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STUDY THE STABILITY OF THE FOURIER TRANSFORM OF THE VARIABLE-ORDER TIME FRACTIONAL SEMI-LINEAR DIFFUSION EQUATION

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Abstract

In this research, we present a finite difference scheme (FDS) aimed at exploring the linear time and space fractional advection equation. To approximate the fractional derivatives, we utilize the fractional Taylor series for $u(x,t)$ at t_j and x_i. Our primary focus lies in constructing a numerical scheme (NS) for the mathematical model. Following this, we delve into an examination of the stability and convergence of our NS. Ultimately, we conduct numerical simulations of the fractional advection equation using the FDM across different fractional parameter values. The outcomes of these simulations demonstrate satisfactory convergence, thereby confirming the efficacy of the proposed algorithm.

Math. Subject Classification: 26A33, 65L12, 65L07, 65L20

Key Words and Phrases: fractional derivative of variable order, finite difference method, numerical scheme, stability, convergence

1. **Introduction**

The variable-order time fractional semi-linear diffusion equation is a partial differential equation that combines elements of fractional calculus operators, diffusion, and nonlinearity. It is expressed in terms of a time fractional derivative with a variable order, a diffusion term, and a semi-linear term. The equation typically takes the form

$$\frac{\partial^{\alpha(x,t)}u(x,t)}{\partial t^{\alpha(x,t)}} = D(x)\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t,u), \tag{1}$$

where $u(x,t)$ is the unknown function representing the quantity being diffused, $D(x)$ is the diffusion coefficient and $f(x,t,u)$ is a semi-linear term, often involving u and its derivatives.

This equation describes the evolution of a quantity u over time and space, where the diffusion process is influenced by a fractional derivative of variable order and is subject to a semi-linear forcing term. The variable-order $\alpha(x,t)$ aspect allows for flexibility in modeling systems with complex dynamics where the diffusion behavior may vary with both time and space.

2. The problem

Consider the variable order time fractional semi-linear diffusion equation

$$\frac{\partial^{\alpha(x,t)} u(x,t)}{\partial t^{\alpha(x,t)}} = a(x,t) \frac{\partial^2 u(x,t)}{\partial x^2} + f(u), \tag{2}$$
$$0 < x < L, \quad 0 < t < T, \quad 0 < \alpha(x,t) \leq 1,$$

with the initial condition

$$u(x,0) = g(x), \tag{3}$$

and boundary conditions

$$u(0,t) = u(L_x,t). \tag{4}$$

3. Discretization

Let $[0, L]$ be the domain of interest, first we discretize this domain. To do this, let us define $x_i = ih$, where $0 \leq i \leq M$, $Mh = L$, $t_j = jk$, $0 \leq j \leq N$, $Nk = T$, where h is the space step length and k is time step size. Suppose that u_i^j be the numerical approximation of $u(x_i, t_j)$, $f(u_i^j) = f(u(x_i, t_j))$ where f_i^j is the numerical approximation of $f(x_i, t_j)$, $a(x_i, t_j) = a_i^j$ and $u(x_i, 0) = g_i$.

4. Development of the scheme

There exist different approaches to define the Caputo fractional order derivative. For simplification, we consider the interval $[0, t]$

$$\frac{\partial^{\alpha(x,t)} u(x,t)}{\partial t^{\alpha(x,t)}} = \begin{cases} \frac{1}{\Gamma(1-\alpha(x,t))} \int_0^t \frac{u_\xi(x,\xi)}{(t-\xi)^{\alpha(x,t)}} d\xi, & \text{if} \quad 0 < \alpha(x,t) < 1, \\ u_t(x,t), \quad \text{if} \quad \alpha(x,t) = 1. \end{cases} \tag{5}$$

The first-order and second spatial derivatives can be approximated by the following expressions

$$\left(\frac{\partial u}{\partial t}\right)_i^{j+1} = \frac{u_i^{j+1} - u_i^{j-1}}{k} + \Delta k, \tag{6}$$

and

$$\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)_i^{j+1} = \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} + \Delta h^2. \tag{7}$$

