -e_3, [e_1, e_4] = e_4, [e_1, e_5] = -e_5, [e_2, e_3] = e_3, [e_2, e_5] = e_5, [e_2, e_6] = e_6 + e_3$.

The corresponding system of geodesic equations is given by:

$$\ddot{w} = 0, \quad \ddot{z} = 0, \quad \ddot{p} = \dot{z}(\dot{p} + \dot{x}) + \dot{q}(\dot{y} - y\dot{w}), \quad \ddot{y} = \dot{y}\dot{w}, \quad \ddot{q} = \dot{q}(\dot{z} - \dot{w}), \quad \ddot{x} = \dot{x}\dot{z}.$$

The symmetry Lie algebra is spanned by the following basis:

$$\begin{aligned}
 e_1 &= D_p & e_2 &= D_x & e_3 &= D_q \\
 e_4 &= xD_p & e_5 &= qD_p + D_y & e_6 &= e^z D_p + D_q \\
 e_7 &= e^w D_y & e_8 &= (z-1)e^z D_p + e^z D_x & & \\
 e_9 &= ye^{-w+z} D_p + e^{z-w} D_q & e_{10} &= D_t & e_{11} &= wD_t \\
 e_{12} &= zD_t & e_{13} &= D_z & e_{14} &= tD_t \\
 e_{15} &= D_w & e_{16} &= -qD_q + yD_y & & \\
 e_{17} &= pD_p + qD_q + xD_x. & & & &
 \end{aligned} \tag{18}$$

The non-zero brackets of the symmetry Lie algebra are given by:

$$\begin{aligned}
 [e_1, e_{17}] &= e_1, & [e_2, e_4] &= e_1, & [e_2, e_{17}] &= e_2, \\
 [e_3, e_5] &= e_1, & [e_3, e_{16}] &= -e_3, & [e_3, e_{17}] &= e_3, \\
 [e_4, e_8] &= -e_6, & [e_5, e_{16}] &= e_5, & [e_6, e_{13}] &= -e_6, \\
 [e_6, e_{17}] &= e_6, & [e_7, e_9] &= e_6, & [e_7, e_{15}] &= -e_7, \\
 [e_7, e_{16}] &= e_7, & [e_8, e_{13}] &= -e_6 - e_8, & [e_8, e_{17}] &= e_8, \\
 [e_9, e_{13}] &= -e_9, & [e_9, e_{15}] &= e_9, & [e_9, e_{16}] &= -e_9, \\
 [e_9, e_{17}] &= e_9, & [e_{10}, e_{14}] &= e_{10}, & [e_{11}, e_{14}] &= e_{11}, \\
 [e_{11}, e_{15}] &= -e_{10}, & [e_{12}, e_{13}] &= -e_{10}, & [e_{12}, e_{14}] &= e_{12}.
 \end{aligned} \tag{19}$$

PROPOSITION 4.4. *The symmetry Lie algebra is a seventeen-dimensional indecomposable solvable Lie algebra. The nilradical is a direct sum of a nine-dimensional nilpotent Lie algebra N_9 spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ and a three-dimensional abelian Lie algebra spanned by e_{10}, e_{11}, e_{12} . The complement is a five-dimensional abelian Lie algebra spanned by $e_{13}, e_{14}, e_{15}, e_{16}, e_{17}$. Hence the symmetry Lie algebra can be identified by $(N_9 \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$, where the non-zero brackets of N_9 are given by*

$$[e_2, e_4] = e_1, [e_3, e_5] = e_1, [e_4, e_8] = -e_6, [e_7, e_9] = e_6. \tag{20}$$

6.32 (a):

$$\begin{aligned}
 [e_4, e_5] &= -e_3, & [e_1, e_4] &= e_4, & [e_1, e_5] &= -e_5, \\
 [e_1, e_6] &= e_3, & [e_2, e_3] &= e_3, & [e_2, e_4] &= ae_4, \\
 [e_2, e_5] &= (1-a)e_5, & [e_2, e_6] &= e_6.
 \end{aligned}$$

The corresponding system of geodesic equations is given by:

$$\begin{aligned}
 \ddot{w} &= 0, & \ddot{z} &= 0, & \ddot{y} &= (a+1)\dot{y}\dot{z}, & \ddot{q} &= -\dot{q}\dot{w}, & \ddot{x} &= \dot{x}\dot{z}, \\
 \ddot{p} &= \dot{q}\dot{y}(1-2a) + y\dot{q}\dot{w}(a-1) + \dot{w}(\dot{x} - q\dot{y}) + q(1-a^2)\dot{y}\dot{z}.
 \end{aligned}$$

