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STRESS-SUM INDEX OF GRAPHS WITH DIAMETER TWO

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Abstract

The process of ranking molecular chemical compounds or large networks is a challenging task as they are degenerate in nature. A novel idea is to apply graph theory-based concepts like centrality measures and topological indices based on them to predict the rank of each molecular chemical graph. The stress of a vertex in a graph is a measure of vertex centrality and is defined as the number of shortest paths that pass through it. Recently, some topological indices based on the stress of vertices in a graph have been defined. In this article, we obtain the stress-sum index of some standard class of graphs with diameter two.

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1. Introduction

Let $G = (V(G), E(G))$ be a finite simple graph with vertex set $V(G) = \{w_1, w_2, \ldots, w_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. If two vertices w_i and w_j are adjacent, we write $w_i \sim w_j$, and the edge between them is denoted by $w_i w_j$.

Let $P = (u = w_0, w_1, \ldots, w_k = v)$ be a u-v path of length k in G with *origin* $u = w_0$ and *terminus* $v = w_k$. The vertices w_i, $1 \leq i \leq k-1$ are called the *internal* vertices of the path P. The length of a shortest u-v path, denoted by $d(u, v)$ is called *distance* between u and v. The *diameter* of G, denoted by $\operatorname{diam}(G)$, is given by $\max\{d(u, v) : u, v \in V\}$. For the remaining graph-theoretic terminologies, we refer to [8].

A vertex centrality measure assigns a real number to each vertex of a graph, and it quantifies the importance or criticality of a vertex from a particular perspective. Different centrality measures describe the importance of a vertex from different points of view. Stress is one such vertex centrality measure studied in [7].

Definition 1.1. [7] Let G be a graph with $V(G) = \{w_1, \ldots, w_n\}$. The *stress* of a vertex w_i is the number of shortest paths in G having w_i as an internal vertex and is denoted by $st(w_i)$. The stress of a graph G is defined by $st(G) = \sum\limits_{i=1}^{n} st(w_i)$.

Stress centrality has many applications in the study of biological networks, social networks etc. Some related works can be found in [1, 4, 5, 6].

Molecular descriptors play a significant role in mathematical chemistry, especially in quantitative structure-property relationship (QSPR), and quantitative structure-activity relationship (QSAR) investigations. Over hundreds of topological graph indices are registered in Chemical Data Bases, which correlate strongly with the physico-chemical properties of molecules. Researchers in the verge of refining the existing topological indices are replacing the usual degrees and eccentricities by various centrality measures.

Motivated by the Zagreb indices, Rajendra et al. [3] introduced two topological indices of graphs called first stress index and second stress index, using stresses of the vertices. Recently, the authors of article [2], have introduced the stress-sum index as follows.

DEFINITION 1.2. [2] The stress-sum index $SS(G)$ of a simple graph G is defined as

$$SS(G) = \sum_{uv \in E(G)} st(u) + st(v).$$

Observe that $\sum_{uv \in E(G)} st(u) + st(v) \neq \sum_{u \in V(G)} (st(u))^2$.

In this article, we obtain the stress-sum index of a few classes of graphs with diameter two.

2. **Graphs with diameter two**

The theorem below gives an expression for the stress centrality of graphs with diameter 2.

THEOREM 2.1. [5] *Suppose $diam(G) = 2$, then for any vertex w in G with $deg(w) \geq 2$,*

$$st(w) = \binom{deg(w)}{2} - m(w),$$

where $m(w)$ denotes the number of edges in the induced subgraph $G[N(w)]$.

Using Theorem 2.1, we obtain stress centrality and hence the stress-sum index of some classes of graphs.

THEOREM 2.2. *Let $G = \overline{P_n}$. Then, $st(G) = (n-3)(n-2)$.*

P r o o f. Let $V(G) = \{w_1, w_2, \ldots, w_n\}$, $deg(w_1) = n-2$. The induced subgraph $G[N(w_1)]$ is the graph, which is obtained from K_{n-2} by removing the edges of P_{n-2}. Hence, $m(w_1) = \binom{n-2}{2} - (n-3)$. Then, $st(w_1) = st(w_n) = \binom{n-2}{2} - \{\binom{n-2}{2} - (n-3)\} = n-3$.