We discretize the variable-order time fractional derivative $\frac{\partial^{\alpha\left(x_i,t_{j+1}\right)}u(x_i,t_{j+1})}{\partial t^{\alpha\left(x_i,t_{j+1}\right)}}$ as follows:

$$
\begin{aligned}
\frac{\partial^{\alpha(x_i,t_{j+1})}u\left(x_i,t_{j+1}\right)}{\partial t^{\alpha(x_i,t_{j+1})}} &= \frac{1}{\Gamma(1-\alpha\left(x_i,t_{j+1}\right))}\int_0^{t_{j+1}}\frac{u_\xi\left(x_i,\xi\right)}{\left(t_{j+1}-\xi\right)^{\alpha(x_i,t_{j+1})}}d\xi \\
&= \frac{1}{\Gamma(1-\alpha\left(x_i,t_{j+1}\right))}\sum_{s=0}^{j}\int_{sk}^{(s+1)k}\frac{u_\xi(x_i,\xi)}{\left(t_{j+1}-\xi\right)^{\alpha(x_i,t_{j+1})}}d\xi \\
&= \frac{1}{\Gamma(1-\alpha\left(x_i,t_{j+1}\right))}\sum_{s=0}^{j}\int_{sk}^{(s+1)k}\left(\frac{\partial u}{\partial \xi}\right)_i^{s+1}\frac{d\xi}{\left(t_{j+1}-\xi\right)^{\alpha(x_i,t_{j+1})}} \\
&= \frac{1}{\Gamma(1-\alpha\left(x_i,t_{j+1}\right))}\sum_{s=0}^{j}\frac{u_i^{s+1}-u_i^{s-1}}{k}\int_{(j-s)k}^{(j-s+1)k}\frac{d\eta}{\eta^{\alpha(x_i,t_{j+1})}} \\
&= \frac{k^{-\alpha(x_i,t_{j+1})}}{\Gamma(2-\alpha\left(x_i,t_{j+1}\right))}\left[u_i^{j+1}-u_i^j+\sum_{n=1}^{j}\left(u_i^{j-n+1}-u_i^{j-n-1}\right)\right. \\
&\quad \times\left.\left[(n+1)^{1-\alpha(x_i,t_{j+1})}-n^{1-\alpha(x_i,t_{j+1})}\right]\right], \qquad (8)
\end{aligned}
$$

where $\eta = t_{j+1}-\xi$.

Then, we have

$$
\begin{aligned}
\frac{\partial^{\alpha(x_i,t_{j+1})}u\left(x_i,t_{j+1}\right)}{\partial t^{\alpha(x_i,t_{j+1})}} &= \frac{k^{-\alpha(x_i,t_{j+1})}}{\Gamma(2-\alpha\left(x_i,t_{j+1}\right))} \\
&\times\left[u_i^{j+1}-u_i^j+\sum_{n=1}^{j}\left(u_i^{j-n+1}-u_i^{j-n-1}\right)\right. \\
&\times\left.\left[(n+1)^{1-\alpha(x_i,t_{j+1})}-n^{1-\alpha(x_i,t_{j+1})}\right]\right]. \qquad (9)
\end{aligned}
$$

Now, using (5) and (8), we obtain the following approximate scheme:

$$
-\theta_i^{j+1}u_{i+1}^{j+1}+\left(1+2\theta_i^{j+1}\right)u_i^{j+1}-\theta_i^{j+1}u_{i-1}^{j+1}-h^2\theta_i^{j+1}f\left(u_i^{j+1}\right) \qquad (10)
$$

$$
= u_i^j-\sum_{n=1}^{j}\left(u_i^{j-n+1}-u_i^{j-n-1}\right)\phi_i^{j+1}\left(n\right),
$$

with the initial condition

$$u_i^0 = g_i, \tag{11}$$

and the boundary conditions

$$u_0^{j+1} = u_1^{j+1}, \quad u_N^{j+1} = u_{N-1}^{j+1}, \tag{12}$$

where

$$\theta_i^{j+1} = a_i^j k^{\alpha_i^{j+1}} h^2 \Gamma\left(2 - \alpha_i^{j+1}\right) \text{ and } \phi_i^{j+1}(n) = (n+1)^{1-\alpha_i^{j+1}} - n^{1-\alpha_i^{j+1}}.$$

5. Stability of the approximate scheme

In this section, we utilize Fourier analysis to investigate the stability of the approximate scheme (9)–(12). We define the following function:

$$u^j(x) = \begin{cases} u_i^j, & if \quad x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}} \quad i = 1, \ldots, M-1, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

The function $u^j(x)$ possesses a Fourier series expansion:

$$u^j(x) = \sum_{p=-\infty}^{+\infty} \xi_j(p) e^{\frac{2\pi p}{L} x}, j = 0, \ldots, N, \tag{14}$$

where $\xi_j(p) = \frac{1}{L} \int_0^L u^j(x) e^{-\frac{2\pi p}{L} x} dx$, and here we denote $u_i^j = \xi_j e^{\mu \theta h i}$ such that $\mu^2 = -1$.

