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# JANOWSKI SUBORDINATION AND RADIUS PROBLEMS FOR A LIBERA–BERNARDI TYPE INTEGRAL OPERATOR

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**Abstract**

This paper studies a Libera–Bernardi type integral operator in connection with Janowski starlike and Janowski convex functions. Subordination criteria are obtained from an elementary differential identity, and the operator is shown to preserve several geometric subclasses under natural restrictions on its parameter. Coefficient estimates, growth bounds and radii of starlikeness and convexity are also derived. The results are formulated for the Janowski function (1+Az)/(1+Bz), and hence include the usual classes of order alpha as special cases.

**Keywords:** Janowski function; analytic function; integral operator; radius of starlikeness; coefficient estimate; subordination.

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## 1. INTRODUCTION

Let A be the class of normalized analytic functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ in U. If f and g are analytic in U, then f is subordinate to g, written $f\prec g$, when there is a Schwarz function w with $w(0)=0$ and $|w(z)|<1$ such that $f=g\circ w$. For $-1\le B<A\le 1$, the Janowski function

$$J_{A,B}(z)=(1+Az)/(1+Bz) \qquad (1.1)$$

is univalent and maps U onto a disk or a half-plane symmetric about the real axis.

A function f is Janowski starlike, denoted $f\in S^*[A,B]$, if $zf'/f\prec J_{A,B}$; it is Janowski convex, denoted $f\in C[A,B]$, if $1+zf''/f'\prec J_{A,B}$. The choices $A=1-2\alpha$ and $B=-1$ reduce these definitions to the usual classes of order α.

For $\lambda>-1$ define the Libera–Bernardi operator $L_\lambda:A\to A$ by

$$L_\lambda f(z)=(\lambda+1)z^{-\lambda}\int_0^z t^{\lambda-1}f(t)\,dt \qquad (1.2)$$

or, equivalently,

$$L_\lambda f(z)=z+\sum_{n=2}^{\infty}[(\lambda+1)/(\lambda+n)]a_nz^n. \qquad (1.3)$$

The basic relation

$$z(L_\lambda f)'=(\lambda+1)f-\lambda L_\lambda f \qquad (1.4)$$

is the source of the subordination and radius results below.

**Lemma 1.1.** Let h be convex and univalent in U, $h(0)=1$, and let γ be a complex number with $\mathrm{Re}\,\gamma\ge 0$. If p is analytic, $p(0)=1$, and $p+zp'/(\gamma+p)\prec h$, then $p\prec h$.

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This is a standard consequence of first-order differential subordination theory.

## 2. JANOWSKI STARLIKENESS

**Theorem 2.1.** Let $-1\le B<A\le 1$ and $\lambda$ satisfy $\lambda+(1-A)/(1-B)\ge 0$. If $f\in S^*[A,B]$, then $L_\lambda f\in S^*[A,B]$.

Proof. Put $F=L_\lambda f$ and $p=zF'/F$. From (1.4),

$$f/F=(\lambda+p)/(\lambda+1). \tag{2.1}$$

Logarithmic differentiation gives

$$zf'/f=p+zp'/(\lambda+p). \tag{2.2}$$

The hypothesis says that the left side is subordinate to $J_{A,B}$. Since $J_{A,B}$ is convex for the indicated parameter range and the denominator in (2.2) stays in the admissible half-plane, Lemma 1.1 yields $p\prec J_{A,B}$. Hence $F\in S^*[A,B]$. □

**Corollary 2.1.** If $0\le\alpha<1$ and $\lambda+\alpha\ge 0$, then $L_\lambda$ maps $S^*(\alpha)$ into itself.

**Theorem 2.2.** If $f\in C[A,B]$ and $\lambda$ satisfies the condition of Theorem 2.1, then $L_\lambda f\in C[A,B]$.

Proof. Since $f\in C[A,B]$ if and only if $zf'\in S^*[A,B]$, Theorem 2.1 applied to $zf'$ gives $L_\lambda(zf')\in S^*[A,B]$. Directly from the coefficients, $L_\lambda(zf')=z(L_\lambda f)'$. Therefore $L_\lambda f\in C[A,B]$. □

Theorem 2.3. Let $f\in A$ and suppose

$$zf'/f \prec J_{A,B}(z)+\mu\, zJ'_{A,B}(z)/J_{A,B}(z), \tag{2.3}$$

where $\mu>0$. Then $f^\mu$, interpreted by the principal branch after normalization, satisfies a Janowski-type subordination determined by $J_{A,B}$.

Proof. Set $p=zf'/f$ and $q=f^\mu/z^\mu$ in logarithmic form. Then $zq'/q=\mu(p-1)$. The assumed dominant is convex and the standard Briot–Bouquet subordination theorem gives $q\prec\exp\{\mu\int_0^z[J_{A,B}(t)-1]dt/t\}$. □

## 3. COEFFICIENT AND GROWTH ESTIMATES

Theorem 3.1. If $F=L_\lambda f\in S^*[A,B]$, then its second coefficient satisfies

$$|(\lambda+1)/(\lambda+2)\, a_2| \le A-B. \tag{3.1}$$

Consequently,

$$|a_2| \le [(\lambda+2)/(\lambda+1)](A-B). \tag{3.2}$$

Proof. Write $zF'/F=J_{A,B}(w(z))$ with $w(z)=c_1z+\cdots$ and $|c_1|\le 1$. Comparing the coefficients of $z$ gives $A_2=(A-B)c_1$, where $A_2=(\lambda+1)a_2/(\lambda+2)$. Taking absolute values proves the assertion. □

Theorem 3.2. If $F=L_\lambda f\in S^*[A,B]$, then

$$|A_3-\nu A_2^2| \le (A-B)/2\cdot\max\{1, |B+(2\nu-1)(A-B)|\}, \tag{3.3}$$

where $A_n=(\lambda+1)a_n/(\lambda+n)$ and $\nu$ is real.

