

PROPERTIES OF THE UNIFORM  
CONVEXITY AND UNIFORM  
SMOOTHNESS OF VARIABLE  
EXPONENT LEBESGUE SPACES

Mykola Yaremenko

Physico-Mathematical Department

The National Technical University of Ukraine

“Igor Sikorsky Kyiv Polytechnic Institute”

37, Prospect Beresteiskyi (former Peremohy)

Kyiv – 03056, UKRAINE

e-mail: [math.kiev@gmail.com](mailto:math.kiev@gmail.com)

**Abstract**

We consider the uniform convexity of variable exponent Lebesgue spaces. We establish that for all  $\varepsilon \in (0, 2)$  and all unit vectors  $u, v \in L^{p(\cdot)}$  such that  $\|u - v\|_{L^{p(\cdot)}} \geq \varepsilon$  we can take numbers  $\delta(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^{\frac{p_m}{p_m-1}}\right)^{\frac{p_S-1}{p_S}}$  if  $1 < p_m \leq p_S < 2$  and  $\delta(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^{p_S}\right)^{\frac{1}{p_m}}$  if  $2 \leq p_m \leq p_S < \infty$  so that  $\left\|\frac{u+v}{2}\right\|_{L^{p(\cdot)}} \leq 1 - \delta(\varepsilon)$ . A uniform convexity of variable Lebesgue space  $L^{p(\cdot)}$  is proven. Some correlations for functions describing uniform convexity and uniform smoothness of  $L^{p(\cdot)}$  spaces were established.

**MSC 2020:** 46E30, 46E35

**Key Words and Phrases:** uniform convexity, uniformly convex space, uniformly rotund space, variable Lebesgue space,  $p(x)$ -Laplacian, variable exponent

### 1. Introduction (history and discussion)

In 1936, James Clarkson introduced the notion of uniform convexity for the norm of Banach spaces, [3]. From a geometrical point of view, uniform convexity means: the center point of the cone base of the equable spherical cone of the unit sphere of the Banach space lying inside its unit sphere and can approach the surface of the sphere only if the radius of the cone base of the spherical cone tends to zero. If we consider the central conic section of the equable spherical cone, then the convexity condition can be reformulated in terms of the plan geometry as follows: the midpoint of the diagonal of the unit equilateral quadrilateral (rhombus) must be inside the point of the unit circle and cannot approach the arc of the circle nisi the length of its diagonal tends to be zero.

In 1940, R.P. Boas simplified the proof of the James Clarkson results by employing the Reisz convexity approach, [1]. In 1955 [5], Olof Hanner revisited the J. Clarkson results and proved that inequality

$$\begin{aligned} (\|u\|_{L^p} + \|v\|_{L^p})^p + \left| \|u\|_{L^p} - \|v\|_{L^p} \right|^p &\leq \|u + v\|_{L^p}^p \\ &+ \|u - v\|_{L^p}^p \leq 2 (\|u\|_{L^p}^p + \|v\|_{L^p}^p) \end{aligned}$$

for  $p \in (1, 2)$ , and

$$\begin{aligned} 2 (\|u\|_{L^p}^p + \|v\|_{L^p}^p) &\leq \|u + v\|_{L^p}^p \\ &+ \|u - v\|_{L^p}^p \leq (\|u\|_{L^p} + \|v\|_{L^p})^p + \left| \|u\|_{L^p} - \|v\|_{L^p} \right|^p \end{aligned}$$

for  $p \in [2, \infty)$ , where J. Clarkson denoted the norm of the real  $\|u\|_{L^p}^p$  the function  $u : R \rightarrow [0, 1]$  given by

$$\|u\|_{L^p}^p = \int_{[0, 1]} |u(t)| dt.$$

In 1970 [6], K.O. Friedrichs rewrote J. Clarkson's inequality in the form

$$\left\| \frac{u + v}{2} \right\|_{L^p}^p + \left\| \frac{u - v}{2} \right\|_{L^p}^p \leq \max \{ \|u\|_{L^p}^p, \|v\|_{L^p}^p \}$$

for  $p \in [2, \infty)$  and for  $p \in (1, 2)$  K.O. Friedrichs wrote

$$\left\| \frac{u + v}{2} \right\|_{L^p}^q + \left\| \frac{u - v}{2} \right\|_{L^p}^q \leq \max \{ \|u\|_{L^p}^q, \|v\|_{L^p}^q \},$$

where  $q = \frac{p}{p-1}$ . In 1990, J. Gao and K.S. Lau [17] introduced and studied some functions describing the convexity and smoothness of Banach spaces. In 2006, H. Hanche-Olsen [10] using the Ball-Carlen-Lieb results (of 1994) [2] further simplified the proof of uniform convexity of  $L^p$ ,  $p \in (1, \infty)$  spaces. H. Hanche-Olsen proved the converse statement: assume  $x$  and  $y$  are unit vectors of Banach space  $X$ , and assume

$$\|x + y\| > 2 - \delta,$$

we chose continuous linear functional  $\varphi \in X^*$  with unit norm such that  $\operatorname{Re}(\varphi(x+y)) > 2 - \delta$ , then  $\operatorname{Re}(\varphi(x)) > 1 - \delta$  and  $\operatorname{Re}(\varphi(y)) > 1 - \delta$ , which implies  $\|x - y\| < \varepsilon$  if positive number  $\delta(\varepsilon) > 0$  has been chosen according to special “thin slices” restrictions. “Thin slices” restrictions demand: for any  $\varepsilon \in (0, 2]$ , there are some  $\delta(\varepsilon) > 0$  such that from  $\varphi \in X^*$ ,  $\|\varphi\| = 1$  follows  $\|x - y\| < \varepsilon$  for all

$$x, y \in \{z \in X : \|z\| = 1 \text{ and } \operatorname{Re}(\varphi(z)) > 1 - \delta\}.$$

