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ON β-ABSORBING PRIMARY SUBMODULES

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Abstract

In this paper, we define the notion of β-absorbing primary submodules. We can see that β-absorbing primary submodules are a generalization of β-absorbing submodules and 2-absorbing submodules. Several properties concerning β-absorbing primary submodules and examples are given.

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Key Words and Phrases: β-absorbing primary ideal, β-absorbing primary submodule

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unital. Prime ideals have an important role in commutative rings. There are several ways to generalize the concept of prime ideals. Badawi [1] defined a nonzero proper ideal I of a ring R to be a 2-absorbing ideal of R if $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

Let R be a ring and $X \subseteq R$. The radical of X with respect to R is to $\sqrt{X} = \{x \in R \mid x^n \in X \text{ for some positive integers } n\}$. If I is an ideal of R, then $\sqrt{I}$ is an ideal of R. In [2], Badawi *et al.* generalized the concept of

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2-absorbing ideals to 2-absorbing primary ideals. A proper ideal I of a ring R is said to be a 2-absorbing primary ideal of R if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Every 2-absorbing ideal is a 2-absorbing primary ideal. However, the converse is not true. For example, $12\mathbb{Z}$ is a 2-absorbing primary ideal of $\mathbb{Z}$ but $12\mathbb{Z}$ is not a 2-absorbing ideal of $\mathbb{Z}$.

Several authors have extended the notion of prime ideals, 2-absorbing ideals and 2-absorbing primary ideals to modules (for example, see [6], [3], [7] and [5]). These are idea of our research.

Recall from [4] that a proper ideal I of R is called a β-absorbing ideal of R if whenever $a, b, c \in R$ with $abc \in I$, then $a(b+b) \in I$ or $a(c+c) \in I$ or $b(c+c) \in I$. We can say that β-absorbing ideals are generalizations of 2-absorbing ideals. Moreover, we give the ideal $8\mathbb{Z}$ of a ring $\mathbb{Z}$ as a example of β-absorbing ideal but is not 2-absorbing ideals.

In Section 2, we extend the notion of β-absorbing ideals to β-absorbing primary ideals. Some of examples are given. In Section 3, we present the concept of β-absorbing primary submodules and study some properties of β-absorbing primary submodules. Section 4 is devoted to the characterizations of β-absorbing primary submodules. Several examples of β-absorbing primary submodules of $\mathbb{Z}$-module $\mathbb{Z}$ are given in Section 5.

2. β-Absorbing primary ideals

We begin this section with the definition of β-absorbing primary ideals. Some examples of β-absorbing primary ideals are also given.

Definition 2.1. A proper ideal I of R is said to be a **β-absorbing *primary* ideal** of R if whenever $a, b, c \in R$ with $abc \in I$, then $a(b+b) \in I$ or $a(c+c) \in \sqrt{I}$ or $b(c+c) \in \sqrt{I}$.

From the definition of β-absorbing ideals and β-absorbing primary ideals, we have every β-absorbing ideal is a β-absorbing primary ideal but the converse does not necessary hold.

Note that for an integer n, $n\mathbb{Z}$ is a β-absorbing primary ideal of $\mathbb{Z}$ if and only if for all $a, b, c \in \mathbb{Z}$, if $n \mid abc$, then $n \mid 2ab$ or $n \mid (2ac)^t$ for some integers t or $n \mid (2bc)^k$ for some integers k.

Example 2.1. In this example, we show that $36\mathbb{Z}$ is a β-absorbing primary ideal of $\mathbb{Z}$ but $36\mathbb{Z}$ is not a β-absorbing ideal of $\mathbb{Z}$.

(1) To show that $36\mathbb{Z}$ is a β-absorbing primary ideal of $\mathbb{Z}$, let $a, b, c \in \mathbb{Z}$ be such that $36 \mid abc$. Note that $36 = 2^2 \cdot 3^2$. Then $3 \mid a$ or $3 \mid b$ or $3 \mid c$. This implies that $2^2 \cdot 3^2 \mid (2ac)^2$ or $2^2 \cdot 3^2 \mid (2bc)^2$.

(2) Next, we prove that $36\mathbb{Z}$ is not a β-absorbing ideal of $\mathbb{Z}$. Given $a = 4$ and $b = c = 3$. Then $36 \mid 4 \cdot 3 \cdot 3$. But $36 \nmid a(b+b)$ and $36 \nmid a(c+c)$ and $36 \nmid b(c+c)$.