The symmetry Lie algebra is spanned by the following basis:

$$\begin{aligned}
 e_1 &= D_t, & e_2 &= tD_p, & e_3 &= D_p, \\
 e_4 &= D_z, & e_5 &= zD_t, & e_6 &= wD_p, \\
 e_7 &= zD_p, & e_8 &= wD_t, & e_9 &= D_x, \\
 e_{10} &= D_y, & e_{11} &= e^{-w}D_q, & e_{12} &= D_w, \\
 e_{13} &= tD_t, & e_{14} &= qD_q - yD_y, & e_{15} &= pD_p + xD_x + yD_y.
 \end{aligned} \tag{21}$$

The non-zero brackets of the symmetry Lie algebra are given by

$$\begin{aligned}
 [e_1, e_2] &= e_3, & [e_1, e_{13}] &= e_1, & [e_2, e_5] &= -e_7, \\
 [e_2, e_8] &= -e_6, & [e_2, e_{13}] &= -e_2, & [e_2, e_{15}] &= e_2, \\
 [e_3, e_{15}] &= e_3, & [e_4, e_5] &= e_1, & [e_4, e_7] &= e_3, \\
 [e_5, e_{13}] &= e_5, & [e_6, e_{12}] &= -e_3, & [e_6, e_{15}] &= e_6, \\
 [e_7, e_{15}] &= e_7, & [e_8, e_{12}] &= -e_1, & [e_8, e_{13}] &= e_8, \\
 [e_8, e_{15}] &= e_8, & [e_{10}, e_{14}] &= -e_{10}, & [e_{10}, e_{15}] &= e_{10}, \\
 [e_{11}, e_{12}] &= e_{11}, & [e_{11}, e_{14}] &= e_{11}.
 \end{aligned} \tag{22}$$

PROPOSITION 4.5. *The symmetry Lie algebra is a fifteen-dimensional indecomposable solvable Lie algebra. The nilradical is a direct sum of an eight-dimensional nilpotent Lie algebra N_8 spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ and a three-dimensional abelian Lie algebra spanned by e_9, e_{10}, e_{11} . The complement is a four-dimensional abelian Lie algebra spanned by $e_{12}, e_{13}, e_{14}, e_{15}$. Hence the symmetry Lie algebra can be identified by $(N_8 \oplus R^3) \rtimes \mathbb{R}^4$, where the non-zero brackets of N_8 are given by*

$$[e_1, e_2] = e_3, [e_2, e_5] = -e_7, [e_2, e_8] = -e_6, [e_4, e_5] = e_1, [e_4, e_7] = e_3. \tag{23}$$

6.33: $[e_4, e_5] = e_3, [e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_2, e_3] = e_3,$
 $[e_2, e_5] = e_5 + e_6, [e_2, e_6] = e_6.$

The corresponding system of geodesic equations is given by:

$$\ddot{w} = 0, \ddot{z} = 0, \ddot{p} = \dot{p}(\dot{w} + \dot{z}) + \dot{q}(\dot{y} - y\dot{w}), \ddot{y} = \dot{y}\dot{w}, \ddot{q} = \dot{q}\dot{z}, \ddot{x} = \dot{z}(\dot{q} + \dot{x}).$$

The symmetry Lie algebra is spanned by the following basis:

$$\begin{aligned}
 e_1 &= D_x, & e_2 &= D_p, \\
 e_3 &= D_q, & e_4 &= qD_x, \\
 e_5 &= qD_p + D_y, & e_6 &= e^z D_x, \\
 e_7 &= e^y D_w, & e_8 &= e^{z+w} D_p, \\
 e_9 &= ye^z D_p + e^z D_q + (z-1)e^z D_x, & e_{10} &= D_t, \\
 e_{11} &= wD_t, & e_{12} &= zD_t, \\
 e_{13} &= D_w, & e_{14} &= D_z, \\
 e_{15} &= tD_t, & e_{16} &= pD_p + yD_y, \\
 e_{17} &= pD_p + qD_q + xD_x.
 \end{aligned} \tag{24}$$

The non-zero brackets of the symmetry Lie algebra are given by

$$\begin{aligned}
 [e_1, e_{17}] &= e_1, & [e_2, e_{16}] &= e_2, & [e_2, e_{17}] &= e_2, \\
 [e_3, e_4] &= e_1, & [e_3, e_5] &= e_2, & [e_3, e_{17}] &= e_3, \\
 [e_4, e_9] &= -e_6, & [e_5, e_{16}] &= e_5, & [e_6, e_{14}] &= -e_6, \\
 [e_6, e_{17}] &= e_6, & [e_7, e_9] &= e_8, & [e_7, e_{13}] &= -e_7, \\
 [e_7, e_{16}] &= e_7, & [e_8, e_{13}] &= -e_8, & [e_8, e_{14}] &= -e_8, \\
 [e_8, e_{16}] &= e_8, & [e_8, e_{17}] &= e_8, & [e_9, e_{14}] &= -e_6 - e_9, \\
 [e_9, e_{17}] &= e_9, & [e_{10}, e_{15}] &= e_{10}, & [e_{11}, e_{13}] &= -e_{10}, \\
 [e_{11}, e_{15}] &= e_{11}, & [e_{12}, e_{14}] &= -e_{10}, & [e_{12}, e_{15}] &= e_{12}.
 \end{aligned} \tag{25}$$