As $deg(w_2) = n-3$, the induced subgraph $G[N(w_2)]$ is the graph which is obtained by K_{n-3} by removing the edges of P_{n-3}. Hence,

$m(w_2) = \binom{n-3}{2} - (n-4)$, and $st(w_2) = st(w_{n-1)} = \binom{n-3}{2} - \{\binom{n-3}{2} - (n-4)\} = n-4$.

Also, $deg(w_3) = n-3$, and $m(w_3) = \binom{n-3}{2} - (n-5)$. Then, $st(w_3) = \binom{n-3}{2} - \{\binom{n-3}{2} - (n-5)\} = n-5$. Similarly, the stress of all the remaining $n-4$ vertices is also $n-5$.

Hence, $st(G) = 2(n-3) + 2(n-4) + (n-4)(n-5)$. □

THEOREM 2.3. *Let $G = \overline{P_n}$. Then,*

$$SS(G) = n^3 - 8n^2 + 23n - 24.$$

P r o o f. Let $V(G) = \{w_1, w_2, \dots, w_n\}$. The edges in which w_1 is one of the end vertices contributes $(n-4)(2n-8) + (4n-13)$ to $SS(G)$. Similarly, the edges in which w_2 is one of the end vertex contributes $(n-5)(2n-9) + (4n-15)$ to $SS(G)$. The edges $w_3w_j, j \geq 5$, contributes $(n-6)(2n-10) + (4n-17)$ to $SS(G)$. Continuing like this the edges $w_{n-4}w_j, j \geq n-2$, contributes $(2n-10) + (4n-17)$ to $SS(G)$. Finally, the edges $w_{n-3}w_{n-1}$ and $w_{n-3}w_n$ contributes $4n-17$ to $SS(G)$. Hence,

$$SS(G) = (n-4)(2n-8) + (4n-13) + (n-5)(2n-9) + (4n-15) + (2n-10)$$

$$\times \sum_{i=1}^{n-6} i + (4n-17)(n-5) + 2n - 8 = n^3 - 8n^2 + 23n - 24.$$

□

THEOREM 2.4. *Let $G = \overline{C_n}$. Then, $st(G) = n(n-4)$.*

P r o o f. Let $V(G) = \{w_1, w_2, \dots, w_n\}$ and $deg(w_i) = n-3$. Then, $m(w_i) = \binom{n-3}{2} - (n-4)$. By Theorem 2.1, $st(w_i) = \binom{n-3}{2} - \left(\binom{n-3}{2} - (n-4)\right)$ $= n-4$. Hence the result follows. □

THEOREM 2.5. *Let $G = \overline{C_n}$. Then, $SS(G) = n(n-3)(n-4)$.*

P r o o f. From Theorem 2.4, if $w \in V(G)$ then $st(w) = n-4$. As $|E(G)| = \dfrac{n(n-3)}{2}$, the result follows. □

THEOREM 2.6. *Let $DW_n = 2C_n + K_1$ be a double wheel graph on $2n+1$ vertices. Then, $st(DW_n) = n(2n-1)$.*

P r o o f. Let $G = DW_n$ and $V(G) = \{u_1, \ldots, u_n, {u_1}', \ldots, {u_n}', u\}$, where $u_i, {u_i}'$ are the vertices of the cycle graph $2C_n$. Note that $deg(u) = 2n$ and $m(u) = 2n$. Hence, by Theorem 2.1, we have $st(u) = \binom{2n}{2} - 2n = n(2n-3)$.

Now, $deg(u_i) = deg(u_i') = 3$ and $m(u_i) = m(u_i') = 2$. Hence, $st(u_i) = st({u_i}') = \binom{3}{2} - 2 = 1$.