THEOREM 5.1. *The implicit finite difference scheme* (9)–(12) *exhibits unconditional stability within the range* $0 < \alpha(x,t) \leq 1$ *if a positive constant* C *exists, satisfying*

$$|\xi_j| \leq C |\xi_0|, \quad j = 1, \ldots, N. \tag{15}$$

P r o o f. We can describe the error function by

$$e_i^j = u_i^j - r_i^j. \tag{16}$$

Now, we start to calculate the error e_i^j, by replacing (16) in (9), then we obtain

$$\begin{aligned} &-\theta_i^{j+1} e_{i+1}^{j+1} + \left(1 + 2\theta_i^{j+1}\right) e_i^{j+1} - \theta_i^{j+1} e_{i-1}^{j+1} \\ &- h^2 \theta_i^{j+1} \Big(f\left(u_i^{j+1}\right) - f\left(r_i^{j+1}\right)\Big) = e_i^j - \sum_{n=1}^{j} \left(e_i^{j-n+1} - e_i^{j-n-1}\right) \phi_i^{j+1}(n), \end{aligned} \tag{17}$$

therefore,

$$\xi_{j+1} = \frac{h^2\theta_i^{j+1}\Big(f\left(u_i^{j+1}\right) - f\left(r_i^{j+1}\right)\Big) + \xi_j - \sum_{s=0}^{j-1}\left(\xi_{s+1} - \xi_{s-1}\right)\phi_i^{j+1}\left(j-s\right)}{\left[1 + 4\theta_i^{j+1}\sin^2\left(\frac{\theta h}{2}\right)\right]}.$$

Now, we assume that f is Lipshitz function, i.e. there is a positive constant K such that

$$\left|f\left(u_i^{j+1}\right) - f\left(r_i^{j+1}\right)\right| \le K_1\left|u_i^{j+1} - r_i^{j+1}\right|. \tag{18}$$

We use to proof by recurrence for $j = 0$, to have

$$\begin{aligned}
|\xi_1| &= \left|\frac{h^2\theta_i^1\Big(f\left(u_i^1\right) - f\left(r_i^1\right)\Big) + \xi_0}{1 + 4\theta_i^1\sin^2\left(\frac{\theta h}{2}\right)}\right| \\
&\le \left|\frac{K_1h^2\theta_i^1\xi_1e^{\mu\theta hi} + \xi_0}{1 + 4\theta_i^1\sin^2\left(\frac{\theta h}{2}\right)}\right| \\
&\le \frac{\left|K_1h^2\theta_i^1\xi_1e^{\mu\theta hi}\right| + |\xi_0|}{\left|1 + 4\theta_i^1\sin^2\left(\frac{\theta h}{2}\right)\right|} \\
&\le \frac{K_1h^2\left|\theta_i^1\right||\xi_1| + |\xi_0|}{\left|1 + 4\theta_i^1\sin^2\left(\frac{\theta h}{2}\right)\right|},
\end{aligned} \tag{19}$$

then

$$\begin{aligned}
|\xi_1| &\le \frac{\left|1 + 4\theta_i^1\sin^2\left(\frac{\theta h}{2}\right)\right|}{\left|1 + 4\theta_i^1\sin^2\left(\frac{\theta h}{2}\right)\right| - K_1h^2\left|\theta_i^1\right|}|\xi_0| && (20)\\
&\le \frac{1 + 4\left|a_i^0\right|k^{\alpha_i^1}h^2}{\left|1 + 4\theta_i^1\sin^2\left(\frac{\theta h}{2}\right)\right| - K_1h^2\left|\theta_i^1\right|}|\xi_0| && (21)\\
&= C_0\left|\xi_0\right|, && (22)
\end{aligned}$$

the constant C_0 is non-negative, where

$$\left|a_i^0\right| \le \frac{1}{k^{\alpha_i^1}\Gamma\left(2 - \alpha_i^1\right)\left(K_1h^2 - 4\right)}. \tag{23}$$