PROPOSITION 4.6. *The symmetry Lie algebra is a seventeen-dimensional indecomposable solvable Lie algebra. The nilradical is a direct sum of a nine-dimensional nilpotent Lie algebra N_9 spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ and a three-dimensional abelian Lie algebra spanned by e_{10}, e_{11}, e_{12} . The complement is a five-dimensional abelian Lie algebra spanned by $e_{13}, e_{14}, e_{15}, e_{16}, e_{17}$. Hence the symmetry Lie algebra can be identified by $(N_9 \oplus R^3) \rtimes \mathbb{R}^5$, where the non-zero brackets of N_9 are given by*

$$[e_3, e_4] = e_1, [e_3, e_5] = e_2, [e_4, e_9] = -e_6, [e_7, e_9] = e_8. \tag{26}$$

6.34 (a):

$$\begin{aligned}
 [e_4, e_5] &= e_3, & [e_1, e_3] &= e_3, & [e_1, e_4] &= e_4, \\
 [e_1, e_5] &= e_6, & [e_2, e_3] &= (1+a)e_3, & [e_2, e_4] &= ae_4, \\
 [e_2, e_5] &= e_5, & [e_2, e_6] &= e_6.
 \end{aligned}$$

The corresponding system of geodesic equations is given by:

$$\begin{aligned}
 \ddot{w} &= 0, & \ddot{z} &= 0, \\
 \ddot{p} &= \dot{p}(a\dot{z} + \dot{z} + \dot{w}) + \dot{q}(y\dot{z} - \dot{y}) + q\dot{y}((1-a)\dot{z} - \dot{w}), & \ddot{y} &= \dot{y}(a\dot{z} + \dot{w}), \\
 \ddot{q} &= \dot{q}\dot{z}, & \ddot{x} &= \dot{q}\dot{w} + \dot{x}\dot{z}.
 \end{aligned}$$

The symmetry Lie algebra is spanned by the following basis

$$\begin{aligned}
 e_1 &= D_x, & e_2 &= D_p, & e_3 &= D_t, \\
 e_4 &= wD_t, & e_5 &= zD_t, & e_6 &= qD_x, \\
 e_7 &= e^z D_x, & e_8 &= e^{(a+1)z+w} D_p, & e_9 &= D_w, \\
 e_{10} &= tD_t, & e_{11} &= D_z, & e_{12} &= pD_p + yD_y, \\
 e_{13} &= pD_p + qD_q + xD_x.
 \end{aligned} \tag{27}$$

The non-zero brackets of the symmetry Lie algebra are given by

$$\begin{aligned}
 [e_1, e_{13}] &= e_1, & [e_2, e_{12}] &= e_2, & [e_2, e_{13}] &= e_2, & [e_3, e_{10}] &= e_3, \\
 [e_4, e_9] &= -e_3, & [e_4, e_{10}] &= e_4, & [e_5, e_{10}] &= e_5, & [e_5, e_{11}] &= -e_3, \\
 [e_7, e_{11}] &= -e_7, & [e_7, e_{13}] &= e_7, & [e_8, e_9] &= -e_8, & [e_8, e_{11}] &= (-a-1)e_8, \\
 [e_8, e_{12}] &= e_8, & [e_8, e_{13}] &= e_8.
 \end{aligned} \tag{28}$$

PROPOSITION 4.7. *The symmetry Lie algebra is a thirteen-dimensional indecomposable solvable Lie algebra. The nilradical is an eight-dimensional abelian Lie algebra spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$. The complement is a five-dimensional abelian Lie algebra spanned by $e_9, e_{10}, e_{11}, e_{12}, e_{13}$. Hence the symmetry algebra is $\mathbb{R}^8 \rtimes \mathbb{R}^5$.*

6.35 ($ab : a^2 + b^2 \neq 0$):

$$\begin{aligned}
 [e_4, e_5] &= e_3, & [e_1, e_4] &= e_5, & [e_1, e_5] &= -e_4, \\
 [e_1, e_6] &= ae_6, & [e_2, e_3] &= 2e_3, & [e_2, e_4] &= e_4, \\
 [e_2, e_5] &= e_5, & [e_2, e_6] &= be_6.
 \end{aligned}$$