Then, $st(G) = n(2n-3) + 2n = n(2n-1)$. □

THEOREM 2.7. *Let $G = DW_n$. Then, $SS(G) = 2n(2n^2 - 6n + 3)$.*

P r o o f. Let $V(G) = \{u_1, u_2, \ldots, u_n, {u_1}', {u_2}', \ldots, {u_n}', u\}$, where $u_i, {u_i}'$ are the vertices of two disjoint cycle graph C_n. From Theorem 2.6, $st(u) = n(2n-3)$ and u is adjacent to all the other $2n$ vertices, each having a stress equal to 1. These edges contributes $(n(2n-3)+1)2n$ to $SS(G)$. The other $2n$ edges, with the end vertices having stress equal to 1 contributes $4n$ to $SS(G)$. Hence the result follows. □

THEOREM 2.8. *Let $F_{m,n} = \overline{K_m} + P_n$ be a generalized fan graph. Then,*

$$st(F_{m,n}) = \frac{mn(n+m-4) + 2(m+n-2)}{2}.$$

P r o o f. Let $G = F_{m,n}$ with $V(G) = \{w_1, w_2, \ldots, w_m, u_1, \ldots, u_n\}$ and $deg(w_i) = n$ and $m(w_i) = n-1$. By Theorem 2.1, we get $st(w_i) = \binom{n}{2} - (n-1) = \frac{(n-1)(n-2)}{2}$. Also, $deg(u_i) = m+1$ for $i = 1, n$ and $m(u_i) = m$. By Theorem 2.1, we get $st(u_i) = \binom{m+1}{2} - m = \frac{m(m-1)}{2}$.

For $2 \le i \le n-1$, $deg(u_i) = m+2$ and $m(u_i) = 2m$. By Theorem 2.1, we get $st(u_i) = \binom{m+2}{2} - 2m = \frac{m^2-m+2}{2}$. Hence,

$$\begin{aligned} st(G) &= m\left(\frac{(n-1)(n-2)}{2}\right) + m(m-1) + (n-2)\left(\frac{m^2-m+2}{2}\right) \\ &= \frac{mn(n+m-4) + 2(m+n-2)}{2}. \end{aligned}$$

□

THEOREM 2.9. *Let $G = F_{m,n}$. Then,*

$$SS(G) = \frac{mn(m+2+n^2+m^2-3n) - 2(m-2n+m^2+4)}{2}.$$

P r o o f. Let $V(G) = \{w_1, w_2, \dots, w_m, u_1, u_2, \dots, u_n\}$ and $deg(w_i) = n$. The edges $w_iu_1, w_iu_n, 1 \leq i \leq m$ contribute

$$((n-1)(n-2) + m(m-1))\, m$$

to $SS(G)$.

Now the edges $w_iu_j, 1 \leq i \leq m, 2 \leq j \leq n-1$ contribute

$$\left[\left(\frac{(n-1)(n-2)}{2} + \frac{m^2-m+2}{2}\right)(n-2)m\right]$$

to $SS(G)$.

Lastly, the edges $u_iu_{i+1}, 1 \leq i \leq n-1$ contribute

$$\left(\frac{m(m-1)}{2} + \frac{m^2-m+2}{2}\right)2 + \left(m^2-m+2\right)(n-3),$$

to $SS(G)$. Hence,

$$SS(G) = \frac{mn(m+2+n^2+m^2-3n) - 2m + 4n - 2m^2 - 8}{2}.$$

□

Let G and Γ be two graphs with vertex sets V and W respectively. The Kronecker product, $G \otimes \Gamma$ is the graph with vertex set $V \times W$ and two vertices (v, w) and (v', w') are adjacent to each other whenever $v \sim v'$ in G and $w \sim w'$ in Γ. The Kronecker product of K_n and K_m is a regular graph of diameter 2 whose stress is given in the next theorem.

THEOREM 2.10. *Let K_n, K_m be complete graphs with n, m, vertices respectively. Then,*

$$st(K_n \otimes K_m) = \frac{nm(n-1)(m-1)(m+n-4)}{2}.$$

P r o o f. Let $G = K_n \otimes K_m$ be a graph on nm vertices and $deg(w) = (n-1)(m-1)$, if $w \in V(G)$. Then,

$$m(w) = \frac{(n-2)(m-2)(n-1)(m-1)}{2}.$$

By Theorem 2.1, we get

$$st(w) = \binom{(n-1)(m-1)}{2} - \frac{(n-2)(m-2)(n-1)(m-1)}{2}$$
$$= \frac{(n-1)(m-1)(m+n-4)}{2}.$$