In 1961, the variable exponent function Lebesgue spaces was introduced by I. Tsenov in his study of problems of the best approximation in Lebesgue spaces [13]. A fairly exhaustive review of the extensive literature on the subject of variable functional spaces can be found in the work of L. Diening, P. Harjulehto, P. Hasto, and M. Ruzicka [4] which is up to date 2010. For some new applications and development see articles by X.L. Fan, Q.H. Zhan, D. Zhao [7, 8,], Y.Q. Fu. K. Ho and I. Sim [9, 11], the uniform convexity of variable exponent Lebesgue spaces  $L^{p(\cdot)}$  was mentioned by X.L. Fan, D. Zhao [7] in 1988 and by Q.M. Zhou, X.P. Xue, B. Ge in 2010, [7].

Let  $\Omega$  be a bounded connected domain in the  $R^n$ . Let  $P(\Omega)$  be a subspace of  $L^1(\Omega)$  such that  $p(\cdot) \in P(\Omega)$ ,  $p(\cdot) : \Omega \rightarrow (1, \infty)$ . We denote  $p_m(\tilde{\Omega}) = \operatorname{ess\,inf}_{x \in \tilde{\Omega}} p(x)$  and  $p_S(\tilde{\Omega}) = \operatorname{ess\,sup}_{x \in \tilde{\Omega}} p(x)$  for fixed  $\tilde{\Omega} \subset \Omega$ . For measurable functions  $f$ , we define a norm by

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \rho_p\left(\frac{f}{\lambda}\right) \leq 1 \right\},$$

where the functional  $\rho_p : L^1(\Omega) \rightarrow R$  is given by

$$\rho_p(f) = \int_{\Omega} |f(x)|^{p(x)} dx$$

for  $\Omega \subseteq R^n$ . The variable exponent Lebesgue space  $L^{p(\cdot)}(\Omega)$  consists of all measurable functions  $f$  such that  $\|f\|_{L^{p(\cdot)}} < \infty$ .

For variable exponent Lebesgue spaces  $L^{p(\cdot)}$  with  $1 < p_m \leq p_S < \infty$ , we can prove a variant of the Holder inequality in the form

$$\int |f(x)g(x)| dx \leq \left(1 + \frac{1}{p_m} - \frac{1}{p_S}\right) \|f\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}}$$

for all  $f \in L^{p(\cdot)}$ ,  $g \in L^{q(\cdot)}$ , where the dual exponent function is given by  $q(x) = \frac{p(x)}{p(x)-1}$ . The Holder constant equals  $k_{Hol} = 1 + \frac{1}{p_m} - \frac{1}{p_S}$ .

A Banach space is said to be uniformly smooth if for all  $\varepsilon \in (0, 1)$  there is  $\delta(\varepsilon) > 0$  such that condition

$$\|x + y\| + \|x - y\| \leq 2 + \varepsilon \|y\|$$

holds for all  $x, y \in X$  so that  $\|x\| = 1$  and  $\|y\| \leq \delta$ . The strong differentiability of the Banach space norm is connected with the uniform smoothness of this Banach space. The necessary condition of Frechet's differentiability of the norm is that the norm is uniformly smooth.

In the present article, we establish that for each  $\varepsilon \in (0, 2)$  and a pair of unit vectors  $u, v \in L^{p(\cdot)}$  such that  $\|u - v\|_{L^{p(\cdot)}} \geq \varepsilon$  there exists

$$\delta(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^{\frac{p_m}{p_m-1}}\right)^{\frac{p_S-1}{p_S}}$$

if  $1 < p_m \leq p_S < 2$  and  $\delta = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^{p_S}\right)^{\frac{1}{p_m}}$ , if  $2 \leq p_m \leq p_S < \infty$  such that  $\left\|\frac{u+v}{2}\right\|_{L^{p(\cdot)}} \leq 1 - \delta(\varepsilon)$ . We also prove that  $L^{p(\cdot)}$  is a uniformly smooth Banach space. To show the nontriviality of these statements, we remind our readers that classical  $L^1([0, 1])$  and  $L^\infty([0, 1])$  are not uniformly convex or uniformly smooth.

## 2. Basic properties of variable exponent Lebesgue spaces

Many basic properties of variable exponential Lebesgue spaces  $L^{p(\cdot)}$  were investigated by X. L. Fan, and D. Zhao [7]. The Lebesgue spaces  $L^{p(\cdot)}$  belong to the Banach spaces, if  $1 < p_m \leq p_S < \infty$  then  $L^{p(\cdot)}$  is reflexive Banach space with its dual  $L^{q(\cdot)}$ ,  $q(x) = \frac{p(x)}{p(x)-1}$ .

**DEFINITION 2.1.** A Banach space  $X$  is called a uniformly convex if for each  $\varepsilon \in (0, 2]$  there are some  $\delta(\varepsilon) > 0$  such that from

$$\|x - y\| \geq \varepsilon$$

follows

$$\frac{1}{2} \|x + y\| \leq 1 - \delta$$

for all unit vectors  $x, y \in X$ .