We assume that the statement is true:

$$|\xi_j| \le C_j\left|\xi_0\right|, \quad j = 1, \ldots, N, \tag{24}$$

and then we prove that the statement is also true

$$|\xi_{j+1}| \leq C_{j+1} |\xi_0|, \quad j = 0, \dots, N-1. \tag{25}$$

First, we have

$$|\xi_{j+1}| =$$

$$\left| \frac{h^2 \theta_i^{j+1} \Big(f\left(u_i^{j+1}\right) - f\left(r_i^{j+1}\right) \Big) + \xi_j - \sum_{s=0}^{j-1} \left(\xi_{s+1} - \xi_{s-1}\right) \phi_i^{j+1} (j-s)}{1 + 4\theta_i^{j+1} \sin^2\left(\frac{\theta h}{2}\right)} \right|$$

$$\leq \frac{h^2 \left|\theta_i^{j+1}\right| \left| f\left(u_i^{j+1}\right) - f\left(r_i^{j+1}\right) \right| + \xi_j + \left| \sum_{s=0}^{j-1} \left(\xi_{s+1} - \xi_{s-1}\right) \phi_i^{j+1} (j-s) \right|}{\left|1 + 4\theta_i^{j+1} \sin^2\left(\frac{\theta h}{2}\right)\right|}$$

$$\leq \frac{K_1 h^2 \left|\theta_i^{j+1}\right| \left|u_i^{j+1} - r_i^{j+1}\right| + |\xi_j| + \left| \sum_{s=0}^{j-1} \xi_{s+1} - \xi_{s-1} \right|}{\left|1 + 4\theta_i^{j+1} \sin^2\left(\frac{\theta h}{2}\right)\right|}$$

$$\leq \frac{K_1 h^2 \left|\theta_i^{j+1}\right| |\xi_{j+1}| + 2|\xi_j|}{\left|1 + 4\theta_i^{j+1} \sin^2\left(\frac{\theta h}{2}\right)\right|} \tag{26}$$

$$\leq \frac{K_1 h^2 \left|\theta_i^{j+1}\right| |\xi_{j+1}| + 2C_j |\xi_0|}{\left|1 + 4\theta_i^{j+1} \sin^2\left(\frac{\theta h}{2}\right)\right|}, \tag{27}$$

and we can conclude that

$$|\xi_{j+1}| \leq \frac{2C_j}{1 - \frac{K_1 h^2 \left|\theta_i^{j+1}\right|}{\left|1 + 4\theta_i^{j+1} \sin^2\left(\frac{\theta h}{2}\right)\right|}} |\xi_0| \tag{28}$$

$$= 2C_j \left(1 + \sum_{\ell=1}^{+\infty} \left(\frac{K_1 h^2 \left|\theta_i^{j+1}\right|}{\left|1 + 4\theta_i^{j+1} \sin^2\left(\frac{\theta h}{2}\right)\right|} \right)^{\ell} \right) |\xi_0|, \tag{29}$$

where $\frac{K_1 h^2 \left|\theta_i^{j+1}\right|}{\left|1 + 4\theta_i^{j+1} \sin^2\left(\frac{\theta h}{2}\right)\right|} < 1$ and $\ell \to +\infty$, and we can obtain

$$|\xi_{j+1}| \leq C_{j+1} |\xi_0|, \tag{30}$$

as result. The approximate scheme (10) is unconditionally stable. □

6. **Test problem and numerical experiments**

In this section, we present numerical results for several values of α for solving variable-order time fractional semi-linear diffusion equation.

Two examples are given. The numerical results reveal that differential finite difference method is very efficient and accurate.

Example 1.

We consider the variable order time fractional semi-linear diffusion equation (2)–(4) described by

$$\frac{\partial^{0.75+0.5\cos(xt)}u\left(x,t\right)}{\partial t^{0.75+0.5\cos(xt)}}=\sin\left(\pi xt\right)u_{xx}+\frac{u}{1+u},\quad 0<x<1,\, 0<t<2, \tag{31}$$

with the initial condition

$$u\left(x,0\right)=e^{-x}\sin\left(x\right), \tag{32}$$

and boundary conditions

$$u\left(0,t\right)=u\left(1,t\right)=0. \tag{33}$$

Then, we obtain the following approximate scheme for equation (2)–(4):

$$-\theta_i^{j+1}u_{i+1}^{j+1}+\left(1+2\theta_i^{j+1}-h^2\theta_i^{j+1}\frac{1}{1+u_i^{j+1}}\right)u_i^{j+1}-\theta_i^{j+1}u_{i-1}^{j+1} \tag{34}$$

$$=u_i^j-\sum_{n=1}^{j}\left(u_i^{j-n+1}-u_i^{j-n-1}\right)\phi_i^{j+1}\left(n\right),$$

with the initial condition

$$u_i^0=e^{-x_i}\sin\left(x_i\right),\quad i=0,\ldots,M, \tag{35}$$

and the boundary conditions

$$u_0^{j+1}=u_1^{j+1},\quad u_N^{j+1}=u_{N-1}^{j+1},\quad j=0,\ldots,N, \tag{36}$$

