International Journal of Applied Mathematics

Volume 37 No. 3 2024, 333–339
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v37i3.4

EVALUATION OF SOME INTEGRALS INVOLVING HYPERBOLIC FUNCTIONS

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Abstract

Some integrals involving hyperbolic functions are evaluated through classical integral transforms. Several integrals that have not appeared in the classical tables of integrals are evaluated.

MSC 2020: 26D15, 30E20

Key Words and Phrases: integrals, hyperbolic functions, Laplace transform, sine and cosine transforms

1. Introduction

Consider a class of integrals of the form

$$A(a;U,V) = \int_0^\infty \frac{U(z)V(z)dz}{\cosh 2\pi z - \cos 2a\pi}, \quad 0 < a < 1, \tag{1}$$

where U is a continuous function on $(0,\infty)$, $V(z) = 1$, $\sinh 2\pi z$, $\cosh \pi z$ or $\sinh \pi z$. The solutions for some special cases of $A(a;U,V)$ are listed in the classical tables of integrals and series [2, 3, 4] either in closed form or infinite

Received: October 28, 2023

series form. For instance, if $V(z) = U(z) = 1$, then the solution of $A(a;1,1)$ in closed form [2, p. 373] is

$$\int_0^\infty \frac{dz}{\cosh 2\pi z - \cos 2a\pi} = \left(\frac{1}{2} - a\right) \csc 2\pi a. \tag{2}$$

If $U(z) = z^{2m}$, then the solution of $A(a; z^{2m}, 1)$ in infinite series form [2, p. 381] is

$$\int_0^\infty \frac{z^{2m} dz}{\cosh 2\pi z - \cos 2a\pi} = \frac{2(2m!)}{(2\pi)^{2m+1}} \frac{1}{\sin 2a\pi} \sum_{k=1}^\infty \frac{\sin 2ka\pi}{k^{2m+1}}, \quad a \neq \frac{1}{2}. \tag{3}$$

In this paper, the integrals of the form $A(a;U,V)$ are evaluated for various cases of U. These evaluations are done through classical integral transforms such as Laplace, Fourier sine and cosine transforms [1]. The solutions of these integrals are given in either closed-form or infinite series.

This paper is organized as follows. The main results are given in Section 2. The integrals of the form $A(a;U,V)$ are evaluated for various values of U in Section 3. The proof of main results are given in Section 4.

2. **Main results**

Suppose that $U(z)$ is a continuous function on $(0,\infty)$ and $t > 0$, then the Fourier sine transform and cosine transforms of $U(z)$ are

$$\mathcal{F}_\mathcal{S}(t;U) = \sqrt{\frac{2}{\pi}} \int_0^\infty U(z) \sin zt dz \quad \text{and} \quad \mathcal{F}_\mathcal{C}(t;U) = \sqrt{\frac{2}{\pi}} \int_0^\infty U(z) \cos zt dz.$$

The Laplace transform of $U(z)$ on $(0,\infty)$ is $\mathcal{L}(t;U) = \int_0^\infty U(z) e^{-zt} dz$, [1].

The following two theorems provide solutions of the integrals of the form $A(a;U,V)$ either in closed-form or infinite series.

Theorem 2.1. *Let* $f_c(t) = \dfrac{\mathcal{F}_\mathcal{C}(t;U)}{1-e^{-t}}$ *and* $f_s(t) = \dfrac{\mathcal{F}_\mathcal{S}(t;U)}{1-e^{-t}}$.

(i) *If* $\mathcal{L}(a;f_c) - \mathcal{L}(1-a;f_c)$ *exists, then*

$$\int_0^\infty \frac{U(z)dz}{\cosh 2\pi z - \cos 2a\pi} = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \dfrac{\mathcal{L}(a;f_c) - \mathcal{L}(1-a;f_c)}{\sin 2a\pi}, & a \neq \frac{1}{2} \\ \dfrac{1}{\pi} \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^\infty \frac{te^{-t/2}}{1-e^{-t}} \mathcal{F}_\mathcal{C}(t,U) dt, & a = \frac{1}{2}. \end{cases} \tag{4}$$