EXAMPLE 2.2. By Example 1.6 and Proposition 3.2 from [4], we have that $30\mathbb{Z}$ is a β-absorbing ideal of $\mathbb{Z}$. Since β-absorbing ideals imply β-absorbing primary ideals, $30\mathbb{Z}$ is a β-absorbing primary ideal of $\mathbb{Z}$. To show that $30\mathbb{Z}$ is not a 2-absorbing primary ideal of $\mathbb{Z}$, let $a = 2, b = 3$ and $c = 5$. Then $30 \mid abc$. Since $30 \nmid ab$ and $30 \nmid (ac)^t$ for all integer t and $30 \nmid (bc)^t$ for all integer t. This shows that $30\mathbb{Z}$ is not a 2-absorbing primary ideal of $\mathbb{Z}$.

EXAMPLE 2.3. If p is a prime number, then $2^m p^n\mathbb{Z}$ is a β-absorbing primary ideal of $\mathbb{Z}$ for all positive integer m and n.

Proof. Assume that m and n are positive integers and p is a prime number. Let $a, b, c \in \mathbb{Z}$ be such that $2^m p^n \mid abc$. Then $p \mid ac$ or $p \mid b$. If $p \mid ac$, then $p^n \mid (ac)^n$. So $2^m p^n \mid 2^m (ac)^n$. Hence $2^m p^n \mid (2ac)^{mn}$. If $p \mid b$, then $p^n \mid b^n$. Thus $p^n \mid (bc)^n$. This impies that $2^m p^n \mid 2^m (bc)^n$. Hence $2^m p^n \mid (2bc)^{mn}$. Therefore $2^m p^n\mathbb{Z}$ is a β-absorbing primary ideal of $\mathbb{Z}$. □

Now we give some preliminary results.

Let $a, b, c \in R$ and I be an ideal of R. Then $a^k(b^k + b^k) = a^k b^k(1+1)$. Moreover, if $a^k(b^k + b^k) \in I$, then $[a(b+b)]^k \in I$, where k is a positive integer. Note that if I is an ideal of R, then $\sqrt{\sqrt{I}} = \sqrt{I}$.

PROPOSITION 2.1. *If I is a β-absorbing primary ideal of R, then $\sqrt{I}$ is a β-absorbing ideal of R.*

Proof. Assume that I is a β-absorbing primary ideal of R. Let $a, b, c \in R$ be such that $abc \in \sqrt{I}$. Then $a^k b^k c^k \in I$ for some positive integers k. Since I is a β-absorbing primary ideal of R, $a^k(b^k + b^k) \in I$ or $a^k(c^k + c^k) \in \sqrt{I}$ or $b^k(c^k + c^k) \in \sqrt{I}$. If $a^k(b^k + b^k) \in I$, then $[a(b+b)]^k \in I$. So $a(b+b) \in \sqrt{I}$. Assume that $a^k(c^k + c^k) \in \sqrt{I}$ or $b^k(c^k + c^k) \in \sqrt{I}$. Then $[a^k(c^k + c^k)]^t \in I$ or $[b^k(c^k + c^k)]^l \in I$ for some positive integers t and l. Hence $[a(c+c)]^{kt} \in I$ or $[b(c+c)]^{kl} \in I$. Therefore $a(c+c) \in \sqrt{I}$ or $b(c+c) \in \sqrt{I}$. This implies that $\sqrt{I}$ is a β-absorbing ideal of R. □

However, the converse of Proposition 2.1 is not true which are illustrated as follows.

EXAMPLE 2.4. If p and q are distinct odd prime numbers, then $4pq\mathbb{Z}$ is not a β-absorbing primary ideal of $\mathbb{Z}$.

P r o o f. Assume that p and q are distinct odd prime numbers. Given $a = p, b = q$ and $c = 4$. Then $4pq \mid abc$. Since $4pq \nmid 2ab$ and $4pq \nmid (2ac)^t$ for all integers t and $4pq \mid (2bc)^t$ for all integers t. This implies that $4pq\mathbb{Z}$ is not a β-absorbing primary ideal of $\mathbb{Z}$. $\square$

Example 2.5. Consider submodules $\sqrt{60\mathbb{Z}}$ and $30\mathbb{Z}$ of a $\mathbb{Z}$-module $\mathbb{Z}$. We know that $\sqrt{60\mathbb{Z}} = 30\mathbb{Z}$. It follows from Example 2.4 that $60\mathbb{Z}$ is not a β-absorbing primary ideal of $\mathbb{Z}$. Then $\sqrt{60\mathbb{Z}}$ is a β-absorbing ideal of $\mathbb{Z}$ but $60\mathbb{Z}$ is not a β-absorbing primary ideal of $\mathbb{Z}$.

Corollary 2.1. *If I is a β-absorbing primary ideal of R, then $\sqrt{I}$ is a β-absorbing primary ideal of R.*

Let R be a ring and M be a left R-module. The commutative ring $R(+)M$ is defined by $R(+)M = R \times M$ with the operations $(a, m)+(b, n) = (a+b, m+n)$ and $(a, m) \cdot (b, n) = (ab, an + bm)$. If I is an ideal of R, then $I(+)R$ is an ideal of $R(+)M$.