In case the exponential function is constant and larger one, i.e.  $p(x) = p > 1$ , we have an explicit expression for general parallelogram inequality [3] in the form

$$(\|u + v\|^r + \|u - v\|^r)^m \leq 2^{r(m-1)} (\|u\|^m + \|v\|^m)^r$$

that holds for all elements  $u, v \in L^p$  and for all  $1 < m \leq p \leq r$  so that  $\frac{r}{r-1} \leq m \leq r$ .

Assume  $p(\cdot) : \Omega \rightarrow [2, \infty)$ ,  $p \in P(\Omega)$ ,  $\Omega \subseteq R^n$  then we have an analog of parallelogram restriction in the integral form

$$\rho(u + v) + \rho(u - v) \geq 2(\rho(u) + \rho(v)),$$

$u, v \in L^{p(\cdot)}$ , this parallelogram condition guarantees the uniform convexity of the variable Lebesgue space  $L^{p(\cdot)}$  as we are going to prove in next lemma.

LEMMA 2.1. *Let  $2 \leq p_m \leq p_S < \infty$ ,  $p_m = \inf_{x \in \Omega} p(x)$ ,  $p_S = \sup_{x \in \Omega} p(x)$ , then  $L^{p(\cdot)}$  is uniformly convex space.*

P r o o f. Let  $u, v \in L^{p(\cdot)}$ .

For a pair of complex numbers  $z_1$  and  $z_2$ , we define a function

$$d = |z_1 + z_2|^t + |z_1 - z_2|^t.$$

Straightforwardly, we have

$$|z_1 + z_2|^t + |z_1 - z_2|^t \geq 2(|z_1|^t + |z_2|^t).$$

We put  $z_1 = \tilde{u}(x)$ ,  $z_2 = \tilde{v}(x)$  and integrate

$$\begin{aligned} & \int_{\Omega} |\tilde{u}(x) + \tilde{v}(x)|^{p(x)} dx + \int_{\Omega} |\tilde{u}(x) - \tilde{v}(x)|^{p(x)} dx \\ & \geq 2 \left( \int_{\Omega} |\tilde{u}(x)|^{p(x)} dx + \int_{\Omega} |\tilde{v}(x)|^{p(x)} dx \right) \end{aligned}$$

for  $\tilde{u}, \tilde{v} \in L^{p(\cdot)}$ . Now, denoting  $\tilde{u} = \frac{(u+v)}{2}$  and  $\tilde{v} = \frac{(u-v)}{2}$ , obtain

$$\begin{aligned} & \int_{\Omega} |u(x)|^{p(x)} dx + \int_{\Omega} |v(x)|^{p(x)} dx \\ & \geq 2 \left( \int_{\Omega} \left| \frac{u+v}{2} \right|^{p(x)} dx + \int_{\Omega} \left| \frac{u-v}{2} \right|^{p(x)} dx \right). \end{aligned}$$

Since vectors  $u$  and  $v$  are unit, i.e.,  $\|u\|_{L^{p(\cdot)}} = 1 = \|v\|_{L^{p(\cdot)}}$  we have

$$\int_{\Omega} \left| \frac{u+v}{2} \right|^{p(x)} dx + \int_{\Omega} \left| \frac{u-v}{2} \right|^{p(x)} dx \leq 1,$$

from the assumption  $\|u - v\| \geq \varepsilon$ , we obtain

$$\int_{\Omega} \left| \frac{u+v}{2} \right|^{p(x)} dx \leq 1 - \left( \frac{\varepsilon}{2} \right)^{p_S},$$

assume  $\delta(\varepsilon) = 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^{p_S} \right)^{\frac{1}{p_m}}$ , so we conclude

$$\left\| \frac{u+v}{2} \right\|_{L^{p(\cdot)}} \leq 1 - \delta.$$

□

LEMMA 2.2. *Let  $\text{mes}(\Omega) < \infty$ . Let  $1 < p_m \leq p_S < 2$ ,  $p_m = \inf_{x \in \Omega} p(x)$ ,  $p_S = \sup_{x \in \Omega} p(x)$ , then  $L^{p(\cdot)}$  is uniformly convex space.*

P r o o f. Applying Clarkson inequality [3], we have

$$\begin{aligned} & \left( \int_{\Omega} \left| \frac{u(x) + v(x)}{2} \right|^{p(x)} dx \right)^{\frac{(p_S-1)}{p_S^2}} \\ & + \left( \int_{\Omega} \left| \frac{u(x) - v(x)}{2} \right|^{p(x)} dx \right)^{\frac{(p_S-1)}{p_S^2}} \leq 1 \end{aligned}$$

with  $q(x) = \frac{p(x)}{p(x)-1}$ . Proceeding similarly to Lemma 2, we obtain

$$\int_{\Omega} \left| \frac{u+v}{2} \right|^{q(x)} dx \leq 1 - \left( \frac{\varepsilon}{2} \right)^{\frac{p_m}{p_m-1}}$$

and taking

$$\delta(\varepsilon) = 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^{\frac{p_m}{p_m-1}} \right)^{\frac{p_S-1}{p_S}},$$

we have

$$\left\| \frac{u+v}{2} \right\|_{L^{p(\cdot)}} \leq 1 - \delta(\varepsilon).$$

□

### 3. Uniform convexity of variable exponent Lebesgue spaces

X. Fan and D. Zhao showed the uniform convexity of  $L^{p(\cdot)}(\Omega)$  applying the theory of spaces with the Luxemburg norm and the following lemma.