(ii) *If* $\mathcal{L}(a;f_s)$ *exists, then*

$$\int_0^\infty \frac{U(z) \sinh 2z\pi}{\cosh 2\pi z - \cos 2a\pi} dz = \frac{1}{\sqrt{2\pi}} \left(\mathcal{L}(a;f_s) + \mathcal{L}(1-a;f_s)\right). \tag{5}$$

THEOREM 2.2. *Let* $g_s(t) = \dfrac{\mathcal{F}_{\mathcal{S}}(t;U)}{1+e^{-t}}$ *and* $g_c(t) = \dfrac{\mathcal{F}_{\mathcal{C}}(t;U)}{1+e^{-t}}$.
(i) *If* $\mathcal{L}(a;g_s) - \mathcal{L}(1-a;g_s)$ *exists, then*

$$\int_0^\infty \frac{U(z)\sinh z\pi}{\cosh 2\pi z - \cos 2a\pi}dz = \begin{cases} \dfrac{1}{2\sqrt{2\pi}}\dfrac{\mathcal{L}(a;g_s) - \mathcal{L}(1-a;g_s)}{\cos a\pi}, & a \neq \frac{1}{2} \\ \dfrac{1}{\pi}\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^\infty \frac{te^{-t/2}}{1+e^{-t}}\mathcal{F}_{\mathcal{S}}(t,U)dt, & a = \frac{1}{2}. \end{cases} \tag{6}$$

(ii) *If* $\mathcal{L}(a;g_c)$ *exists, then*

$$\int_0^\infty \frac{U(z)\cosh z\pi}{\cosh 2\pi z - \cos 2a\pi}dz = \frac{1}{2\sqrt{2\pi}}\frac{\mathcal{L}(a;g_c) + \mathcal{L}(1-a;g_c)}{\sin a\pi}. \tag{7}$$

3. **Examples**

Several integrals of the form $A(a,U,V)$ can be evaluated using main results given in Section 2. Some of them are listed as examples in this section.

EXAMPLE 3.1. If $U(z) = \dfrac{1}{\mu^2+z^2}$ for $\mu > 0$, then $\mathcal{F}_{\mathcal{C}}(t,U) = \sqrt{\dfrac{\pi}{2}}\dfrac{e^{-\mu t}}{\mu}$. Using (4) and (7), we find for $a \neq \frac{1}{2}$

$$\int_0^\infty \frac{1}{\cosh 2\pi z - \cos 2a\pi}\frac{dz}{\mu^2+z^2} = \frac{1}{2\mu}\frac{\psi(\mu+1-a) - \psi(\mu+a)}{\sin 2a\pi},$$

$$\int_0^\infty \frac{1}{\cosh^2 \pi z}\frac{dz}{z^2+\mu^2} = \frac{1}{\pi\mu}\zeta(2,\mu+1/2)$$

and

$$\int_0^\infty \frac{\cosh z\pi}{\cosh 2\pi z - \cos 2a\pi}\frac{dz}{z^2+\mu^2} = \frac{1}{4\mu\sin a\pi}\left(\beta(a+\mu) + \beta(\mu+1-a)\right),$$

where $\psi(.)$ is digamma function, $\zeta(.,.)$ is the Hurwitz zeta function and $\beta(.)$ is incomplete beta function, [2].

EXAMPLE 3.2. Let $U(z) = z^{-3/2}$ and $\dfrac{z}{(\mu^2+z^2)^2}$ for $\mu > 0$. Using (5), we obtain

$$\int_0^\infty \frac{\sinh 2z\pi}{\cosh 2\pi z - \cos 2a\pi}\frac{dz}{z\sqrt{z}} = \frac{1}{\sqrt{2}}\left(\zeta\left(\frac{3}{2},a\right) + \zeta\left(\frac{3}{2},1-a\right)\right)$$

and

$$\int_0^\infty \frac{\sinh 2z\pi}{\cosh 2\pi z - \cos 2a\pi} \frac{zdz}{(z^2+\mu^2)^2} = \frac{1}{4\mu}\left(\zeta\left(2, a+\mu\right) + \zeta\left(2, 1-a+\mu\right)\right).$$

