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PROJECTIVE EIGENVALUE BOUNDS

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Abstract

In this paper, we derive expressions for the bounds of the eigenvalues of real symmetric matrices. We use symmetric projection operators and

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also consider situations when some of the eigenvalues may be known. These bounds are based on the trace of the matrix and its Frobenius norm. They are relatively easy and inexpensive to compute.

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1. **Introduction**

The knowledge of the eigenvalue distribution of a matrix $\mathbf{A}$ is crucial in almost all branches of science and engineering. For real symmetric matrices, this distribution is limited to $\mathcal{R}$. Some recent applications have been to search engines [5] and the crypto correlation matrix [3], which assists in the creation of a crypto portfolio. The eigenvectors form a canonical basis in which the associated linear operator is easily represented. The conditioning of a symmetric linear system depends on the ration $\frac{|\lambda_n|}{|\lambda_1|}$, λ_n is the eigenvalue of largest absolute magnitude and λ_1 is the eigenvalue of least absolute magnitude. Usually the accurate location of the eigenvalues is synonymous with the computation of the associated eigenvectors. For large dense matrices, such computation is greatly facilitated by the location of the spectrum $\sigma(\mathbf{A})$. Some simple, yet effective methods, based on matrix entries are the Gerschgorin disks and ovals of Cassini [2]. The power methods and its variants, together with th Rayleigh quotient [6] can locate the dominant eigenpairs effectively. Bounds based on only the traces of the matrix and the traces of its powers, have been studied in great details, [8, 9, 10, 11]. Here we generalize the trace bounds to cases where few eigenvalues may be known, and show how the remainder of the spectrum may be bounded. Our approach makes uses of projection operators and we illustrate its effect with few examples.

2. **Theory**

Let $\boldsymbol{\lambda} = (\lambda_{\sigma_i}), i = 1, 2, \cdots, n$ be the eigenvalues of a real symmetric matrix $\mathbf{A}$, where $\sigma_i \in S_n = \{1, 2, \cdots, n\}$. Also let $S_k = \{1, 2, \cdots, k\}$, $S_{k_r} = S_k \cup \{r\}$, where $r \in S_{n-k} = S_n - S_k$ and $S^r_{n-k-1} = S_{n-k} - \{r\}$.

LEMMA 2.1. *Let $\sigma_i \neq \sigma_j, i \neq j$ and choose k and $r \in S_{n-k}$. Let $\mathbf{e}_{\sigma_i}$ denote the standard basis vector in $\mathbf{R}^n$, with 1 in the $\sigma_{i_{th}}$ position.*

Define $\mathbf{P} \in \mathbf{R}^{n\times n}$ *by*

$$\mathbf{P} = \mathbf{I} - \sum_{i\in S_{k_r}} \mathbf{e}_{\sigma_i}\mathbf{e}_{\sigma_i}^t - \frac{1}{n-k-1}\sum_{i\in S^r_{n\text{-}k\text{-}1}}\sum_{j\in S^r_{n\text{-}k\text{-}1}} \mathbf{e}_{\sigma_i}\mathbf{e}_{\sigma_j}^t. \tag{1}$$

Then the following is true:

(a) $\mathbf{P}$ *is idempotent and symmetric*

(b) *rank* $(\mathbf{P}) = n-k-2$

(c) *an orthonormal basis for the nullspace* $N(\mathbf{P})$ *is given by*

$$\left\{\{\mathbf{e}_{\sigma_i}\}_{i\in S_{k_r}},\ \frac{1}{\sqrt{n-k-1}}\sum_{i\in S^r_{n\text{-}k\text{-}1}} \mathbf{e}_{\sigma_i}\right\} \tag{2}$$

(d) $\mathbf{R}^n = R(\mathbf{P}) \oplus N(\mathbf{P})$ *is an orthogonal decomposition of* $\mathbf{R}^n$*, where* $R(\mathbf{P})$ *denotes the range of* $\mathbf{P}$*.*

P r o o f.

(a) The symmetry part is obvious. Let

$$\begin{aligned}\mathbf{P}_{k+1} &= \sum_{i\in S_{k_r}} \mathbf{e}_{\sigma_i}\mathbf{e}_{\sigma_i}^t \\ &= \sum_{i\in S_{k_r}} \mathbf{P}_{\sigma_i},\end{aligned}$$

where $\mathbf{P}_{\sigma_i} = \mathbf{e}_{\sigma_i}\mathbf{e}_{\sigma_i}^t$ and

$$\mathbf{P}_{n-k-1} = \frac{1}{n-k-1}\sum_{i\in S^r_{n\text{-}k\text{-}1}}\sum_{j\in S^r_{n\text{-}k\text{-}1}} \mathbf{e}_{\sigma_i}\mathbf{e}_{\sigma_j}^t.$$