Theorem 2.1. *Let I be a proper ideal of R. Then the following statements hold.*

(1) I is a β-absorbing ideal of R if and only if $I(+)M$ is a β-absorbing ideal of $R(+)M$.
(2) I is a β-absorbing primary ideal of R if and only if $I(+)M$ is a β-absorbing primary ideal of $R(+)M$.

P r o o f. (1) The proof is obvious.

(2) $(\rightarrow)$ Assume that I is a β-absorbing primary ideal of R. Let $a, b, c \in R$ and $m, n, p \in M$ be such that $(a, m)(b, n)(c, p) \in I(+)M$. We have the fact that $(abc, abp + acn + bcm) \in I(+)M$. Then $abc \in I$. Since I is a β-absorbing primary ideal of R, $a(b + b) \in I$ or $a(c + c) \in \sqrt{I}$ or $b(c + c) \in \sqrt{I}$. If $a(b + b) \in I$, then $(a, m)[(b, n) + (b, n)] \in I(+)M$. Next, assume that $a(c + c) \in \sqrt{I}$ or $b(c + c) \in \sqrt{I}$. Then $a^t(c + c)^t \in I$ or $b^k(c + c)^k \in I$, where t and k are integers. This implies that $(a, m)[(c, p) + (c, p)] \in \sqrt{I(+)M}$ or $(b, n)[(c, p) + (c, p)] \in \sqrt{I(+)M}$. Therefore $I(+)M$ is a β-absorbing primary ideal of $R(+)M$.

$(\leftarrow)$ Assume that $I(+)M$ is a β-absorbing primary ideal of $R(+)M$. Let $a, b, c \in R$ be such that $abc \in I$. Then $(a, 0)(b, 0)(c, 0) = (abc, 0) \in I(+)M$. Since $I(+)M$ is a β-absorbing primary ideal of $R(+)M$, we have that $(a, 0)[(b, 0)+(b, 0)] \in I(+)M$ or $(a, 0)[(c, 0) + (c, 0)] \in \sqrt{I(+)M}$ or $(b, 0)[(c, 0) + (c, 0)] \in \sqrt{I(+)M}$. This obtains $a(b + b) \in I$ or $a(c + c) \in \sqrt{I}$ or $b(c + c) \in \sqrt{I}$. Hence I is a β-absorbing primary ideal of R. $\square$

3. β-Absorbing primary submodules

In this section, we generalize the concept of β-absorbing submodules to β-absorbing primary submodules. Recall from [4], a proper submodule P of M is call a β-absorbing submodule of M if for any element $r, s \in R$ and $m \in M$ such that $rsm \in P$, we have $rs + rs \in (P : M)$ or $r(m + m) \in P$ or $s(m + m) \in P$.

Definition 3.1. Let M be an R-module and P be a proper submodule of M. Then P is called a ***β-absorbing primary submodule*** of M if for any elements $r, s \in R$ and $m \in M$ such that $rsm \in P$, we have $rs+rs \in \sqrt{(P : M)}$ or $r(m + m) \in P$ or $s(m + m) \in P$.

Note that every β-absorbing submodule of a module is a β-absorbing primary submodule. However, a β-absorbing primary submodule need not be β-absorbing submodules. (See Example 5.2 and Example 5.3 in Section 5). Now, we give some properties of β-absorbing primary submodules.

Proposition 3.1. *If P is a β-absorbing primary submodule of M and K is a submodule of M, then $P \cap K$ is a β-absorbing primary submodule of K.*

Proof. Assume that P is a β-absorbing primary submodule of M and K is a submodule of M. Let $r, s \in R$ and $m \in K$ be such that $rsm \in P \cap K$. Since $rsm \in P$ and P is a β-absorbing primary submodule of M, $rs+rs \in \sqrt{(P : M)}$ or $r(m + m) \in P$ or $s(m + m) \in P$. Assume that $rs + rs \in \sqrt{(P : M)}$. Then $(rs + rs)^k \in (P : M)$ for some positive integer k. Then $(rs + rs)^k K \subseteq K$ and $(rs + rs)^k K \subseteq (rs + rs)^k M \subseteq P$. This implies that $rs + rs \in \sqrt{(P \cap K : K)}$. Next, assume that $r(m+m) \in P$ or $s(m+m) \in P$. Since $m \in K$, $r(m+m) \in P \cap K$ or $s(m + m) \in P \cap K$. Therefore $P \cap K$ is a β-absorbing primary submodule of K. □

Let $r \in R$ and N be a submodule of a left R-module M. We define $(N :_M r)$ by $(N :_M r) = \{m \in M \mid rm \in N\}$. Then $(N :_M r)$ is a submodule of M containing N.