where:

$$\theta_i^{j+1}=$$

$$0.01\left(0.02\right)^{0.75-0.5\cos(x_it_{j+1})}\Gamma(1.25+0.5\cos\left(x_it_{j+1}\right))\sin\left(\pi x_it_{j+1}\right), \tag{37}$$
$$i=0,\ldots,M,\quad j=0,\ldots,N,$$

and

$$\phi_i^{j+1}\left(n\right)=\left(n+1\right)^{0.25-0.5\cos(x_it_{j+1})}-n^{0.25-0.5\cos(x_it_{j+1})}. \tag{38}$$

According to Theorem 5.1 and the figures below, the approximate scheme (34)–(36) is unconditionally stable.

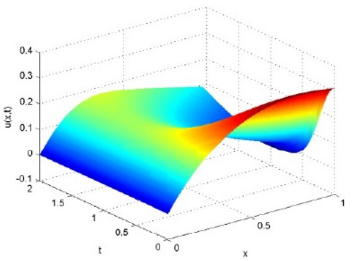

Figure 1. Numerical solution of the system (34)–(38) where $0<x<1$ and $0<t<2$

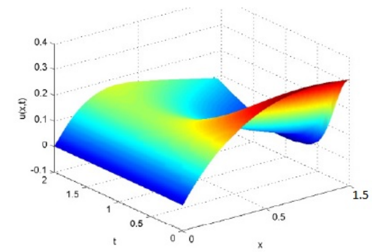

Figure 2. Numerical solution of the system (34)–(38) where $0<x<1.5$ and $0<t<2$

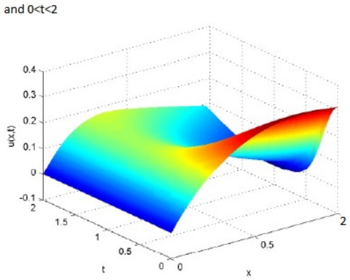

Figure 3. Numerical solution of the system (34)–(38) where $0<x<2$ and $0<t<2$

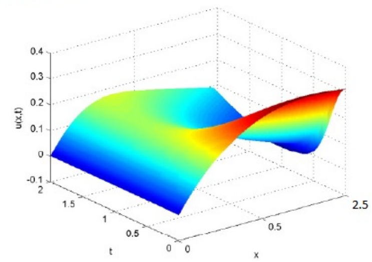

Figure 4. Numerical solution of the system (34)–(38) where $0<x<2.5$ and $0<t<2$

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References

[1] T.F. Nonnenmacher, R. Metzler, On the Riemann-Liouville fractional calculus and some recent applications, *Fractals*, **3**, No 3, 557–566 (1995).

[2] T. Mahmood, M. Rahman, M. Arfan, S.-I. Kayani, M. Sun, Mathematical study of Algae as a bio-fertilizer using fractal-fractional dynamic model, *Math. Comput. Simul.*, **203**, 207–222 (2023).

[3] Z. Chen, P. Qiu, X.-J. Yang, Y. Feng, J. Liu, A new fractional derivative model for the anomalous diffusion problem, *Therm. Sci.*, **23**, Suppl. 3, 1005–1011 (2019).

[4] C. Chen, F. Liu, K. Burrage, Finite difference methods and a Fourier analysis for the fractional reaction-subdiffusion equation, *Appl. Math. Comput.*, **198**, No 2, 754–769 (2008).

[5] G.A. Birajdar, D.B. Dhaigude, An implicit numerical method for semilinear fractional diffusion equation, In: *Proc. of the International Conference on Mathematical Sciences, Chennai, India*, 674–678 (2014).

[6] Y. Lin, X. Chuanju, Finite difference/spectral approximations for the time-fractional diffusion equation, *J. Comput. Phys.*, **225**, No 2, 1533–1552 (2007).

[7] R.A. Oderinu, J.A. Owolabi, M. Taiwo, Approximate solutions of linear time-fractional differential equations, *J. of Mathematics and Computer Sci.*, **29**, No 1, 60–72 (2023).

[8] H.-M. Zhang, F. Liu, Numerical simulation of the Riesz fractional difusion equation with a nonlinear source term, *J. Appl. Math. Comput.*, **26**, No 1-2, 1–14 (2008).

[9] P. Kumar, O.P. Agrawal, An approximate method for numerical solution of fractional diferential equations, *Signal Process.*, **86**, No 10, 2602–2610 (2006).