Clearly $\mathbf{P}_{\sigma_i}$ is a projector onto the $\mathbf{e}_{\sigma_i}$ axis. Furthermore, these are orthogonal projectors. Now

$$\begin{aligned}
&\mathbf{P}^2_{n-k-1}\\
&= \frac{1}{(n-k-1)^2}\left(\sum_{i\in S^r_{n\text{-}k\text{-}1}}\sum_{j\in S^r_{n\text{-}k\text{-}1}}\mathbf{e}_{\sigma_i}\mathbf{e}^t_{\sigma_j}\right)^2\\
&= \frac{1}{(n-k-1)^2}\sum_{i\in S^r_{n\text{-}k\text{-}1}}\sum_{q\in S_{n\text{-}k\text{-}1}}\sum_{j\in S^r_{n\text{-}k\text{-}1}}\sum_{p\in S^r_{n\text{-}k\text{-}1}}\mathbf{e}_{\sigma_i}\left(\mathbf{e}^t_{\sigma_j}\mathbf{e}_{\sigma_p}\right)\mathbf{e}^t_{\sigma_q}\\
&= \frac{1}{(n-k-1)^2}\sum_{i\in S^r_{n\text{-}k\text{-}1}}\sum_{q\in S^r_{n\text{-}k\text{-}1}}\mathbf{e}_{\sigma_i}\mathbf{e}^t_{\sigma_q}\sum_{j\in S^r_{n\text{-}k\text{-}1}}(1)\\
&= \frac{1}{(n-k-1)}\sum_{i\in S^r_{n\text{-}k\text{-}1}}\sum_{q\in S^r_{n\text{-}k\text{-}1}}\mathbf{e}_{\sigma_i}\mathbf{e}^t_{\sigma_q}\\
&= \mathbf{P}_{n-k-1}.
\end{aligned}$$

It is easily verified that $\mathbf{P}_{\sigma_i}\mathbf{P}_{n-k-1} = \mathbf{P}_{\sigma_i}\mathbf{P}_{n-k-1} = \mathbf{0}$. Thus $\left\{\{\mathbf{P}_{\sigma_i}\}_{i\in S_{k_r}}, \mathbf{P}_{n-k-1}\right\}$ is a set of mutually orthogonal projectors, and the sum of such projectors is again a projector.

(b)

$$\begin{aligned}
\operatorname{rank}(\mathbf{I}) &= \operatorname{rank}\left(\mathbf{P} + \sum_{i\in S_{k_r}}\mathbf{P}_{\sigma_i} + \mathbf{P}_{n-k-1}\right)\\
&\le \operatorname{rank}(\mathbf{P}) + \sum_{i\in S_{k_r}}\operatorname{rank}(\mathbf{P}_{\sigma_i}) + \operatorname{rank}(\mathbf{P}_{n-k-1}).
\end{aligned}$$

Clearly $\mathbf{P}_{\sigma_i}$ are rank one matrices. Since $\mathbf{e}_{\sigma_i}\mathbf{e}^t_{\sigma_j}$ is the matrix with one in the ij_{th} position, it follows that $\sum^n_{i\in S_{mn\text{-}k11}}\sum_{j\in S_{n\text{-}k\text{-}1}}\mathbf{e}_{\sigma_i}\mathbf{e}^t_{\sigma_j}$ is the $(n-k-1)\times(n-k-1)$ matrix of all ones, thus $\mathbf{P}_{n-k-1}$ also has rank one. We therefore conclude that rank$(\mathbf{P}) \ge n-k-2$. Hence nullity$(\mathbf{P}) \le k+2$. However it is easily shown that $\mathbf{P}$ annihilates the $k+2$ independent vectors in (2), which implies that $\mathbf{P}$ has rank $n-k-2$.

(c) This is shown in (b).

(d) It follows from the elementary theory of projections that $\mathbf{R}^n = R(\mathbf{P}) \oplus N(\mathbf{P})$. This is an orthogonal decomposition as

$\langle \mathbf{Pz}, \mathbf{y} \rangle = \langle \mathbf{z}, \mathbf{Py} \rangle = \mathbf{0}$, where $\mathbf{Pz} \in R(\mathbf{P})$ and $\mathbf{y} \in N(\mathbf{P})$. Here $\langle ., . \rangle$ denotes the standard inner product on $\mathbf{R}^n$.

□

DEFINITION 2.1. Denote by $S_\lambda^k = \{\lambda_{\sigma_1}, \lambda_{\sigma_2}, \cdots \lambda_{\sigma_k}\}$ the the set of k known eigenvalues of $\mathbf{A}$ and $S_\lambda^{n-k} = \sigma(\mathbf{A}) - S_\lambda^k$, the set of unknown eigenvalues.