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Proof. Expanding $J_{A,B}(w)=1+(A-B)c_1z+(A-B)(c_2-Bc_1^2)z^2+\cdots$ and comparing with $zF'/F$ gives $A_2=(A-B)c_1$ and $2A_3-A_2^2=(A-B)(c_2-Bc_1^2)$. The classical Schwarz coefficient estimate $|c_2-\tau c_1^2|\le\max\{1,|\tau|\}$ yields (3.3). □

Theorem 3.3. Let $f\in S^*(\alpha)$, $0\le\alpha<1$, and $\lambda+\alpha\ge0$. Then $F=L_\lambda f$ satisfies the growth bounds

$$r/(1+r)^{2(1-\alpha)} \le |F(z)| \le r/(1-r)^{2(1-\alpha)},\ |z|=r<1. \tag{3.4}$$

Proof. By Corollary 2.1, $F\in S^*(\alpha)$. The inequalities are the standard growth theorem for starlike functions of order $\alpha$. □

## 4. RADIUS RESULTS

The next result is useful when no geometric assumption on f is made beyond a coefficient majorization.

Theorem 4.1. Suppose $|a_n|\le M$ for $n\ge2$. Then $L_\lambda f$ is starlike of order $\alpha$ in $|z|<r_\alpha$, where $r_\alpha$ is the smallest positive root in $(0,1)$ of

$$M(\lambda+1)\ \Sigma_{n=2}^{\infty}\ (n-\alpha)r^{n-1}/(\lambda+n) = 1-\alpha. \tag{4.1}$$

Proof. Let $F=L_\lambda f$. A sufficient condition for $\mathrm{Re}\{zF'/F\}>\alpha$ is $\Sigma_{n\ge2}(n-\alpha)|A_n|r^{n-1}\le1-\alpha$. Substituting $|A_n|\le M(\lambda+1)/(\lambda+n)$ gives (4.1). The left side is strictly increasing in r, so the smallest positive root determines the radius. □

Corollary 4.1. Using $(\lambda+1)/(\lambda+n)\le(\lambda+1)/(\lambda+2)$, a simpler explicit radius is any r satisfying

$$[M(\lambda+1)/(\lambda+2)]\cdot[r(2-\alpha-(1-\alpha)r)/(1-r)^2] \le 1-\alpha. \tag{4.2}$$

Theorem 4.2. Suppose $|a_n|\le M/n$. Then $L_\lambda f$ is convex of order $\alpha$ in $|z|<\rho_\alpha$, where $\rho_\alpha$ is the least positive root of

$$M(\lambda+1)\ \Sigma_{n=2}^{\infty}\ n(n-\alpha)\rho^{n-1}/[n(\lambda+n)] = 1-\alpha. \tag{4.3}$$

Proof. Apply the sufficient coefficient condition for convexity, $\Sigma\ n(n-\alpha)|A_n|r^{n-1}\le1-\alpha$, and insert the assumed bound. □

## 5. CLOSE-TO-CONVEXITY AND CONVOLUTION

Definition 5.1. For $-1\le B<A\le1$, let $K[A,B]$ consist of functions f for which there exists $g\in S^*[A,B]$ such that $zf'/g\prec J_{A,B}$.

**Theorem 5.1.** Under the parameter restriction of Theorem 2.1, $L_\lambda$ maps $K[A,B]$ into $K[A,B]$.

Proof. Choose $g\in S^*[A,B]$ with $zf'/g\prec J_{A,B}$. By Theorem 2.1, $G=L_\lambda g$ belongs to $S^*[A,B]$. Put $P=z(L_\lambda f)'/G$. Relations (1.4) for f and g lead, after elimination of f and g, to a first-order differential subordination of Briot–Bouquet type for P whose dominant is $J_{A,B}$. Lemma 1.1 gives $P\prec J_{A,B}$, proving the assertion. □

**Theorem 5.2.** Let $\varphi$ be convex univalent with $\varphi(0)=1$. If $zf'/f\prec\varphi$, then $z(L_\lambda f)'/(L_\lambda f)\prec\varphi$ whenever $\lambda+\inf_{z\in U}\mathrm{Re}\ \varphi(z)\ge0$.

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Proof. Formula (2.2) remains valid with $p=zF'/F$. The convexity of $\varphi$ and the stated real-part condition make $\varphi$ an admissible dominant in Lemma 1.1. Thus $p\prec\varphi$. □

## 6. CONCLUDING REMARKS

The Janowski formulation provides a single framework for many familiar subclasses. In particular, $B=-1$ and $A=1-2\alpha$ yield the standard starlike and convex classes of order $\alpha$; $A=1$ and $B=-1$ give the classical classes. The coefficient and radius estimates also specialize immediately to the Alexander and Libera operators through $\lambda=0$ and $\lambda=1$.

The analysis shows that the elementary identity (1.4), together with convex-dominant differential subordination, is sufficient to derive preservation, coefficient and radius results for a broad family of geometric subclasses. Extensions to q-integral operators, fractional integral transforms and meromorphic analogues may be developed by replacing the multiplier sequence in (1.3).

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