The corresponding system of geodesic equations is given by:

$$\begin{aligned}
 \ddot{w} &= 0, & \ddot{z} &= 0, & \ddot{p} &= 2\dot{p}(\dot{z} + p\dot{w}) + \dot{q}(q\dot{w} + x\dot{z}) - \dot{x}(x\dot{w} + q\dot{z}), \\
 \ddot{q} &= \dot{q}\dot{z} - \dot{x}\dot{w}, & \ddot{x} &= \dot{q}\dot{w} + \dot{x}\dot{z}, & \ddot{y} &= \dot{y}(b\dot{z} + a\dot{w}).
 \end{aligned} \tag{29}$$

The symmetry Lie algebra is spanned by the following basis:

$$\begin{aligned}
 e_1 &= D_t, & e_2 &= zD_t, & e_3 &= D_w - \frac{a}{b}D_z, & e_4 &= zD_t + \frac{a}{b}wD_t, \\
 e_5 &= e^{aw+bz}D_y, & e_6 &= D_y, & e_7 &= tD_t, & e_8 &= D_z, \\
 e_9 &= yD_y.
 \end{aligned} \tag{30}$$

The non-zero brackets of the symmetry Lie algebra are given by

$$\begin{aligned}
 [e_1, e_7] &= e_1, & [e_2, e_3] &= \frac{a}{b}e_1, & [e_2, e_7] &= e_2, & [e_2, e_8] &= -e_1, \\
 [e_4, e_7] &= e_4, & [e_4, e_8] &= -e_1, & [e_5, e_8] &= -be_5, & [e_5, e_9] &= e_5, \\
 [e_6, e_9] &= e_6.
 \end{aligned} \tag{31}$$

PROPOSITION 4.8. *The symmetry Lie algebra is a nine-dimensional indecomposable solvable Lie algebra. The nilradical is a six-dimensional decomposable solvable Lie algebra spanned by $e_1, e_2, e_3, e_4, e_5, e_6$. The complement is an abelian three-dimensional abelian Lie algebra spanned by e_7, e_8, e_9 . Hence the symmetry algebra is $(A_{3,1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^3$.*

6.36: $[e_4, e_5] = e_3, [e_1, e_4] = e_5, [e_1, e_5] = -e_4, [e_2, e_3] = 2e_3, [e_2, e_4] = e_4, [e_2, e_5] = e_5, [e_2, e_6] = 2e_6 + e_3.$

The corresponding system of geodesic equations is given by:

$$\begin{aligned} \ddot{w} &= 0, \quad \ddot{z} = 0, \quad \ddot{p} = \dot{z}(2\dot{y} - x\dot{q} + 2\dot{p}) - \dot{w}(q\dot{q} + x\dot{x}), \\ \ddot{q} &= \dot{q}\dot{z} - \dot{x}\dot{w}, \quad \ddot{x} = \dot{q}\dot{w} + \dot{x}\dot{z}, \quad \ddot{y} = 2\dot{y}\dot{z}. \end{aligned}$$

The symmetry Lie algebra is spanned by the following basis

$$\begin{aligned} e_1 &= e^{2z}D_p, & e_2 &= D_p, \\ e_3 &= (2z - 1)e^{2z}D_p + e^{2z}D_y, & e_4 &= D_y, \\ e_5 &= yD_p, & e_6 &= D_w, \\ e_7 &= D_t, & e_8 &= wD_t, \\ e_9 &= zD_t, & e_{10} &= qD_p + D_x, \\ e_{11} &= tD_t, & e_{12} &= D_z, \\ e_{13} &= pD_p + \frac{q}{2}D_q + \frac{x}{2}D_x + yD_y. \end{aligned} \tag{32}$$

The non-zero brackets of the symmetry Lie algebra are given by

$$\begin{aligned} [e_1, e_{12}] &= -2e_1, & [e_1, e_{13}] &= e_1, & [e_2, e_{13}] &= e_2, & [e_3, e_5] &= e_1, \\ [e_3, e_{12}] &= -2e_1 - 2e_3, & [e_3, e_{13}] &= e_3, & [e_4, e_5] &= e_2, & [e_4, e_{13}] &= e_4, \\ [e_6, e_8] &= e_7, & [e_7, e_1] &= e_7, & [e_8, e_{11}] &= e_8, & [e_9, e_{11}] &= e_9, \\ [e_9, e_{12}] &= -e_7, & [e_{10}, e_{13}] &= \frac{1}{2}e_8. \end{aligned} \tag{33}$$

PROPOSITION 4.9. *The symmetry Lie algebra is a thirteen-dimensional indecomposable solvable Lie algebra. The nilradical is ten-dimensional decomposable Lie algebra spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$. The complement is a three-dimensional abelian Lie algebra spanned by e_{11}, e_{12}, e_{13} . Hence the symmetry algebra is $(A_{5,1} \oplus A_{3,1} \oplus \mathbb{R}^2) \rtimes \mathbb{R}^3$, where $A_{5,1}$ and $A_{3,1}$ are the five and three dimensional Lie algebras in Winternitz list.*