LEMMA 2.2. *Let m_{n-k} denote the average of the set S_λ^{n-k} and v_{n-k} denote the variance and choose $r \in S_{n-k}$, then*

$$|\lambda_{\sigma_r} - m_{n-k}| \leq v_{n-k}\sqrt{n-k-1}. \tag{3}$$

P r o o f. Firstly, it is clear that

$$m_{n-k} = \frac{trace(\mathbf{A}) - \sum_{i \in S_k} \lambda_{\sigma_i}}{n-k}, \tag{4}$$

by definition. Let $\boldsymbol{\lambda}^{n-k} = [\lambda_{\sigma_{k+1}}, \lambda_{\sigma_{k+2}}, \cdots, \lambda_{\sigma_n}]$ and $\mathbf{e}^{n-k} = [1, 1, \cdots, 1]$ be the vector of length $n-k$, then

$$\begin{aligned} v_{n-k}^2 &= \frac{1}{n-k}\langle \boldsymbol{\lambda}^{n-k} - m_{n-k}\mathbf{e}^{n-k}, \boldsymbol{\lambda}^{n-k} - m_{n-k}\mathbf{e}^{n-k} \rangle \\ &= \frac{\langle \boldsymbol{\lambda}^{n-k}, \boldsymbol{\lambda}^{n-k} \rangle}{n-k} - 2m_{n-k}\frac{\langle \boldsymbol{\lambda}^{n-k}, \mathbf{e}^{n-k} \rangle}{n-k} + m_{n-k}^2\frac{\langle \mathbf{e}^{n-k}, \mathbf{e}^{n-k} \rangle}{n-k} \\ &= \frac{tr(\mathbf{A}^2) - \sum_{i \in S_k} \lambda_{\sigma_i}^2}{n-k} - m_{n-k}^2. \end{aligned} \tag{5}$$

From the Pythagorean theorem [7] in an inner product space we have

$$\begin{aligned} \|\boldsymbol{\lambda}\|_2^2 &= \|\mathbf{P}\boldsymbol{\lambda}\|_2^2 + \|(\mathbf{I} - \mathbf{P})\boldsymbol{\lambda}\|_2^2 \\ &\geq \|(\mathbf{I} - \mathbf{P})\boldsymbol{\lambda}\|_2^2 \\ &= \|\mathbf{P}_{k+1}\boldsymbol{\lambda} + \mathbf{P}_{n-k-1}\boldsymbol{\lambda}\|_2^2 \\ &= \|\mathbf{P}_{k+1}\boldsymbol{\lambda}\|_2^2 + \|\mathbf{P}_{n-k-1}\boldsymbol{\lambda}\|_2^2. \end{aligned} \tag{6}$$

Now

$$\begin{aligned} \mathbf{P}_{k+1}\boldsymbol{\lambda} &= \sum_{i \in S_{k_r}} \mathbf{P}_{\sigma_i}\boldsymbol{\lambda} \\ &= \sum_{i \in S_{k_r}} \langle \boldsymbol{\lambda}, \mathbf{e}_{\sigma_i} \rangle \mathbf{e}_{\sigma_i}, \end{aligned}$$

thus

$$\|\mathbf{P}_{k+1}\boldsymbol{\lambda}\|_2^2 = \sum_{i\in S_{k_r}} \lambda_{\sigma_i}^2. \tag{7}$$

Also

$$\begin{aligned}\mathbf{P}_{n-k-1}\boldsymbol{\lambda} &= \frac{1}{n-k-1}\sum_{i\in S^r_{n\text{-}k\text{-}1}}\sum_{j\in S_{n\text{-}k\text{-}1}} \langle \boldsymbol{\lambda}, \mathbf{e}_{\sigma_j}\rangle \mathbf{e}_{\sigma_i}\\ &= \frac{1}{n-k-1}\sum_{j\in S^r_{n\text{-}k\text{-}1}} \lambda_{\sigma_j} \sum_{i\in S^r_{n\text{-}k\text{-}1}} \mathbf{e}_{\sigma_i},\end{aligned}$$

thus

$$\|\mathbf{P}_{n-k-1}\boldsymbol{\lambda}\|_2^2 = \frac{1}{(n-k-1)^2}\left(\sum_{j\in S^r_{n\text{-}k\text{-}1}} \lambda_{\sigma_j}\right)^2 \left\|\sum_{i\in S^r_{n\text{-}k\text{-}1}} \mathbf{e}_{\sigma_i}\right\|_2^2. \tag{8}$$

$$= \frac{1}{n-k-1}\left(\sum_{j\in S^r_{n\text{-}k\text{-}1}} \lambda_{\sigma_j}\right)^2. \tag{9}$$