LEMMA 3.1. (Fan and Zhao [7]) *Let  $1 < p_m \leq p_S < \infty$ , then for each fixed  $\varepsilon \in (0, 1)$ , there exists  $\delta(\varepsilon) \in (0, 1)$  such that the inequality*

$$\left( \frac{1+\alpha}{2} \tau \right)^{p(x)-p(\varepsilon)} \leq \frac{(1-\delta(\varepsilon))}{2} \left( 1 + \tau^{p(\alpha x)-p(\varepsilon)} \right)$$

*holds for all  $0 \leq \tau$  and all  $\alpha \in [0, \varepsilon]$ .*

A straightforward application of the Fan-Zhao lemma yields the following theorem.

THEOREM 3.1. (Fan and Zhao [7]) *Let  $1 < p_m \leq p_S < \infty$  then  $L^{p(\cdot)}(\Omega)$  is uniformly convex.*

If the exponential function is constant the Fan-Zhao theorem easily follows from the parallelogram inequalities. In the general situation, we have the following theorem.

**THEOREM 3.2.** *Let  $\varepsilon \in (0, 2)$  and  $u, v \in L^{p(\cdot)}$  such that  $\|u\|_{L^{p(\cdot)}} = 1 = \|v\|_{L^{p(\cdot)}}$  and  $\|u - v\|_{L^{p(\cdot)}} \geq \varepsilon$ . Then, the inequality*

$$\left\| \frac{u + v}{2} \right\|_{L^{p(\cdot)}} \leq 1 - \delta(\varepsilon)$$

*holds with  $\delta(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^{\frac{p_m}{p_m-1}}\right)^{\frac{p_S-1}{p_S}}$  if  $1 < p_m \leq p_S < 2$ ; and with  $\delta(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^{p_S}\right)^{\frac{1}{p_m}}$  if  $2 \leq p_m \leq p_S < \infty$ .*

**P r o o f.** The statement of the theorem follows from Lemmas 2.1 and 2.2.

Employing the convexity argument, we obtain that assume  $\Omega \subset R^n$ ,  $mes(\Omega) < \infty$ , then the estimate

$$\|f\|_{L^{p(\cdot)}} \leq mes(\Omega) + 1$$

holds for all  $u \in L^{p(\cdot)}$ . □

We are going to use the Harald Hanche-Olsen ideas to present an alternative qualitative proving of Theorem 3.2.

**LEMMA 3.2.** (analog of the H. Hanche-Olsen) *Let  $(\Omega, \mu)$  be a probability space. Let  $p(\cdot) \in P(\Omega)$ ,  $p(\cdot) : \Omega \rightarrow (1, \infty)$ ,  $1 < p_m \leq p_S < \infty$ , then for each fixed  $\varepsilon \in (0, 1)$ , there exists  $\delta(\varepsilon) \in (0, 1)$  such that the inequality*

$$\|z - 1\|_{L^{p(\cdot)}} < \varepsilon$$

*holds for all functions  $z \in L^1$  such that  $\|z\|_{L^{p(\cdot)}} = 1$  and  $Re \int_{\Omega} z d\mu > 1 - \delta$ .*

**P r o o f.** We introduce the function

$$\varphi(u) = |u|^{p(\cdot)} - 1 + p(\cdot)(1 - Re(u)).$$

So, for each  $\varepsilon \in (0, 1)$ , there are some numbers  $a > 1$  such that from  $\{|u - 1| \geq \varepsilon\}$  follows  $|u - 1|^{p(\cdot)} \leq a\varphi(u)$ . We calculate

$$\begin{aligned} \int_{\Omega} |z - 1|^{p(\cdot)} d\mu &= \int_{\Omega \setminus \{w \in \Omega : |z-1| < \varepsilon\}} |z - 1|^{p(\cdot)} d\mu \\ &+ \int_{\{w \in \Omega : |z-1| < \varepsilon\}} |z - 1|^{p(\cdot)} d\mu \leq \varepsilon^{p_m} + a\delta p_S, \end{aligned}$$

therefore, taking  $\delta = \frac{\varepsilon^{p_m}}{ap_S}$  we obtain the inequality  $\|z - 1\|_{L^{p(\cdot)}} < 2^{p_m^{-1}} \varepsilon$ , which proves the lemma. □

Now, we present another proof of Theorem 3.2 employing the analog of the H. Hanche-Olsen lemma.

**P r o o f.** Let  $u, v \in L^{p(\cdot)}$ ,  $\|u\|_{L^{p(\cdot)}} = 1 = \|v\|_{L^{p(\cdot)}}$  such that  $\|u + v\|_{L^{p(\cdot)}} > 2 - \delta$ . If  $\tilde{u} = \frac{u+v}{\|u+v\|_{L^{p(\cdot)}}}$ , then there exists an element  $\tilde{v} \in L^{q(\cdot)}$  such that  $\tilde{u}\tilde{v} = |\tilde{u}|^{p(\cdot)} = |\tilde{v}|^{q(\cdot)}$  so that  $\|\tilde{u}\|_{L^{p(\cdot)}} = 1 = \|\tilde{v}\|_{L^{q(\cdot)}}$ . We have

$$\int_{\Omega} |u + v|^{p(\cdot)} \tilde{v} d\mu = \|u + v\|_{L^{p(\cdot)}} \int_{\Omega} \tilde{u}\tilde{v} d\mu = \|u + v\|_{L^{p(\cdot)}} > 2 - \delta.$$