6.37 (a):

$$\begin{aligned} [e_4, e_5] &= e_3, & [e_1, e_4] &= e_5, & [e_1, e_5] &= -e_5, \\ [e_2, e_3] &= 2e_3, & [e_2, e_4] &= e_4 + ae_5, & [e_2, e_5] &= -ae_4 + e_5, \\ [e_2, e_6] &= 2e_5, & [e_1, e_6] &= e_3. \end{aligned}$$

The corresponding system of geodesic equations is given by:

$$\ddot{w} = 0, \ddot{z} = 0, \ddot{y} = (p^2 + q^2)(\dot{w} - a\dot{z})\dot{z} + (a^2 - 1)(p\dot{q} - q\dot{p})\dot{z} + (aq - p)\dot{w}\dot{p} - (ap + q)\dot{w}\dot{q}, \ddot{p} = \dot{p}\dot{z} + (\dot{q} - q\dot{z})(a\dot{z} - \dot{w}), \ddot{q} = \dot{q}\dot{z} + (\dot{p} - p\dot{z})(\dot{w} - a\dot{z}), \ddot{x} = 2\dot{x}\dot{z}.$$

The symmetry Lie algebra is spanned by the following basis

$$\begin{aligned} e_1 &= tD_y, & e_2 &= D_y, & e_3 &= D_t, \\ e_4 &= D_w, & e_5 &= wD_y, & e_6 &= zD_y, \\ e_7 &= wD_t, & e_8 &= zD_t, & e_9 &= D_x, \\ e_{10} &= qD_p - pD_q, & e_{11} &= e^{2z}D_x, & e_{12} &= tD_t, \\ e_{13} &= D_z, & e_{14} &= xD_x, & e_{15} &= pD_p + qD_q + 2yD_y. \end{aligned} \quad (34)$$

The non-zero brackets of the symmetry Lie algebra are given by

$$\begin{aligned} [e_1, e_3] &= -e_2, & [e_1, e_7] &= -e_5, & [e_1, e_8] &= -e_6, & [e_1, e_{12}] &= -e_1, \\ [e_1, e_{15}] &= 2e_1, & [e_2, e_{15}] &= 2e_2, & [e_3, e_{12}] &= e_3, & [e_4, e_5] &= e_2, \\ [e_4, e_7] &= e_3, & [e_5, e_{15}] &= 2e_5, & [e_6, e_{13}] &= -e_2, & [e_6, e_{15}] &= 2e_6, \\ [e_7, e_{12}] &= e_7, & [e_8, e_{12}] &= e_8, & [e_8, e_{13}] &= -e_3, & [e_9, e_{14}] &= e_9, \\ [e_{11}, e_{13}] &= -2e_{11}, & [e_{11}, e_{14}] &= e_{11}. \end{aligned} \quad (35)$$

PROPOSITION 4.10. *The symmetry Lie algebra is a fifteen-dimensional indecomposable solvable Lie algebra. The nilradical is an eleven-dimensional decomposable Lie algebra spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}$. The complement is a four-dimensional abelian Lie algebra spanned by $e_{12}, e_{13}, e_{14}, e_{15}$. Hence the symmetry algebra is $(N_8 \oplus \mathbb{R}^3) \rtimes \mathbb{R}^4$, where N_8 is an eight-dimensional nilpotent Lie algebra given by the following non-zero brackets:*

$$[e_1, e_3] = -e_2, [e_1, e_7] = -e_5, [e_1, e_8] = -e_6, [e_4, e_5] = e_2, [e_4, e_7] = e_3. \quad (36)$$

6.38: $[e_4, e_5] = e_3, [e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_2, e_3] = e_3, [e_2, e_5] = e_5, [e_1, e_2] = e_6.$

The corresponding system of geodesic equations is given by:

$$\begin{aligned} \ddot{w} &= 0, & \ddot{z} &= 0, \\ \ddot{p} &= (\dot{z} + \dot{w})(\dot{p} - xe^{-z}\dot{q}) - e^{-z}\dot{q}\dot{x}, & \ddot{x} &= \dot{x}(2\dot{z} + \dot{w}) - x\dot{z}(\dot{z} + \dot{w}), \\ \ddot{q} &= \dot{q}\dot{z}, & \ddot{y} &= \dot{z}\dot{w}. \end{aligned}$$