Using (7) and (9) in (6) results in

$$\begin{aligned}&\sum_{i\in S_n} \lambda_{\sigma_i}^2\\ &\geq \sum_{i\in S_{k_r}} \lambda_{\sigma_i}^2 + \frac{1}{n-k-1}\left(\sum_{i\in S_{n\text{-}k\text{-}1}} \lambda_{\sigma_i}\right)^2 tr(\mathbf{A}^2) - \sum_{i\in S_k} \lambda_{\sigma_i}^2\\ &\geq \lambda_{\sigma_r}^2 + \frac{1}{n-k-1}\left(tr(\mathbf{A}) - \sum_{i\in S_k} \lambda_{\sigma_i} - \lambda_{\sigma_r}\right)^2.\end{aligned} \tag{10}$$

It follows that

$$\begin{aligned}&(n-k-1)\left(tr(\mathbf{A}^2) - \sum_{i\in S_k} \lambda_{\sigma_i}^2\right)\\ &\geq (n-k)\lambda_{\sigma_r}^2 - 2\lambda_{\sigma_r}\left(tr(\mathbf{A}) - \sum_{i\in S_k} \lambda_{\sigma_i}\right)\\ &+ \left(tr(\mathbf{A}) - \sum_{i\in S_k} \lambda_{\sigma_i}\right)^2.\end{aligned} \tag{11}$$

Thus, we get

$$\begin{aligned}
&\frac{n-k-1}{n-k}\left(tr(\mathbf{A}^2)-\sum_{i\in S_k}\lambda_{\sigma_i}^2\right)\\
&\geq \lambda_{\sigma_r}^2 - 2\lambda_{\sigma_r}\left(\frac{tr(\mathbf{A})-\sum_{i\in S_k}\lambda_{\sigma_i}}{n-k}\right)\\
&+(n-k)\left(\frac{tr(\mathbf{A})-\sum_{i\in S_k}\lambda_{\sigma_i}}{n-k}\right)^2\\
&=\lambda_{\sigma_r}^2-2\lambda_{\sigma_r}m_{n-k}+(n-k)m_{n-k}^2\\
&=(\lambda_{\sigma_r}-m_{n-k})^2+(n-k-1)m_{n-k}^2, \qquad (12)
\end{aligned}$$

$$(\lambda_{\sigma_r}-m_{n-k})^2 \leq (n-k-1)\left(\frac{tr(\mathbf{A}^2)-\sum_{i\in S_k}\lambda_{\sigma_i}^2}{n-k}-m_{n-k}^2\right). \qquad (13)$$

Taking the square root in (13) completes the proof. □

Note that there is no need to evaluate $\mathbf{A}^2$ as it is equal to the Frobenius norm $\|\mathbf{A}\|_F^2$.

THEOREM 2.1. *Let $\lambda_{\sigma_m}=\min S_\lambda^{n-k}$ and $\lambda_{\sigma_M}=\max S_\lambda^{n-k}$. Upper and lower bounds for λ_{σ_M} and λ_{σ_m} are given by*

$$\lambda_{\sigma_M} \leq m_{n-k}+v_{n-k}\sqrt{n-k-1} \qquad (14)$$

$$\lambda_{\sigma_m} \geq m_{n-k}-v_{n-k}\sqrt{n-k-1}. \qquad (15)$$

P r o o f. Let $r=M$ and $r=m$ in Lemma 2.2. □

THEOREM 2.2. *Lower and upper bounds for λ_{σ_M} and λ_{σ_m} are given by*

$$\lambda_{\sigma_M} \geq m_{n-k}+\frac{v_{n-k}}{\sqrt{n-k-1}} \qquad (16)$$

$$\lambda_{\sigma_m} \leq m_{n-k}-\frac{v_{n-k}}{\sqrt{n-k-1}}. \qquad (17)$$

P r o o f. We use the fact that for the set of real numbers S_λ^{n-k}, the variance v_{n-k} satisfies the inequality [1].

$$v_{n-k}^2 \leq [\lambda_{\sigma_M} - m_{n-k}][m_{n-k} - \lambda_{\sigma_m}]. \tag{18}$$

We prove only (16) as (17) is proved similarly. From (18) and (15) we have

$$\begin{aligned} \lambda_{\sigma_M} &\geq m_{n-k} + \frac{v_{n-k}^2}{m_{n-k} - \lambda_{\sigma_m}} \\ &\geq m_{n-k} + \frac{v_{n-k}}{\sqrt{n-k-1}}. \end{aligned}$$

□

When $k = 0$ in Theorem 2.1 and Theorem 2.2 we obtain the classic result of Wolkowicz and Styan [11].