Now, we write  $Re \int_{\Omega} v\tilde{v} d\mu \leq 1$  so that  $Re \int_{\Omega} u\tilde{v} d\mu > 1 - \delta$ . We choose  $\delta$  as in the H. Hanché-Olsen lemma and obtain  $\|u - \tilde{u}\|_{L^{p(\cdot)}} \leq \varepsilon$  and  $\|v - \tilde{u}\|_{L^{p(\cdot)}} \leq \varepsilon$ , thus  $\|u - v\|_{L^{p(\cdot)}} < 2\varepsilon$ , which proves uniform convexity of variable Lebesgue spaces  $L^{p(\cdot)}$ .  $\square$

#### 4. Uniform smoothness of variable exponent Lebesgue spaces

Now, we consider the uniform smoothness of variable Lebesgue spaces, which is in some way a dual approach to uniform convexity.

**DEFINITION 4.1.** A Banach space  $X$  is called a uniformly smooth if for each  $\varepsilon > 0$  there are some  $\delta(\varepsilon) > 0$  such that, for all  $x, y \in X$ , from  $\|x\| = 1$  and  $\|y\| \leq \delta$  follows

$$\|x + y\| + \|x - y\| \leq 2 + \varepsilon \|y\|.$$

If the norm of the Banach space is differentiable then the identity

$$\lim_{\|y\| \rightarrow 0} \frac{1}{\|y\|} \left( \left\| \frac{x+y}{2} \right\| + \left\| \frac{x-y}{2} \right\| - \|x\| \right) = 0$$

must hold for all  $x, y \in X$ .

**DEFINITION 4.2.** Let  $X$  be a Banach space. A function defined by

$$\gamma_X(\tau) = \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}$$

is called the modulus of smoothness of the norm of the Banach space  $X$ .

**REMARK 4.1.** In the theory of variable exponent Lebesgue spaces, the letter  $\rho$  stands for  $p(\cdot)$ -modular of the  $L^{p(\cdot)}$ -space.

Straightforwardly, a Banach space is uniformly smooth if and only if  $\lim_{\tau \rightarrow 0} \frac{\gamma_X(\tau)}{\tau} = 0$ .

Now, we are going to prove the theorem from the analysis of abstract Banach spaces.

**THEOREM 4.1.** (Lindenstrauss) *Let  $X$  be a separable reflexive Banach space then  $X$  is uniformly convex if and only if its dual  $X^*$  is uniformly smooth.*

**P r o o f.** First, assume the Banach space  $X$  is uniformly convex and show that its dual  $X^*$  is uniformly smooth. Let  $\tau > 0$ ,  $\zeta > 0$  and a pair of unit functionals  $x^*, y^* \in X^*$ , then there exists  $\varepsilon(\tau) \in [0, 1]$  such that

$$\begin{aligned} & \left\| \frac{x^* + \tau y^*}{2} \right\|_{X^*} + \left\| \frac{x^* - \tau y^*}{2} \right\|_{X^*} - 1 \\ & \leq \tau \varepsilon - \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\|_X : \|x - y\|_X < 2\varepsilon \right\} + \zeta, \end{aligned}$$

indeed, we can choose vectors  $x(\tau), y(\tau) \in X^*$  so that the inequalities

$$\langle x^* + \tau y^*, x(\tau) \rangle \geq \|x^* + \tau y^*\|_{X^*} - \zeta$$

and

$$\langle x^* - \tau y^*, y(\tau) \rangle \geq \|x^* - \tau y^*\|_{X^*} - \zeta$$

be satisfied. So, we put

$$\varepsilon(\tau) = \left\| \frac{x(\tau) + y(\tau)}{2} \right\|_X \in [0, 1]$$

and

$$\begin{aligned} & \left\| \frac{x(\tau) + y(\tau)}{2} \right\|_X \\ & \leq 1 - \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\|_X : \|x - y\|_X < 2\varepsilon \right\}, \end{aligned}$$

and obtain

$$\begin{aligned} & \left\| \frac{x^* + \tau y^*}{2} \right\|_{X^*} + \left\| \frac{x^* - \tau y^*}{2} \right\|_{X^*} \\ & \leq \frac{1}{2} (\langle x^* + \tau y^*, x(\tau) \rangle + \langle x^* - \tau y^*, y(\tau) \rangle) + \zeta \\ & = \frac{1}{2} (\langle x^*, x(\tau) + y(\tau) \rangle + \tau \langle y^*, x(\tau) - y(\tau) \rangle) + \zeta \\ & \leq 1 - \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\|_X : \|x - y\|_X < 2\varepsilon \right\} + \tau \varepsilon(\tau) + \zeta. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \gamma_{X^*}(\tau) \\ & \leq - \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\|_X : \|x - y\|_X < 2\varepsilon \right\} + \tau \varepsilon(\tau) + \zeta \\ & \leq \zeta + \sup_{\varepsilon \in [0, 1]} \left\{ \tau \varepsilon(\tau) - \inf_{x, y} \left\{ 1 - \left\| \frac{x + y}{2} \right\|_X : \|x - y\|_X < 2\varepsilon \right\} \right\}, \end{aligned}$$

so

$$\begin{aligned} & \gamma_{X^*}(\tau) \\ & \leq \sup_{\varepsilon \in [0, 1]} \left\{ \tau \varepsilon(\tau) - \inf_{x, y} \left\{ 1 - \left\| \frac{x + y}{2} \right\|_X : \|x - y\|_X < 2\varepsilon \right\} \right\} \end{aligned}$$

since  $\zeta > 0$  can be arbitrarily small.