The symmetry Lie algebra is spanned by the following basis:

$$\begin{aligned}
e_1 &= D_y, & e_2 &= D_t, \\
e_3 &= wD_y, & e_4 &= wD_t, \\
e_5 &= D_w - xD_x - D_z, & e_6 &= D_p, \\
e_7 &= D_q, & e_8 &= (w+z)D_y, \\
e_9 &= (w+z)D_t, & e_{10} &= e^{w+z}D_p, \\
e_{11} &= tD_t - \frac{1}{2}(wz - 2y)D_y, & e_{12} &= pD_p + qD_q, \\
e_{13} &= pD_p + xD_x, & e_{14} &= -pD_p + D_z + \frac{w}{2}D_y, \\
e_{15} &= tD_y, & e_{16} &= tD_t + \frac{1}{2}(wz - 2y)D_y, \\
e_{17} &= (wz - 2y)D_t.
\end{aligned} \tag{37}$$

The non-zero brackets of the symmetry Lie algebra are given by

$$\begin{aligned}
[e_1, e_{11}] &= e_1, & [e_1, e_{16}] &= -e_1, & [e_1, e_{17}] &= -2e_2, \\
[e_2, e_{11}] &= e_2, & [e_2, e_{15}] &= e_1, & [e_2, e_{16}] &= e_2, \\
[e_3, e_5] &= -e_1, & [e_3, e_{11}] &= e_3, & [e_3, e_{16}] &= -e_3, \\
[e_3, e_{17}] &= -2e_4, & [e_4, e_5] &= -e_2, & [e_4, e_{11}] &= e_4, \\
[e_4, e_{15}] &= e_3, & [e_4, e_{16}] &= e_4, & [e_5, e_{11}] &= e_3 - \frac{1}{2}e_8, \\
[e_5, e_{14}] &= \frac{1}{2}e_1, & [e_5, e_{16}] &= -e_3 + \frac{1}{2}e_8, & [e_5, e_{17}] &= -2e_4 + e_9, \\
[e_6, e_{12}] &= e_6, & [e_6, e_{13}] &= e_6, & [e_6, e_{14}] &= -e_6, \\
[e_7, e_{12}] &= e_7, & [e_8, e_{11}] &= e_8, & [e_8, e_{14}] &= -e_1, \\
[e_8, e_{16}] &= -e_8, & [e_8, e_{17}] &= -2e_9, & [e_9, e_{11}] &= e_9, \\
[e_9, e_{14}] &= -e_2, & [e_9, e_{15}] &= e_8, & [e_9, e_{16}] &= e_9, \\
[e_{10}, e_{12}] &= e_{10}, & [e_{10}, e_{13}] &= e_{10}, & [e_{10}, e_{14}] &= -2e_{10}, \\
[e_{15}, e_{16}] &= -2e_{15}, & [e_{15}, e_{17}] &= -2e_{16}, & [e_{16}, e_{17}] &= -2e_{17}.
\end{aligned} \tag{38}$$

PROPOSITION 4.11. *The symmetry Lie algebra is a seventeen-dimensional indecomposable Lie algebra. The nilradical is a ten-dimensional decomposable Lie algebra spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$. The complement is a four-dimensional abelian Lie algebra spanned by $e_{11}, e_{12}, e_{13}, e_{14}$. Hence the symmetry algebra is $((A_{5,1} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^4 \rtimes sl(2, \mathbb{R}))$, where $A_{5,1}$ is a five-dimensional Lie algebra listed by Winternitz.*

6.39:

$$\begin{aligned}
[e_4, e_5] &= e_3, & [e_1, e_4] &= e_5, & [e_1, e_5] &= -e_4, \\
[e_2, e_3] &= 2e_3, & [e_2, e_4] &= e_4, & [e_2, e_5] &= e_5, \\
[e_1, e_2] &= e_6.
\end{aligned}$$

The corresponding system of geodesic equations is given by:

$$\begin{aligned}\ddot{w} &= 0, \quad \ddot{z} = 0, \quad \ddot{p} = \dot{w}\dot{z}, \\ \ddot{x} &= \cos(2w)\dot{x}\dot{w} + (y(\sin^2(w) + 1) - x\sin(2w))\dot{z}\dot{w} - \dot{y}\dot{z}, \\ \ddot{y} &= \dot{y}\dot{w} + \dot{x}\dot{z} + x\cos^2(w)\dot{z}\dot{w} + \sin(2w)\dot{x}\dot{w}, \\ \ddot{q} &= 2\dot{q}\dot{w} + (y^2 - x^2\cos(2w))\dot{z}\dot{w} + y\cos(2w)\dot{x}\dot{w} \\ &\quad - x(\dot{y}\dot{w} + \dot{x}\dot{z}) - y\dot{y}\dot{z} - x\sin(2w)\dot{w}(y\dot{z} + \dot{x}).\end{aligned}$$