Note that for $r \in R$, $r \in (N : M)$ if and only if $(N :_M r) = M$.

Proposition 3.2. *If P is a β-absorbing primary submodule of M and an element $r \in R\backslash(P : M)$, then $(P :_M r)$ is a β-absorbing primary submodule of M.*

Proof. Assume that P is a β-absorbing primary submodule of M and an element $r \in R\backslash(P : M)$. Let $a, b \in R$ and $m \in M$ be such that $abm \in (P :_M$

$r)$. Then $ab(rm) \in P$. Since P is a β-absorbing primary submodule of M, $ab+ab \in \sqrt{(P:M)}$ or $a(rm+rm) \in P$ or $b(rm+rm) \in P$. If $a(rm+rm) \in P$ or $b(rm+rm) \in P$, then $a(m+m) \in (P :_M r)$ or $b(m+m) \in (P :_M r)$. Assume that $ab+ab \in \sqrt{(P:M)}$. Then $(ab+ab)^t \in (P:M)$ for some positive integer t. Then $(ab+ab)^t M \subseteq P$. This implies that $r(ab+ab)^t m \in P$ for all $m \in M$. It means $(ab+ab)^t m \in (N :_M r)$ for all $m \in M$. Hence $(ab+ab)^t M \subseteq (N :_M r)$. We have that $(ab+ab) \in \sqrt{((N :_M r) : M)}$. Therefore, this completes the proof. □

Let $m \in M$ and N be a submodule of a left R-module M. We define $(N :_R m)$ by $(N :_R m) = \{r \in R \mid rm \in N\}$. Note that for each $m \in M$, $m \in N$ if and only if $(N :_R m) = R$.

Proposition 3.3. *If P is a β-absorbing primary submodule of M and $m \notin P$, then $(P :_R m)$ is a β-absorbing primary submodule of M.*

P r o o f. Assume that P is a β-absorbing primary submodule of M and $m \notin P$. Let $a, b, c \in R$ be such that $abc \in (P :_R m)$. Then $abcm \in P$. By assumption, $ab+ab \in \sqrt{(P:M)}$ or $a(cm+cm) \in P$ or $b(cm+cm) \in P$. Assume that $ab+ab \in \sqrt{(P:M)}$. Then $(ab+ab)^t M \subseteq P$ for some positive integer t. This implies that $(ab+ab)^t m \in P$. Hence $(ab+ab)^t \in (P :_R m)$. This shows that $ab+ab \in \sqrt{(P :_R m)}$. If $a(cm+cm) \in P$ or $b(cm+cm) \in P$, then $a(c+c) \in (P :_R m)$ or $b(c+c) \in (P :_R m)$. Therefore $(P :_R m)$ is a β-absorbing primary submodule of M. □

Proposition 3.4. *If P is a β-absorbing primary submodule of M, then $(P:M)$ is a β-absorbing primary ideal of R, and hence $\sqrt{(P:M)}$ is a β-absorbing primary ideal of R.*

P r o o f. Assume that P is a β-absorbing primary submodule of M. Let $a, b, c \in R$ be such that $abc \in (P:M)$. Suppose that $a(b+b) \notin (P:M)$ and $b(c+c) \notin \sqrt{(P:M)}$. Then $b(c+c) \notin (P:M)$. There are $m_1, m_2 \in M$ such that $a(b+b)m_1 \notin P$ and $b(c+c)m_2 \notin P$. Since $ac(bm_1+bm_2) \in P$ and P is a β-absorbing primary submodule of M, $ac+ac \in \sqrt{(P:M)}$ or $a(bm_1+bm_2+bm_1+bm_2) \in P$ or $c(bm_1+bm_2+bm_1+bm_2) \in P$. We consider two cases.

Case 1. $a(bm_1+bm_2+bm_1+bm_2) \in P$.

Since $a(b+b)m_1 \notin P$, $ab(m_2+m_2) \notin P$. Since $ac(bm_2) \in P$ and P is a β-absorbing primary submodule of M, $ac+ac \in \sqrt{(P:M)}$.

Case 2. $c(bm_1+bm_2+bm_1+bm_2) \in P$.

Since $b(c+c)m_2 \notin P$, $cb(m_1+m_1) \notin P$. Since $ac(bm_1) \in P$ and P is a β-absorbing primary submodule of M, $ac+ac \in \sqrt{(P:M)}$.

Hence $(P:M)$ is a β-absorbing primary ideal of R.