Thus, for any  $\tilde{\varepsilon} > 0$  we take

$$\tau < \left( \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\|_X : \|x-y\|_X < 2\tilde{\varepsilon} \right\} \right)^{-1},$$

then from

$$\begin{aligned} & \frac{\gamma_{X^*}(\tau)}{\tau} \\ & \leq \sup_{\tilde{\varepsilon} \in [0, 1]} \left\{ \tilde{\varepsilon} - \tau^{-1} \inf_{x, y} \left\{ 1 - \left\| \frac{x+y}{2} \right\|_X : \|x-y\|_X < 2\tilde{\varepsilon} \right\} \right\} \xrightarrow{\tau \rightarrow 0} 0. \end{aligned}$$

Thus,  $X^*$  is uniformly smooth.

Second, assume  $X^*$  is uniformly smooth and show that  $X$  is uniformly convex. Let  $\varepsilon \in (0, 1)$  and let  $x, y \in X$  be unit vectors such that  $\left\| \frac{x-y}{2} \right\| \geq \varepsilon$ . Then, we choose a pair of unit linear functionals  $x^*, y^* \in X^*$  such that

$$\langle x^*, x+y \rangle \geq \|x+y\| - 2\zeta$$

and

$$\langle y^*, x-y \rangle \geq \|x-y\| - 2\zeta.$$

For all  $\tau \geq 0$ , we have

$$\begin{aligned} \gamma_{X^*}(\tau) & \geq \left\| \frac{x^* + \tau y^*}{2} \right\|_{X^*} + \left\| \frac{x^* - \tau y^*}{2} \right\|_{X^*} - 1 \\ & \geq \frac{1}{2} (\langle x^* + \tau y^*, x \rangle + \langle x^* - \tau y^*, y \rangle) - 1 \\ & = \frac{1}{2} (\langle x^*, x+y \rangle + \tau \langle y^*, x-y \rangle) - 1 \\ & \geq \left\| \frac{x+y}{2} \right\|_X + \tau \left\| \frac{x-y}{2} \right\|_X - 1 - 2\zeta \\ & \geq - (1 + \left\| \frac{x+y}{2} \right\|_X) + \tau\varepsilon - 2\zeta, \end{aligned}$$

so, we deduce

$$\gamma_{X^*}(\tau) + \left( 1 + \left\| \frac{x+y}{2} \right\|_X \right) \geq \tau\varepsilon - 2\zeta.$$

Since  $X^*$  is uniformly smooth we have that there exists  $\tilde{\tau} > 0$  such that

$$\frac{\gamma_{X^*}(\tau)}{\tau} \leq \frac{\varepsilon}{2}$$

for all  $\tau < \tilde{\tau}$ , so that

$$\delta = \inf_{x, y} \left\{ 1 - \left\| \frac{x+y}{2} \right\|_X : \|x-y\| < 2\varepsilon \right\} \geq \frac{\varepsilon}{2}\tau > 0,$$

and for such  $\delta$  we have  $\frac{1}{2}\|x+y\| \leq 1 - \delta$ , which prove uniform convexity of  $X$ .  $\square$

Now, we can formulate the theorem about the uniform smoothness of variable Lebesgue space.

**THEOREM 4.2.** (uniform smoothness of  $L^{p(\cdot)}$ ) *Let  $1 \leq p_m \leq p_S < \infty$ ,  $p_m = \inf_{x \in \Omega} p(x)$ ,  $p_S = \sup_{x \in \Omega} p(x)$  then  $L^{p(\cdot)}$  is uniformly smooth space.*

**P r o o f.** Since the space  $L^{q(\cdot)}$  with  $q(x) = \frac{p(x)}{p(x)-1}$  is uniformly convex,  $L^{p(\cdot)}$  is uniformly smooth by the Lindenstrauss theorem.  $\square$

### 5. Some functions describing uniform smoothness of variable exponent Lebesgue spaces

Finally, we consider some functions related to so-called moduli of convexity and smoothness of the Banach spaces.

For  $\varepsilon \in [0, 1]$ , we introduce functions

$$M(x, \varepsilon) = \inf \{ \max \{ \|x + y\|, \|x - y\| \} : \|y\| = \varepsilon \}$$

and

$$m(x, \varepsilon) = \sup \{ \min \{ \|x + y\|, \|x - y\| \} : \|y\| = \varepsilon \}$$

for all  $x, y \in X$ ,  $\|x\| = 1$ ,  $\|y\| = \varepsilon$ .

For all  $x, y \in X$ ,  $\|x\| = 1$ ,  $\|y\| = \varepsilon$ ,  $\varepsilon \in [0, 1]$ , we denote values

$$\alpha_1(\varepsilon) = \inf \{ M(x, \varepsilon) : \|x\| = 1 \},$$

$$\alpha_2(\varepsilon) = \sup \{ M(x, \varepsilon) : \|x\| = 1 \}$$

and

$$\beta_1(\varepsilon) = \inf \{ m(x, \varepsilon) : \|x\| = 1 \},$$

$$\beta_2(\varepsilon) = \sup \{ m(x, \varepsilon) : \|x\| = 1 \}.$$

**THEOREM 5.1.** (J. Gao and K.S. Lau) *Let  $\varepsilon \in [0, 1]$ , then*

$$\alpha_1(\varepsilon) \leq \alpha_2(\varepsilon) \leq \beta_2(\varepsilon)$$

and

$$\alpha_1(\varepsilon) \leq \beta_1(\varepsilon) \leq \beta_2(\varepsilon).$$

A similar theorem was proven by J. Gao and K.S. Lau in [17].