The symmetry Lie algebra is spanned by the following basis

$$\begin{aligned}e_1 &= D_p & e_2 &= D_t \\ e_3 &= zD_p & e_4 &= zD_t \\ e_5 &= D_z & e_6 &= wD_t \\ e_7 &= wD_p & e_8 &= D_q \\ e_9 &= e^{2w}D_q & e_{10} &= tD_t - \frac{1}{2}(zw - 2p)D_p \\ e_{11} &= 2qD_q + xD_x + yD_y & e_{12} &= tD_t + \frac{1}{2}(wz - 2p)D_p \\ e_{13} &= tD_p & e_{14} &= (wz - 2p)D_t.\end{aligned}\tag{39}$$

The non-zero brackets of the symmetry Lie algebra are given by

$$\begin{aligned}[e_1, e_{10}] &= e_1, & [e_1, e_{12}] &= -e_1, & [e_1, e_{14}] &= -2e_2, \\ [e_2, e_{10}] &= e_2, & [e_2, e_{12}] &= e_2, & [e_2, e_{13}] &= e_1, \\ [e_3, e_5] &= -e_1, & [e_3, e_{10}] &= e_3, & [e_3, e_{12}] &= -e_3, \\ [e_3, e_{14}] &= -2e_4, & [e_4, e_5] &= -e_2, & [e_4, e_{10}] &= e_4, \\ [e_4, e_{12}] &= e_4, & [e_4, e_{13}] &= e_3, & [e_5, e_{10}] &= -\frac{1}{2}e_7, \\ [e_5, e_{12}] &= \frac{1}{2}e_7, & [e_5, e_{14}] &= e_6, & [e_6, e_{10}] &= e_6, \\ [e_6, e_{12}] &= e_6, & [e_6, e_{13}] &= e_7, & [e_7, e_{10}] &= e_7, \\ [e_7, e_{12}] &= -e_7, & [e_7, e_{14}] &= -2e_6, & [e_8, e_{11}] &= 2e_8, \\ [e_9, e_{11}] &= 2e_9, & [e_{12}, e_{13}] &= 2e_{13}, & [e_{12}, e_{14}] &= -2e_{14}, \\ [e_{13}, e_{14}] &= -2e_{12}.\end{aligned}\tag{40}$$

PROPOSITION 4.12. *The symmetry Lie algebra is a fourteen-dimensional indecomposable Lie algebra. The nilradical is a nine-dimensional decomposable Lie algebra spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$. The complement is a two-dimensional abelian Lie algebra spanned by e_{10}, e_{11} . The Levi decomposition also gives $sl(2, \mathbb{R})$ spanned by e_{12}, e_{13}, e_{14} . Hence the symmetry algebra is $(A_{5,1} \oplus \mathbb{R}^4) \rtimes \mathbb{R}^2 \rtimes sl(2, \mathbb{R})$.*

6.40: $[e_4, e_5] = e_3, [e_1, e_4] = e_5, [e_1, e_5] = -e_4, [e_2, e_6] = e_6,$
 $[e_1, e_2] = e_3.$

The corresponding system of geodesic equations is given by:

$$\ddot{w} = 0, \ddot{z} = 0, \ddot{q} = \dot{w}(x\dot{x} + y\dot{y}) + 2\dot{z}\dot{w}, \ddot{y} = \dot{x}\dot{w}, \ddot{x} = -\dot{y}\dot{w}, \ddot{p} = \dot{p}\dot{z}.$$

The symmetry Lie algebra is spanned by the following basis:

$$\begin{aligned} e_1 &= tD_q, & e_2 &= D_q, & e_3 &= D_p, & e_4 &= wD_q, & e_5 &= zD_q, \\ e_6 &= wD_t, & e_7 &= zD_t, & e_8 &= yD_q + D_x, & e_9 &= -xD_q + D_y, \\ e_{10} &= (-y\cos(w) + x\sin(w))D_q + \cos(w)D_x + \sin(w)D_y, \\ e_{11} &= (-x\cos(w) - y\sin(w))D_q + \sin(w)D_x - \cos(w)D_y, \\ e_{12} &= D_p, & e_{13} &= e^z D_p, & e_{14} &= D_z + wD_q, & e_{15} &= D_w + zD_q, \\ e_{16} &= tD_t, & e_{17} &= pD_p, & e_{18} &= yD_x - xD_y, \\ e_{19} &= (-2wz + 2q)D_q + xD_x + yD_y. \end{aligned} \tag{41}$$