LEMMA 2.3. *Let* $\mathbf{Q}$ *be a symmetric orthogonal projector and* $\mathbf{u}, \mathbf{v} \notin N(\mathbf{Q})$ *then*

$$|\langle \mathbf{Q}\mathbf{u}, \mathbf{v} \rangle| \leq \langle \mathbf{Q}\mathbf{u}, \mathbf{u} \rangle^{\frac{1}{2}} \langle \mathbf{Q}\mathbf{v}, \mathbf{v} \rangle^{\frac{1}{2}}. \tag{19}$$

P r o o f. Use the Cauchy-Schwarz inequality and the fact that $\mathbf{Q}$ is a symmetric projector to get

$$\begin{aligned} |\langle \mathbf{Q}\mathbf{u}, \mathbf{v} \rangle| &= |\langle \mathbf{Q}^2\mathbf{u}, \mathbf{v} \rangle| \\ &= |\langle \mathbf{Q}\mathbf{u}, \mathbf{Q}\mathbf{v} \rangle| \\ &\leq \langle \mathbf{Q}\mathbf{u}, \mathbf{Q}\mathbf{u} \rangle^{\frac{1}{2}} \langle \mathbf{Q}\mathbf{v}, \mathbf{Q}\mathbf{v} \rangle^{\frac{1}{2}} \\ &= \langle \mathbf{Q}\mathbf{u}, \mathbf{u} \rangle^{\frac{1}{2}} \langle \mathbf{Q}\mathbf{v}, \mathbf{v} \rangle^{\frac{1}{2}}. \end{aligned} \tag{20}$$

□

LEMMA 2.4. *Choose* $\lambda_{\sigma_p}, \lambda_{\sigma_q} \in S_\lambda^{n-k}$, $p \neq q$ *then*

$$|\lambda_{\sigma_q} - m_{n-k-1}^p| \leq \sqrt{n-k-2} \left[\frac{tr(\mathbf{A}^2) - \sum_{i \in S_k} \lambda_{\sigma_i}^2}{n-k-1} - (m_{n-k-1}^p)^2 \right]^{\frac{1}{2}}, \tag{21}$$

where

$$m_{n-k-1}^p = \frac{trace(\mathbf{A}) - \sum_{i \in S_k} \lambda_{\sigma_i} - \lambda_{\sigma_p}}{n-k-1}.$$

P r o o f. Define the matrix $\mathbf{Q}$ of order $n-k$ by

$$\mathbf{Q} = \mathbf{I} - \frac{(\mathbf{e}^{n-k} - \mathbf{e}^{n-k}_{\sigma_p})(\mathbf{e}^{n-k} - \mathbf{e}^{n-k}_{\sigma_p})^t}{n-k-1}. \tag{22}$$

It is easily verified that $\mathbf{Q}$ is a symmetric orthogonal projector.

$$\begin{aligned}\langle \mathbf{Q}\boldsymbol{\lambda}^{n-k}, \mathbf{e}^{n-k}_{\sigma_q}\rangle &= \lambda_{\sigma_q} - \frac{1}{n-k-1}\sum_{i\in S^p_{n\text{-}k\text{-}1}} \lambda_{\sigma_i} \\ &= \lambda_{\sigma_q} - \frac{tr(\mathbf{A}) - \sum_{i\in S_k}\lambda_{\sigma_i} - \lambda_{\sigma_p}}{n-k-1} \\ &= \lambda_{\sigma_q} - m^p_{n-k-1},\end{aligned} \tag{23}$$

$$\begin{aligned}\langle \mathbf{Q}\boldsymbol{\lambda}^{n-k}, \boldsymbol{\lambda}^{n-k}\rangle &= \sum_{i\in S_{n\text{-}k}} \lambda^2_{\sigma_i} - \frac{1}{n-k-1}\left(\sum_{i\in S^p_{n\text{-}k\text{-}1}} \lambda_{\sigma_i}\right)^2 \\ &= tr(\mathbf{A}^2) - \sum_{i\in S_k}\lambda^2_{\sigma_i} - \frac{(tr(\mathbf{A}) - \sum_{i\in S_k}\lambda_{\sigma_i} - \lambda_{\sigma_p})^2}{n-k-1} \\ &= tr(\mathbf{A}^2) - \sum_{i\in S_k}\lambda^2_{\sigma_i} - (n-k-1)(m^p_{n-k-1})^2,\end{aligned} \tag{24}$$

$$\begin{aligned}\langle \mathbf{Q}\mathbf{e}^{n-k}_{\sigma_q}, \mathbf{e}^{n-k}_{\sigma_q}\rangle &= 1 - \frac{1}{n-k-1} \\ &= \frac{n-k-2}{n-k-1}.\end{aligned} \tag{25}$$