**THEOREM 5.2.** *Let  $L^{p(\cdot)}(\Omega)$ ,  $\mu(\Omega) = 1$ , and let  $\varepsilon \in [0, 1]$ . Then, the following estimates are true:*

1) if  $2 \leq p_m \leq p_S < \infty$ , then

$$(1 + \varepsilon^{p_S})^{\frac{1}{p_m}} \leq \alpha_1(\varepsilon),$$

$$\alpha_1(\varepsilon) \leq (1 + \varepsilon^{p_m})^{\frac{1}{p_S}},$$

$$\alpha_2(\varepsilon) \leq (1 + \varepsilon^{p_m})^{\frac{1}{p_m}},$$

$$\left( \frac{(1+\varepsilon)^{p_S} + (1-\varepsilon)^{p_m}}{2} \right)^{\frac{1}{p_S}} \leq \beta_1(\varepsilon),$$

$$\left( \frac{(1+\varepsilon)^{p_m} + (1-\varepsilon)^{p_S}}{2} \right)^{\frac{1}{p_m}} \geq \beta_2(\varepsilon);$$

2) if  $1 < p_m \leq p_S < 2$ , then

$$\alpha_1(\varepsilon) \geq \left( \frac{(1+\varepsilon)^{p_m} + (1-\varepsilon)^{p_S}}{2} \right)^{\frac{1}{p_S}},$$

$$\beta_2(\varepsilon) \leq \left( \frac{(1+\varepsilon)^{p_S} + (1-\varepsilon)^{p_m}}{2} \right)^{\frac{1}{p_m}},$$

$$\beta_1(\varepsilon) \geq (1 + \varepsilon^{p_S})^{\frac{1}{p_m}},$$

$$\beta_2(\varepsilon) \leq (1 + \varepsilon^{p_m})^{\frac{1}{p_S}}.$$

**P r o o f.** 1) Assume  $u, v \in L^{p(\cdot)}$ ,  $\|u\|_{L^{p(\cdot)}} = 1$ ,  $\|v\|_{L^{p(\cdot)}} = \varepsilon$  with  $2 \leq p_m \leq p_S < \infty$ , we have

$$2(1 + \varepsilon^{p_S}) \leq \int_{\Omega} |u(x) + v(x)|^{p(x)} dx + \int_{\Omega} |u(x) - v(x)|^{p(x)} dx$$

$$\geq 2(\max\{\|u+v\|, \|u-v\|\})^{p_S},$$

so,  $(1 + \varepsilon^{p_S})^{\frac{1}{p_m}} \leq \alpha_1(\varepsilon)$ . Applying parallelogram inequality, we obtain  $\alpha_1(\varepsilon) \leq (1 + \varepsilon^{p_m})^{\frac{1}{p_S}}$ .

For arbitrary small  $\zeta > 0$ , there is an estimate

$$(1 + \zeta)^{\frac{1}{p_S}} < 1 + 2p_S\zeta,$$

we take a number  $\delta \in (0, 1)$ , so that

$$\int_{B(\delta)} |u(x)|^{p(x)} dx < \varepsilon^{p_m} \zeta^{p_m}$$

and denote

$$v(x) \equiv \begin{cases} \varepsilon \delta^{-\frac{1}{p_m}}, & |x| \leq \delta, \\ 0, & |x| \geq \delta, \end{cases}$$

we calculate

$$\begin{aligned} & \rho(u+v) \\ & \leq 1 + \int_{B(\delta)} \left| u(x) + \varepsilon \delta^{-\frac{1}{p_m}} \right|^{p(x)} dx \\ & \leq 1 + \int_{B(\delta)} \varepsilon^{p_m} \delta^{-1} \left| 1 + \varepsilon^{-1} \delta^{\frac{1}{p_m}} u(x) \right|^{p(x)} dx \\ & \leq 1 + \varepsilon^{p_m} \delta^{-1} \int_{B(\delta)} \left| 1 + \varepsilon^{-1} \delta^{\frac{1}{p_m}} u(x) \right|^{p(x)} dx \end{aligned}$$

and

$$\begin{aligned} & \rho(u+v) \\ & \leq 1 + \varepsilon^{p_m} \delta^{-1} \left( \delta^{\frac{1}{p_S}} + \varepsilon^{-1} \delta^{\frac{1}{p_S}} \left( \int_{B(\delta)} |u(x)|^{p(x)} dx \right)^{\frac{1}{p_S}} \right)^{p_m} \\ & \leq 1 + \left( 1 + \varepsilon \frac{\zeta}{\varepsilon} \right)^{p_S} \varepsilon^{p_m} = 1 + \varepsilon^{p_m} (1 + \zeta)^{p_S}. \end{aligned}$$