The non-zero brackets of the symmetry Lie algebra are given by

$$\begin{aligned} [e_1, e_3] &= -e_2, & [e_1, e_6] &= -e_4, & [e_1, e_7] &= -e_5, & [e_1, e_{16}] &= -e_1, \\ [e_1, e_{19}] &= 2e_1, & [e_2, e_{19}] &= 2e_2, & [e_3, e_{16}] &= e_3, & [e_4, e_{15}] &= -e_2, \\ [e_4, e_{19}] &= 2e_4, & [e_5, e_{14}] &= -e_2, & [e_5, e_{19}] &= 2e_5, & [e_6, e_{15}] &= -e_3, \\ [e_6, e_{16}] &= e_6, & [e_7, e_{14}] &= -e_3, & [e_7, e_{16}] &= e_7, & [e_8, e_9] &= -2e_2, \\ [e_8, e_{18}] &= -e_9, & [e_8, e_{19}] &= e_8, & [e_9, e_{18}] &= e_8, & [e_9, e_{19}] &= e_9, \\ [e_{10}, e_{11}] &= -2e_2, & [e_{10}, e_{15}] &= e_{11}, & [e_{10}, e_{18}] &= e_{11}, & [e_{10}, e_{19}] &= e_{10}, \\ [e_{11}, e_{15}] &= -e_{10}, & [e_{11}, e_{18}] &= -e_{10}, & [e_{11}, e_{19}] &= e_{11}, & [e_{12}, e_{17}] &= e_{12}, \\ [e_{13}, e_{14}] &= -e_{13}, & [e_{13}, e_{17}] &= e_{13}. \end{aligned} \tag{42}$$

PROPOSITION 4.13. *The symmetry Lie algebra is a nineteen-dimensional indecomposable solvable Lie algebra. The nilradical is a thirteen-dimensional decomposable Lie algebra spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}$. The complement is a six-dimensional abelian Lie algebra spanned by $e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}$. Hence the symmetry algebra is $(N_{11} \oplus \mathbb{R}^2) \rtimes \mathbb{R}^6$, where N_{11} is an eleven-dimensional nilpotent Lie algebra given by the following non-zero brackets:*

$$\begin{aligned} [e_1, e_3] &= -e_2, & [e_1, e_6] &= -e_4, \\ [e_1, e_7] &= -e_5, & [e_8, e_9] &= -2e_2, \\ [e_{10}, e_{11}] &= -2e_2. \end{aligned} \tag{43}$$

5. Conclusions and future work

In this work, we have investigated the symmetry Lie algebra of the geodesic equations of the canonical connection on a Lie group corresponding to the thirteen classes of Lie algebra $A_{6,28} - A_{6,40}$ in [1]. Each of these thirteen Lie algebras has the property that its nilradical is non-abelian and its complement is non-abelian as well. For each Lie algebra, we list the nonzero brackets of the given Lie algebra, the geodesic equations, a basis for the symmetry Lie algebra in terms of vector fields and the non-zero brackets of the symmetry Lie algebra. We finally identify its nilradical, solvable complement, and semi-simple factor; a summary of our results is given in Table 1. In future work, we would like to construct the Lie invariance condition for any n -dimensional Lie algebra with r -codimensional abelian nilradical. For this general n -dimensional case, we will investigate the symmetry Lie algebra by integration the system of PDE's coming for the Lie invariance condition and establish results about the dimension of the symmetry Lie algebra.

Lie Algebras	Dimension	Identification
$A_{6,28}$	9	$(A_{3,1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^3$
$A_{6,29}$	17	$(A_{3,1} \oplus A_{3,1} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^6$
$A_{6,30}$	14	$\mathbb{R}^9 \rtimes \mathbb{R}^5$
$A_{6,31}$	17	$(N_9 \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$
$A_{6,32}$	15	$(N_8 \oplus \mathbb{R}^3) \rtimes \mathbb{R}^4$
$A_{6,33}$	18	$(N_9 \oplus \mathbb{R}^3) \rtimes \mathbb{R}^5$
$A_{6,34}$	18	$\mathbb{R}^8 \rtimes \mathbb{R}^5$
$A_{6,35}$	18	$(A_{3,1} \oplus \mathbb{R}^3) \rtimes \mathbb{R}^3$
$A_{6,36}$	18	$(A_{5,1} \oplus A_{3,1} \oplus \mathbb{R}^2) \rtimes \mathbb{R}^3$
$A_{6,37}$	15	$(N_8 \oplus \mathbb{R}^3) \rtimes \mathbb{R}^4$
$A_{6,38}$	17	$((A_{5,1} \oplus \mathbb{R}^5) \rtimes \mathbb{R}^4) \rtimes sl(2, \mathbb{R})$
$A_{6,39}$	14	$(A_{5,1} \oplus \mathbb{R}^4) \rtimes \mathbb{R}^2 \rtimes sl(2, \mathbb{R})$
$A_{6,40}$	19	$(N_{11} \oplus \mathbb{R}^2) \rtimes \mathbb{R}^6$

TABLE 1. Six-dimensional Lie algebras and identification of the symmetry algebra

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