Moreover, it follows from Corollary 2.1 that if $(P:M)$ is a β-absorbing primary ideal of R, then $\sqrt{(P:M)}$ is a β-absorbing primary ideal of R. $\square$

Let R_1 and R_2 be commutative ring with identity and M_i be a unital R_i-module where $i=1,2$. Then $M_1 \times M_2$ is an $(R_1 \times R_2)$-module by the operation $(m_1,m_2)+(n_1,n_2)=(m_1+n_1,m_2+n_2)$ and $(r_1,r_2)(m_1,m_2)=(r_1m_1,r_2m_2)$. Next, some properties of β-absorbing primary submodules of $M_1 \times M_2$ are studied.

Proposition 3.5. *Let $R=R_1 \times R_2$ and $M=M_1 \times M_2$ and let N_2 be an R_2-submodule of M_2. Then $M_1 \times N_2$ is a β-absorbing primary submodule of $M_1 \times M_2$ if and only if N_2 is a β-absorbing primary submodule of M_2.*

Proof. ($\rightarrow$) Assume that $M_1 \times N_2$ is a β-absorbing primary submodule of $M_1 \times M_2$. Let $r,s \in R_2$ and $m \in M_2$ be such that $rsm \in N_2$. Then $(0,r)(0,s)(0,m)=(0,rsm) \in M_1 \times N_2$. Since $M_1 \times N_2$ is a β-absorbing primary submodule of $M_1 \times M_2$, $(0,rs+rs) \in \sqrt{(M_1 \times N_2 : M_1 \times M_2)}$ or $(0,r(m+m)) \in M_1 \times N_2$ or $(0,s(m+m)) \in M_1 \times N_2$. If $(0,r(m+m)) \in M_1 \times N_2$ or $(0,s(m+m)) \in M_1 \times N_2$, then $r(m+m) \in N_2$ or $s(m+m) \in N_2$. Assume that $(0,rs+rs) \in \sqrt{(M_1 \times N_2 : M_1 \times M_2)}$. Then we obtain that $(0,rs+rs)^t \in (M_1 \times N_2 : M_1 \times M_2)$ for some integers t. Hence $(0,(rs+rs)^t)(M_1 \times M_2) \subseteq M_1 \times N_2$. Let $x \in M_2$. Then $(0,(rs+rs)^t x)=(0,(rs+rs)^t)(0,x) \in M_1 \times N_2$. So $(rs+rs)^t x \in N_2$. This implies that $(rs+rs)^t M_2 \subseteq N_2$. Therefore $rs+rs \in \sqrt{(N_2 : M_2)}$. This complete the result N_2 is a β-absorbing primary submodule of M_2.

($\leftarrow$) Assume that N_2 is a β-absorbing primary submodule of M_2. Next, we let $(r_1,r_2),(s_1,s_2) \in R_1 \times R_2$ and $(m,n) \in M_1 \times M_2$ be such that $(r_1,r_2)(s_1,s_2)(m,n) \in M_1 \times N_2$. Then $r_2s_2n \in N_2$. Since N_2 is a β-absorbing primary submodule of M_2, $r_2s_2+r_2s_2 \in \sqrt{(N_2:M_2)}$ or $r_2(n+n) \in N_2$ or $s_2(n+n) \in N_2$. Hence $(r_1,r_2)(s_1,s_2)+(r_1,r_2)(s_1,s_2) \in \sqrt{(M_1 \times N_2 : M_1 \times M_2)}$ or $(r_1,r_2)((m,n)+(m,n)) \in M_1 \times N_2$ or $(s_1,s_2)((m,n)+(m,n)) \in M_1 \times N_2$. Therefore $M_1 \times N_2$ is a β-absorbing primary submodule of $M_1 \times M_2$. $\square$

Proposition 3.6. *Let $R=R_1 \times R_2$ and $M=M_1 \times M_2$ and let N_2 be an R_2-submodule of M_2. If $\{0\} \times N_2$ is a β-absorbing primary submodule of M, then N_2 is a β-absorbing primary submodule of M_2.*

Proof. Assume that $\{0\} \times N_2$ is a β-absorbing primary submodule of M. Let $r,s \in R_2$ and $m \in M_2$ be such that $rsm \in N_2$. Then $(0,r)(0,s)(0,m) \in \{0\} \times N_2$. Since $\{0\} \times N_2$ is a β-absorbing primary submodule of M, we have $(0, rs+rs) \in \sqrt{(\{0\} \times N_2 : M_1 \times M_2)}$ or $(0, r(m+m)) \in \{0\} \times N_2$ or $(0, s(m+m)) \in \{0\} \times N_2$. Hence $rs+rs \in \sqrt{(N_2 : M_2)}$ or $r(m+m) \in N_2$ or $s(m+m) \in N_2$. Therefore N_2 is a β-absorbing primary submodule of M_2. $\square$

We give Examples 5.5 in Section 5 to show the converse of Proposition 3.6 need not be true.