Use Lemma 2.3 with $\mathbf{u} = \boldsymbol{\lambda}^{n-k}$ and $\mathbf{v} = \mathbf{e}^{n-k}_{\sigma_q}$ to obtain the result. □

THEOREM 2.3.

$$\lambda_{\sigma_m} \geq m^M_{n-k-1} - \sqrt{n-k-2}\left[\frac{tr(\mathbf{A}^2) - \sum_{i\in S_k}\lambda^2_{\sigma_i}}{n-k-1} - (m^M_{n-k-1})^2\right]^{\frac{1}{2}} \tag{26}$$

$$\lambda_{\sigma_M} \leq m^m_{n-k-1} + \sqrt{n-k-2}\left[\frac{tr(\mathbf{A}^2) - \sum_{i\in S_k}\lambda^2_{\sigma_i}}{n-k-1} - (m^m_{n-k-1})^2\right]^{\frac{1}{2}}. \tag{27}$$

P r o o f. Set $q = m, p = M$ and $q = M, p = m$ respectively in Lemma 2.4 to obtain the result. □

3. **Results**

We shall consider three examples, all matrices are taken from [4]. For ease of explanation we shall assume that $\sigma_i = i$ and that the eigenvalues are arranged in the order

$$\lambda_1 \le \lambda_2 \le \cdots \lambda_{n-1} \le \lambda_n.$$

For all examples the eigenvalues are summarized in the first table. The second table lists the bounds on the extreme eigenvalues from S_λ^{n-k} using equations (14)-(17). The third table lists the upper bound on the maximum eigenvalue from S_λ^{n-k} , when additionally the minimal one is known, as well as the minimum eigenvalue from S_λ^{n-k} when, additionally the maximal one is known. Here we use equations (26)-(27).

Example 3.1.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & -3 & 1 & 5 \\ 3 & 1 & 6 & -2 \\ 4 & 5 & -2 & -1 \end{bmatrix}$$

$\boldsymbol{\lambda}$	$\boldsymbol{\sigma}(\mathbf{A})$
λ_1	-8.028578
λ_2	-1.573191
λ_3	5.668864
λ_4	7.932905

Table 1. $\sigma(\mathbf{A})$-Example 3.1

Example 3.2.

$$\mathbf{A} = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 6 & 0 & 4 & 3 \\ 3 & 0 & 7 & 6 & 5 \\ 2 & 4 & 6 & 8 & 7 \\ 1 & 3 & 5 & 7 & 9 \end{bmatrix}$$

k	S_λ^k	S_λ^{n-k}	Bounds
0	$\emptyset$	$\{\lambda_1, \cdots, \lambda_4\}$	$\lambda_1 \in [-9.885771, -2.628590]$
			$\lambda_4 \in [\ \ 4.628590,\ 11.885771]$
1	$\{\lambda_1\}$	$\{\lambda_2, \cdots, \lambda_4\}$	$\lambda_2 \in [-1.724177,\ \ 1.142674]$
			$\lambda_4 \in [\ \ 6.876378,\ \ 9.743229]$
1	$\{\lambda_4\}$	$\{\lambda_1, \cdots, \lambda_3\}$	$\lambda_1 \in [-9.223537, -5.267252]$
			$\lambda_3 \in [\ \ 2.645316,\ \ 6.601600]$
2	$\{\lambda_1, \lambda_4\}$	$\{\lambda_2, \lambda_3\}$	$\lambda_2 \in [-1.573191, -1.573191]$
			$\lambda_3 \in [\ \ 5.668864,\ \ 5.668864]$

Table 2. Bounds using (14)-(17)

k	S_λ^k	S_λ^{n-k}	Known	Bounds
0	$\emptyset$	$\{\lambda_1, \cdots, \lambda_4\}$	λ_1	$\lambda_4 \leq 12.718567$
			λ_4	$\lambda_1 \geq -11.536560$
1	$\{\lambda_1\}$	$\{\lambda_2, \cdots, \lambda_4\}$	λ_2	$\lambda_4 \leq 8.388000$
			λ_4	$\lambda_2 \geq -4.628789$
1	$\{\lambda_4\}$	$\{\lambda_1, \cdots, \lambda_3\}$	λ_1	$\lambda_3 \leq 8.781400$
			λ_3	$\lambda_1 \geq -9.947341$
2	$\{\lambda_1, \lambda_4\}$	$\{\lambda_2, \lambda_3\}$	λ_3	$\lambda_4 \leq 5.668864$
			λ_4	$\lambda_2 \geq -1.573191$

Table 3. Bounds using (26)-(27)