Hence, the result follows. □

THEOREM 2.11. *Let $G = K_n \otimes K_m$. Then,*

$$SS(G) = \frac{mn(m-1)^2(n-1)^2(m+n-4)}{2}.$$

P r o o f. The proof follows from Theorem 2.10, by noting that

$$st(w) = \frac{(n-1)(m-1)(m+n-4)}{2}, \quad \text{if} \ \ w \in V(G)$$

and

$$|E(G)| = nm(n-1)(m-1).$$

□

The line graph of G is denoted by $L(G)$ is defined as corresponding to every edge of G there is a vertex in $L(G)$ and two vertices in $L(G)$ are adjacent if the corresponding edges of G have a vertex in common. The line graph of K_n and $K_{p,q}$ are regular graph with diameter two whose stress is given in next Theorems 2.12 and 2.14.

THEOREM 2.12. *Let $G = L(K_n)$. Then, $st(G) = \frac{n(n-1)(n-2)(n-3)}{2}$.*

P r o o f. In G there are $\binom{n}{2}$ vertices and $deg(w) = 2n-4$, if $w \in V(G)$. Note that $m(w) = 2\binom{n-2}{2} + (n-2)$. By Theorem 2.1, we get $st(w) = \binom{2n-4}{2} - \{2\binom{n-2}{2} + (n-2)\} = (n-2)(n-3)$. Hence, the proof follows. □

THEOREM 2.13. *Let $G = L(K_n)$. Then,*

$$SS(G) = 2n(n-1)(n-2)^2(n-3).$$

P r o o f. The proof follows from Theorem 2.12, by noting that $st(w) = (n-2)(n-3)$, if $w \in V(G)$ and $|E(G)| = n(n-1)(n-2)$. □

THEOREM 2.14. *Let $G = K_{p,q}$. Then,*

$$st(L(G)) = pq(pq-p-q+1).$$

P r o o f. Let $V(G) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_q\}$. Let $w \in L(G)$. Then, $deg(w) = p+q-2$, $m(w) = \binom{p-1}{2} + \binom{q-1}{2}$. By Theorem 2.1, we get $st(w) = \binom{p+q-2}{2} - \{\binom{p-1}{2} + \binom{q-1}{2}\} = pq - p - q + 1$. Hence, $st(L(G)) = pq(pq - p - q + 1)$. □

THEOREM 2.15. *Let $G = L(K_{p,q})$. Then,*

$$SS(G) = pq(q + p - 2)(pq - p - q + 1).$$

P r o o f. The proof follows from Theorem 2.14, by noting that

$$st(w) = pq - p - q + 1, \text{ if } w \in V(G) \text{ and } |E(G)| = \frac{pq(p + q - 2)}{2}.$$

□

A bistar graph is the graph obtained by joining the center vertices of $K_{1,n}$ and $K_{1,m}$ by an edge and it is denoted by $B_{n,m}$.

THEOREM 2.16. *Let $G = B_{n,m}$. Then, $st(L(G)) = nm$.*

P r o o f. Let $V(L(G)) = \{v_1, v_2, \ldots, v_m, v, u_1, u_2, \ldots, u_n\}$. Observe that $deg(v_i) = m$, $deg(u_j) = n$, $m(u_i) = \binom{n}{2}$ and $m(v_i) = \binom{m}{2}$ for $1 \leq i \leq m, 1 \leq j \leq n$. By Theorem 2.1, $st(u_i) = st(v_i) = 0$. Also, $deg(v) = m + n$ and $m(v) = \binom{n}{2} + \binom{m}{2}$. Again by Theorem 2.1, we get $st(v) = st(L(G)) = nm$. □

THEOREM 2.17. *Let $G = L(B_{n,m})$. Then, $SS(G) = nm(n + m)$.*

P r o o f. Let $V(G) = \{v_1, v_2, \ldots, v_m, v, u_1, u_2, \ldots, u_n\}$. From Theorem 2.16, $st(v) = nm$. As u is adjacent to all the other vertices with stress equal to zero, the result follows. □

3. **Conclusion**

This article explores one of the novel centrality based topological indices of many classes of derived graphs. There are a huge number of graph operations still open, for which this centrality measure as well as the topological index based on them can be computed.

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