4. Characterization of β-absorbing primary submodules

For a subset H of a group G, the symbol $\beta(H) = \{\, h+h \,|\, h \in H \,\}$ and $\alpha(H) = \{\, h \,|\, h+h \in H \,\}$. If H is a subgroup, then $\beta(H) \subseteq H \subseteq \alpha(H)$. Moreover, if N is a submodule of a module M, then $\alpha(N)$ and $\beta(N)$ are submodules of M.

THEOREM 4.1. *Let $r, s \in R$ and P be a submodule of an R-module M. Then the following statements are equivalent,*

(1) P is a β-absorbing primary submodule of M.

(2) If $rs+rs \notin \sqrt{(P : M)}$, then $(P :_M rs) \subseteq \alpha((P :_M r)) \cup \alpha((P :_M s))$.

(3) If $rs+rs \notin \sqrt{(P : M)}$, then $(P :_M rs) \subseteq \alpha((P :_M r))$ or $(P :_M rs) \subseteq \alpha((P :_M s))$.

P r o o f. (1) $\to$ (2) Assume that P is a β-absorbing primary submodule of M and $rs+rs \notin \sqrt{(P : M)}$. Let $m \in (P :_M rs)$. Then $rsm \in P$. Since P is a β-absorbing primary submodule of M and $rs+rs \notin \sqrt{(P : M)}$, $r(m+m) \in P$ or $s(m+m) \in P$. Hence $m \in \alpha((P :_M r))$ or $m \in \alpha((P :_M s))$. Therefore $(P :_M rs) \subseteq \alpha((P :_M r)) \cup \alpha((P :_M s))$.

(2) $\to$ (3) This part is clear because $(P :_M rs), \alpha((P :_M r))$ and $\alpha((P :_M s))$ are submodules of M.

(3) $\to$ (1) Assume that (3) holds. To show that P is a β-absorbing primary submodule of M, let $r, s \in R$ and $m \in M$ be such that $rsm \in P$ and $rs+rs \notin \sqrt{(P : M)}$. Then $(P :_M rs) \subseteq \alpha((P :_M r))$ or $(P :_M rs) \subseteq \alpha((P :_M s))$. Since $rsm \in P$, $m \in (P :_M rs)$. Hence $m \in \alpha((P :_M r))$ or $m \in \alpha((P :_M s))$. Therefore $r(m+m) \in P$ or $s(m+m) \in P$. We complete the proof P is a β-absorbing primary submodule of M. $\square$

THEOREM 4.2. *A proper submodule P of an R-module M is β-absorbing primary if and only if for all submodules N of M and for all $r, s \in R$,*

if $rsN \subseteq P$, then $rs+rs \in \sqrt{(P : M)}$ or $r\beta(N) \subseteq P$ or $s\beta(N) \subseteq P$.

P r o o f. ($\to$) Assume that P is a β-absorbing primary submodule of M. Let $r, s \in R$ and N be a submodule of M such that $rsN \subseteq P$ and $r\beta(N) \nsubseteq P$ and $s\beta(N) \nsubseteq P$. There are elements $n_1, n_2 \in N$ such that $r(n_1+n_1) \notin P$

and $s(n_2+n_2) \notin P$. Since $rsN \subseteq P$, $rs(n_1+n_2) \in P$. Since P is β-absorbing primary, $rs+rs \in \sqrt{(P:M)}$ or $r(n_1+n_2+n_1+n_2) \in P$ or $s(n_1+n_2+n_1+n_2) \in P$. Suppose that $r(n_1+n_2+n_1+n_2) \in P$. Since $r(n_1+n_1) \notin P$, $r(n_2+n_2) \notin P$. Since $rsn_2 \in P$, $rs+rs \in \sqrt{(P:M)}$. Similarly, if $s(n_1+n_2+n_1+n_2) \in P$, then $rs+rs \in \sqrt{(P:M)}$.

($\leftarrow$) Let $r, s \in R$ and $m \in P$ be such that $rsm \in P$. Then $rs(Rm) = R(rsm) \subseteq RP \subseteq P$. This implies that $rs+rs \in \sqrt{(P:M)}$ or $r\beta(Rm) \subseteq P$ or $s\beta(Rm) \subseteq P$. Since $m \in Rm$, $m+m \in \beta(Rm)$. Hence $rs+rs \in \sqrt{(P:M)}$ or $r(m+m) \in P$ or $s(m+m) \in P$. Therefore P is a β-absorbing primary submodule of M. □

Compare the next results with ([4], Lemma 2.2 - 2.4 and Proprosition 2.5).

LEMMA 4.1. *Let I and J be ideals of R and P be a β-absorbing primary submodule of M. The following statements hold.*

(1) If $a \in R$ and $m \in M$ and $aIm \subseteq P$, then $a(m+m) \in P$ or $I(m+m) \subseteq P$ or $aI \subseteq \alpha(\sqrt{(P:M)})$.