$\boldsymbol{\lambda}$	$\boldsymbol{\sigma}(\mathbf{A})$
λ_1	-1.096595
λ_2	1.327046
λ_3	4.848950
λ_4	7.513724
λ_5	22.406875

Table 4. $\sigma(\mathbf{A})$-Example3.2

Example 3.3.

$$\mathbf{A} = \begin{bmatrix} 5 & 1 & -2 & 0 & -2 & 5 \\ 1 & 6 & -3 & 2 & 0 & 6 \\ -2 & -3 & 8 & -5 & -6 & 0 \\ 0 & 2 & -5 & 5 & 1 & -2 \\ -2 & 0 & -6 & 1 & 6 & -3 \\ 5 & 6 & 0 & -2 & -3 & 8 \end{bmatrix}$$

k	S_λ^k	S_λ^{n-k}	Bounds
0	$\emptyset$	$\{\lambda_1, \cdots, \lambda_5\}$	$\lambda_1 \in [-9.492423, \ 2.876894]$
			$\lambda_5 \in [\ 11.123106, 23.492423]$
1	$\{\lambda_1\}$	$\{\lambda_2, \cdots, \lambda_5\}$	$\lambda_2 \in [-4.887800, \ 4.386832]$
			$\lambda_5 \in [\ 13.661465, 22.936098]$
1	$\{\lambda_5\}$	$\{\lambda_1, \cdots, \lambda_4\}$	$\lambda_1 \in [-2.549432, \ 1.249044]$
			$\lambda_4 \in [\ 5.047519, \ 8.845994]$
2	$\{\lambda_1, \lambda_5\}$	$\{\lambda_2, \cdots, \lambda_4\}$	$\lambda_2 \in [\ 0.979951, \ 2.771595]$
			$\lambda_4 \in [\ 6.354885, \ 8.146529]$

TABLE 5. Bounds using (14)-(17)

k	S_λ^k	S_λ^{n-k}	Known	Bounds
0	$\emptyset$	$\{\lambda_1, \cdots, \lambda_5\}$	λ_1	$\lambda_5 \leq 22.968474$
			λ_5	$\lambda_1 \geq -17.075838$
1	$\{\lambda_1\}$	$\{\lambda_2, \cdots, \lambda_5\}$	λ_2	$\lambda_5 \leq 22.569334$
			λ_5	$\lambda_2 \geq -14.079507$
1	$\{\lambda_5\}$	$\{\lambda_1, \cdots, \lambda_4\}$	λ_1	$\lambda_4 \leq 8.256699$
			λ_4	$\lambda_1 \geq -5.346366$
2	$\{\lambda_1, \lambda_5\}$	$\{\lambda_2, \cdots, \lambda_4\}$	λ_2	$\lambda_4 \leq 7.810994$
			λ_4	$\lambda_2 \geq -2.509232$

TABLE 6. Bounds using (26)-(27)

$\boldsymbol{\lambda}$	$\boldsymbol{\sigma}(\mathbf{A})$
λ_1	-1.598734
λ_2	-1.598734
λ_3	4.455990
λ_4	4.455990
λ_5	16.142745
λ_6	16.142745

TABLE 7. $\sigma(\mathbf{A})$-Example3.3

k	S_λ^k	S_λ^{n-k}	Bounds
0	$\emptyset$	$\{\lambda_1, \cdots, \lambda_6\}$	$\lambda_1 \in [-10.132119, \ 3.040243]$
			$\lambda_6 \in [\ 9.626424, 22.798785]$
1	$\{\lambda_1\}$	$\{\lambda_2, \cdots, \lambda_6\}$	$\lambda_2 \in [\ -6.217639, \ 4.385400]$
			$\lambda_6 \in [\ 11.454093, 22.057133]$
1	$\{\lambda_6\}$	$\{\lambda_1, \cdots, \lambda_5\}$	$\lambda_1 \in [\ -8.585829, \ 1.132131]$
			$\lambda_5 \in [\ 7.610771, 17.328731]$
2	$\{\lambda_1, \lambda_6\}$	$\{\lambda_2, \cdots, \lambda_5\}$	$\lambda_2 \in [\ -5.270745, \ 2.152417]$
			$\lambda_5 \in [\ 9.575578, 16.998740]$