Therefore, we have the inequalities

$$\rho(u+v) \leq 1 + \varepsilon^{p_m} (1 + 2p_S \zeta)$$

and

$$\rho(u-v) \leq 1 + \varepsilon^{p_m} (1 + 2p_S \zeta),$$

so, the inequality

$$\max \{ \|u+v\|, \|u-v\| \} \leq (1 + \varepsilon^{p_m} (1 + 2p_S \zeta))^{\frac{1}{p_m}}$$

holds for all small  $\zeta > 0$ . We obtain the estimate

$$\alpha_2(\varepsilon) \leq (1 + \varepsilon^{p_m})^{\frac{1}{p_m}}.$$

Applying parallelogram inequalities, straightforwardly, we deduce

$$\left( \frac{(1+\varepsilon)^{p_S} + (1-\varepsilon)^{p_m}}{2} \right)^{\frac{1}{p_S}} \leq \beta_1(\varepsilon),$$

hence

$$(\min \{ \|u+v\|, \|u-v\| \})^{p_S} \geq \frac{(1+\varepsilon)^{p_S} + (1-\varepsilon)^{p_m}}{2}.$$

Similarly, we have

$$\left( \frac{(1+\varepsilon)^{p_m} + (1-\varepsilon)^{p_S}}{2} \right)^{\frac{1}{p_m}} \geq \beta_2(\varepsilon).$$

2) We assume  $u, v \in L^{p(\cdot)}$ ,  $\|u\|_{L^{p(\cdot)}} = 1$ ,  $\|v\|_{L^{p(\cdot)}} = \varepsilon$  with  $1 < p_m \leq p_S < 2$ . Applying parallelogram inequalities, we have

$$\alpha_1(\varepsilon) \geq \left( \frac{(1+\varepsilon)^{p_m} + (1-\varepsilon)^{p_S}}{2} \right)^{\frac{1}{p_S}}.$$

Since  $\|u\|_{L^{p(\cdot)}} = 1$  we can take  $\sigma \in (0, 1)$  so that

$$\int_{B(\sigma)} |u(x)|^{p(x)} dx = \int_{\Omega \setminus B(\sigma)} |u(x)|^{p(x)} dx = \frac{1}{2}.$$

We define a function

$$v(x) \equiv \begin{cases} -\varepsilon u(x), & |x| \leq \sigma, \\ \varepsilon u(x), & |x| > \sigma, \end{cases}$$

so that  $\|v\|_{L^{p(\cdot)}} = \varepsilon$ . We calculate

$$\begin{aligned} & \rho(u+v) \\ &= \int_{B(\sigma)} |u(x) + v(x)|^{p(x)} dx + \int_{\Omega \setminus B(\sigma)} |u(x) + v(x)|^{p(x)} dx \\ &= \int_{B(\sigma)} (1-\varepsilon)^{p(x)} |u(x)|^{p(x)} dx \\ &\quad + \int_{\Omega \setminus B(\sigma)} (1+\varepsilon)^{p(x)} |u(x)|^{p(x)} dx \\ &\leq \frac{(1-\varepsilon)^{p_m} + (1+\varepsilon)^{p_S}}{2}. \end{aligned}$$

Similarly, we estimate

$$\rho(u-v) \leq \frac{(1-\varepsilon)^{p_m} + (1+\varepsilon)^{p_S}}{2}.$$

Straightforwardly, we write

$$\beta_2(\varepsilon) \leq \left( \frac{(1+\varepsilon)^{p_S} + (1-\varepsilon)^{p_m}}{2} \right)^{\frac{1}{p_m}}.$$

Next, we can write

$$\beta_2(\varepsilon) \leq (1 + \varepsilon^{p_m})^{\frac{1}{p_S}}$$

by the parallelogram inequalities.

We will prove that

$$\beta_1(\varepsilon) \geq (1 + \varepsilon^{p_S})^{\frac{1}{p_m}}.$$

Indeed, for each small,  $\zeta > 0$  there is a  $\delta \in (0, 1)$  such that

$$\int_{B(\delta)} |u(x)|^{p(x)} dx < \zeta$$

and

$$\int_{\Omega} |u(x)|^{p(x)} dx \geq 1 - \zeta.$$

We define a function

$$v(x) \equiv \begin{cases} \varepsilon \delta^{-\frac{1}{p_m}}, & |x| \leq \delta, \\ 0, & |x| \geq \delta, \end{cases}$$

and calculate

$$\begin{aligned} & \rho(u+v) \\ &= \int_{\Omega} |u(x) + v(x)|^{p(x)} dx \\ &= \int_{B(\delta)} \left| u(x) + \varepsilon \delta^{-\frac{1}{p_m}} \right|^{p(x)} dx + \int_{\Omega} |u(x)|^{p(x)} dx \\ &\geq 1 - \zeta + \int_{B(\delta)} \left| u(x) + \varepsilon \delta^{-\frac{1}{p_m}} \right|^{p(x)} dx. \end{aligned}$$

Next, we have

$$\int_{B(\delta)} \left| u(x) + \varepsilon \delta^{-\frac{1}{p_m}} \right|^{p(x)} dx \geq \left( \varepsilon - \zeta^{\frac{1}{p_S}} \right)^{p_S},$$

so that

$$\rho(u+v) \geq 1 - \zeta + \left(\varepsilon - \zeta^{\frac{1}{p_S}}\right)^{p_m}$$

and

$$\rho(u-v) \geq 1 - \zeta + \left(\varepsilon - \zeta^{\frac{1}{p_S}}\right)^{p_m},$$

therefore, we obtain  $\beta_1(\varepsilon) \geq (1 + \varepsilon^{p_S})^{\frac{1}{p_m}}$ . □

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