(2) If $m \in M$ and $IJm \subseteq P$, then $I(m+m) \subseteq P$ or $J(m+m) \subseteq P$ or $IJ \subseteq \alpha(\sqrt{(P:M)})$.

(3) If N is a submodule of M and $IJN \subseteq P$, then $I(m+m) \subseteq P$ for all $m \in M$ or $J(m+m) \subseteq P$ for all $m \in M$ or $IJ \subseteq \alpha(\sqrt{(P:M)})$.

P r o o f. This proof is similar to Lemma 2.2 - 2.4 in [4]. □

THEOREM 4.3. *A proper submodule P of an R-module M is β-absorbing primary if and only if for all submodule N of M and for all ideals I and J of R,*

if $IJN \subseteq P$, then $IJ \subseteq \alpha(\sqrt{(P:M)})$ or $I\beta(N) \subseteq P$ or $J\beta(N) \subseteq P$.

P r o o f. This proof is similar to Proposition 2.5 in [4]. □

5. Examples of β-absorbing primary submodules of $\mathbb{Z}$

In [7], consider $\mathbb{Z}$ as an $\mathbb{Z}$-module, $n\mathbb{Z}$ is a 2-absorbing submodule of $\mathbb{Z}$ if and only if $n = 0, p, p^2$ or pq, where p and q are prime integers.

EXAMPLE 5.1. [4] Let $n \in \mathbb{Z}$. Then $n\mathbb{Z}$ is a β-absorbing submodule of an $\mathbb{Z}$-module $\mathbb{Z}$ if and only if $n = 0, 32, p, pq, 2^3p$ or $2pq$, where p and q are prime numbers.

The followings are examples of β-absorbing primary submodules but are not β-absorbing submodules.

Example 5.2. Consider $\mathbb{Z}$ as an $\mathbb{Z}$-module, we have $2^k\mathbb{Z}$ is a β-absorbing primary submodule of $\mathbb{Z}$ for all positive integers k. Moreover, if $k \geq 6$, then $2^k\mathbb{Z}$ is not a β-absorbing submodule of $\mathbb{Z}$.

P r o o f. Let k be a positive integer and $r, s, m \in \mathbb{Z}$ be such that $rsm \in 2^k\mathbb{Z}$. It is clear that $(2rs)^k\mathbb{Z} \subseteq 2^k\mathbb{Z}$. Then $(rs+rs)^k \in (2^k\mathbb{Z} : \mathbb{Z})$. This implies $rs + rs \in \sqrt{(2^k\mathbb{Z} : \mathbb{Z})}$. Therefore $2^k\mathbb{Z}$ is a β-absorbing primary submodule of $\mathbb{Z}$. Moreover, it follows from Example 5.1 that $2^k\mathbb{Z}$ is not a β-absorbing submodule of $\mathbb{Z}$ where $k \geq 6$. □

For a prime number p, mathematical induction can be used to prove that for all positive integers k and integers a and b, if $p^k \mid ab$ and $p^k \nmid b$, then $p \mid a$.

Example 5.3. Consider $\mathbb{Z}$ as an $\mathbb{Z}$-module, if p is a prime number, then $p^k\mathbb{Z}$ is a β-absorbing primary submodule of $\mathbb{Z}$ for all positive integers k.

P r o o f. Without loss of generality, assume that p is an odd prime and k is a positive integer with $k \geq 3$. Let $r, s, m \in \mathbb{Z}$ be such that $p^k \mid rsm$. If $p^k \mid m$, then $p^k \mid 2rm$ or $p^k \mid 2sm$. Next, if $p^k \nmid m$, then $p \mid rs$. Hence $p^k \mid (2rs)^k$. This means $p^k\mathbb{Z}$ is a β-absorbing primary submodule of $\mathbb{Z}$. □

Example 5.1 and Example 5.3 obtain $p^k\mathbb{Z}$ is not a β-absorbing submodule of $\mathbb{Z}$ but $p^k\mathbb{Z}$ is a β-absorbing primary submodule of $\mathbb{Z}$, where $k \geq 3$ and p is odd prime.

By mathematical induction, we have the fact that for a prime number p and for all positive integers k and integers a and b, if $p^k \mid ab$ and $p \nmid b$, then $p^k \mid a$.

Example 5.4. In $\mathbb{Z}$ as an $\mathbb{Z}$-module, if p is a prime number, then $2^2p^k\mathbb{Z}$ is a β-absorbing primary submodule of $\mathbb{Z}$ for all positive integers k.