TABLE 8. Bounds using (14)-(17)

k	S_λ^k	S_λ^{n-k}	Known	Bounds
0	$\emptyset$	$\{\lambda_1, \cdots, \lambda_6\}$	λ_1	$\lambda_6 \leq \ 22.129266$
			λ_6	$\lambda_1 \geq -15.028592$
1	$\{\lambda_1\}$	$\{\lambda_2, \cdots, \lambda_6\}$	λ_2	$\lambda_6 \leq \ 20.514657$
			λ_6	$\lambda_2 \geq -12.008429$
1	$\{\lambda_6\}$	$\{\lambda_1, \cdots, \lambda_5\}$	λ_1	$\lambda_5 \leq \ 17.084490$
			λ_5	$\lambda_1 \geq -13.502411$
2	$\{\lambda_1, \lambda_6\}$	$\{\lambda_2, \cdots, \lambda_5\}$	λ_2	$\lambda_5 \leq \ 16.251340$
			λ_5	$\lambda_2 \geq -11.346977$

TABLE 9. Bounds using (26)-(27)

EXAMPLE 3.4.

$$\mathbf{A} = \begin{bmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The last two rows of Tables 2 and 3 give exact results as they are a special case for which $n-k=2$, $m_{n-k} = \frac{\lambda_2+\lambda_3}{2}$ and $v_{n-k} = \frac{\lambda_2-\lambda_3}{2}$. For all examples, we note that the second tables give very good bounds for the minimum and maximum of the set S_λ^{n-k}. In fact these bounds are

$\boldsymbol{\lambda}$	$\boldsymbol{\sigma}(\mathbf{A})$
λ_1	0.261295
λ_2	0.299557
λ_3	0.381966
λ_4	0.558365
λ_5	1.000000
λ_6	2.618034
λ_7	22.880783

TABLE 10. $\sigma(\mathbf{A})$-Example3.4

k	S_λ^k	S_λ^{n-k}	Bounds
0	$\emptyset$	$\{\lambda_1, \cdots, \lambda_7\}$	$\lambda_1 \in [-14.973666, \ 0.837722]$
			$\lambda_7 \in [\ 7.162278, 22.973666]$
1	$\{\lambda_1\}$	$\{\lambda_2, \cdots, \lambda_7\}$	$\lambda_2 \in [-13.718375, \ 0.954819]$
			$\lambda_7 \in [\ 8.291416, 22.964610]$
1	$\{\lambda_7\}$	$\{\lambda_1, \cdots, \lambda_6\}$	$\lambda_1 \in [\ -0.995682, \ 0.483426]$
			$\lambda_6 \in [\ 1.222980, \ 2.702088]$
2	$\{\lambda_1, \lambda_7\}$	$\{\lambda_2, \cdots, \lambda_6\}$	$\lambda_2 \in [\ -0.744603, \ 0.542538]$
			$\lambda_6 \in [\ 1.400631, \ 2.687772]$

TABLE 11. Bounds using (14)-(17)

k	S_λ^k	S_λ^{n-k}	Known	Bounds
0	$\emptyset$	$\{\lambda_1, \cdots, \lambda_7\}$	λ_1	$\lambda_7 \leq \ 22.966161$
			λ_7	$\lambda_1 \geq -20.115668$
1	$\{\lambda_1\}$	$\{\lambda_2, \cdots, \lambda_7\}$	λ_2	$\lambda_7 \leq \ 22.954377$
			λ_7	$\lambda_2 \geq -19.565442$
1	$\{\lambda_7\}$	$\{\lambda_1, \cdots, \lambda_6\}$	λ_1	$\lambda_6 \leq \ 2.703612$
			λ_6	$\lambda_1 \geq \ -1.902861$
2	$\{\lambda_1, \lambda_7\}$	$\{\lambda_2, \cdots, \lambda_6\}$	λ_2	$\lambda_5 \leq \ 2.690438$
			λ_6	$\lambda_2 \geq \ -1.755287$

TABLE 12. Bounds using (26)-(27)

tighter as the set S_λ^k increases in size. However the same is not true for results from the third tables. Here as additionally the maximum of S_λ^{n-k}

is known, results for the lower bound of the minimum of S_λ^{n-k} are rather poor. However if additionally the the minimum of S_λ^{n-k} is known, then the upper bound on its maximum is greatly improved. Thus formulae (14),(17) and (27) are useful in this case. As there are innumerable combinations for choosing the set S_λ^k, we have not investigated all. The fundamental reason is that it is fairly well known that variants of the power method can be used to calculate the maximum and minimum eigenvalues of **A**. Having found the latter, the bounds from the second tables for example may be used to great advantage to further calculate the next two extreme eigenvalues of **A**, by say shifted power iteration. This is useful especially for matrices of large dimensions.

4. Conclusion

We have provided relatively simple bounds for the extremal eigenvalues of real symmetric matrices. In the event that some eigenvalues are known, we have shown how this information may be used to further bound the extremal eigenvalues of the remaining unknown set. These results are useful for matrices of large dimensions, as they may be employed a starting values, in various numerical schemes to determine accurately the remaining eigenvalues of interest.

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