P r o o f. Let p and k be positive integers. By Example 5.2, if $p = 2$, then $2^2p^k\mathbb{Z}$ is a β-absorbing primary submodule of $\mathbb{Z}$. Assume that p is odd prime number. Let $r, s, m \in \mathbb{Z}$ be such that $2^2p^k \mid rsm$. If $p \mid rs$, then $p^k \mid (rs)^k$. So $(2p)^{2k} \mid (2rs)^{2k}$. Hence $2^2p^k \mid (2rs)^{2k}$. Next, let $p \nmid rs$. Since $p^k \mid rsm$ and $p \nmid rs$, $p^k \mid m$. Since $2 \mid rsm$, $2 \mid r$ or $2 \mid s$ or $2 \mid m$. If $2 \mid m$ and $p^k \mid m$, then $2p^k \mid m$. So $2^2p^k \mid 2rm$. If $2 \mid r$ or $2 \mid s$, then $2p^k \mid rm$ or $2p^k \mid sm$. Thus $2^2p^k \mid 2rm$ or $2^2p^k \mid 2sm$. All of above conclude that if p is prime, then $2^2p^k\mathbb{Z}$ is a β-absorbing primary submodule of $\mathbb{Z}$ for all positive integer k. □

The next example is a counterexample for the converse of Proposition 3.6.

EXAMPLE 5.5. It follows from Example 5.4 that $36\mathbb{Z}$ is a β-absorbing primary submodule of $\mathbb{Z}$. To show that $\{0\} \times 36\mathbb{Z}$ is not a β-absorbing primary submodule of $\mathbb{Z} \times \mathbb{Z}$, let $r = s = (1,3)$ and $m = (0,4)$. Then $rsm = (1,3)(1,3)(0,4) = (0,36) \in \{0\}\times 36\mathbb{Z}$. We have $(rs+rs)^t = (2,18)^t \notin \{0\}\times 36\mathbb{Z}$ for all integers t and $r(m+m) = s(m+m) = (0,24) \notin \{0\} \times 36\mathbb{Z}$. Therefore $\{0\} \times 36\mathbb{Z}$ is not a β-absorbing primary submodule of $\mathbb{Z} \times \mathbb{Z}$.

The following example obtains that $P \times Q$ may be not β-absorbing primary submodule, even though P and Q are β-absorbing primary submodules.

EXAMPLE 5.6. We know every β-absorbing submodule of a module is a β-absorbing primary submodule. By Example 5.1, $30\mathbb{Z}$ is a β-absorbing primary submodule of $\mathbb{Z}$. It is clear that $36\mathbb{Z}$ is a β-absorbing primary submodule of $\mathbb{Z}$. Next, we will show that $30\mathbb{Z} \times 36\mathbb{Z}$ is not a β-absorbing primary submodule of $\mathbb{Z} \times \mathbb{Z}$. Let $r = (2,4), s = (3,3)$ and $m = (5,3)$. Then $rsm = (30,36) \in 30\mathbb{Z} \times 36\mathbb{Z}$. We see that $r(m+m) = (20,24) \notin 30\mathbb{Z} \times 36\mathbb{Z}$ and $s(m+m) = (30,18) \notin 30\mathbb{Z} \times 36\mathbb{Z}$. Since $30 \nmid 12^t$ for all integers t, $(rs+rs)^t = (12,24)^t \notin 30\mathbb{Z} \times 36\mathbb{Z}$ for all integers t. Thus $(12,24) = rs+rs \notin \sqrt{(30\mathbb{Z} \times 36\mathbb{Z} : \mathbb{Z} \times \mathbb{Z})}$. This shows that $30\mathbb{Z} \times 36\mathbb{Z}$ is not a β-absorbing primary submodule of $\mathbb{Z} \times \mathbb{Z}$.

EXAMPLE 5.7. In $\mathbb{Z}$ as an $\mathbb{Z}$-module, let p be a prime number. Then the following statements hold:

(1) $2^3p\mathbb{Z}$ and $2^3p^2\mathbb{Z}$ are β-absorbing primary submodules of $\mathbb{Z}$.
(2) If $l \geq 4$ and $p \neq 2$, then $2^lp^2\mathbb{Z}$ is not a β-absorbing primary submodule of $\mathbb{Z}$.

Proof. (1) and (2) are straightforward. □

Example 2.3 and Example 5.7 give the difference between a β-absorbing primary ideal of $\mathbb{Z}$ and a β-absorbing primary submodule of $\mathbb{Z}$ as follows.

EXAMPLE 5.8. In $\mathbb{Z}$ as a ring, $2^43^2\mathbb{Z}$ is a β-absorbing primary ideal of $\mathbb{Z}$ but $2^43^2\mathbb{Z}$ is not a β-absorbing primary submodule of $\mathbb{Z}$ as an $\mathbb{Z}$-module.

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