Example 3.3. If $U(z) = \dfrac{z}{\mu^2+z^2}$ for $\mu > 0$, then $\mathcal{F}_{\mathcal{S}}(t, U) = \sqrt{\dfrac{\pi}{2}} e^{-\mu t}$. Using (6), we find that

$$\int_0^\infty \frac{\sinh z\pi}{\cosh 2\pi z - \cos 2a\pi} \frac{z}{\mu^2+z^2} dz$$
$$= \begin{cases} \dfrac{1}{4} \dfrac{\beta(\mu+a) - \beta(\mu+1-a)}{\cos a\pi}, & a \neq \frac{1}{2} \\ \dfrac{2}{\pi} \displaystyle\sum_{k=0}^\infty \frac{(-1)^k}{(2\mu+2k+1)^2}, & a = \frac{1}{2} \end{cases}.$$

Example 3.4. Let $U(z) = \begin{cases} \dfrac{1}{\sqrt{1-z^2}}, & 0 < z < 1 \\ 0, & z > 1, \end{cases}$ in (4).

Using the solution of the integrals $\displaystyle\int_0^1 \frac{\cos \beta z}{\sqrt{1-z^2}} dz = \frac{\pi}{2} J_0(\beta)$, [2, p. 435], and $\displaystyle\int_0^\infty e^{-\alpha x} J_0(x) dx = \frac{1}{\sqrt{\alpha^2+1}}$ for $\alpha, \beta > 0$, [2, p. 694], we find that

$$\int_0^1 \frac{1}{\cosh 2\pi z - \cos 2a\pi} \frac{dz}{\sqrt{1-z^2}}$$
$$= \begin{cases} \dfrac{1}{2} \dfrac{1}{\sin 2\pi a} \sum_{k=0}^\infty \left(\frac{1}{\sqrt{(a+k)^2+1}} - \frac{1}{\sqrt{(1-a+k)^2+1}} \right), & a \neq \frac{1}{2}, \\ \dfrac{2}{\pi} \sum_{k=0}^\infty \frac{2k+1}{((2k+1)^2+4)^{\frac{3}{2}}}, & a = \frac{1}{2}, \end{cases}$$

where $J_0(.)$ is the Bessel function of first kind of order zero.

4. Proof of main results

The proof of the main results requires the following infinite series representations of $\pi \cot \pi x$ and $\pi \csc \pi x$ [2], for $x \neq 0, \pm 1, \pm 2, \ldots$:

$$\pi \cot \pi x = \sum_{k=-\infty}^{\infty} \frac{1}{x+k} \tag{8}$$

and

$$\pi \csc \pi x = \sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{x+k}. \tag{9}$$

4.1. **Proof of Theorem 2.1.** If $V(z) = 1$ and $a \neq \frac{1}{2}$, then the integral $A(a; U, 1)$ in (1) can be written by using elementary trigonometric addition identities as follows:

$$A(a; U, 1) = \frac{1}{2\sin 2a\pi} \int_0^\infty U(z) \left(\cot \pi(a - zi) + \cot \pi(a + zi)\right) dz \tag{10}$$

Applying the cotangent series found in (8), we find that

$$\begin{aligned} A(a; U, 1) &= \frac{1}{2\pi} \frac{1}{\sin 2a\pi} \int_0^\infty U(z) \sum_{k=-\infty}^{\infty} \left(\frac{1}{a - zi + k} + \frac{1}{a + zi + k} \right) \\ &= \frac{1}{2\pi} \frac{1}{\sin 2a\pi} \int_0^\infty \sum_{k=0}^{\infty} \left(\frac{1}{a - iz + k} - \frac{1}{1 - a + iz + k} \right. \\ &\qquad \left. + \frac{1}{a + iz + k} - \frac{1}{1 - a - iz + k} \right) U(z) dz. \end{aligned} \tag{11}$$

After simplification, (11) becomes

$$\begin{aligned} A(a; U, 1) &= \frac{1}{\pi} \frac{1}{\sin 2a\pi} \\ &\times \sum_{k=0}^{\infty} \int_0^\infty \left(\frac{a+k}{(a+k)^2 + z^2} - \frac{1-a+k}{(1-a+k)^2 + z^2} \right) U(z) dz. \end{aligned} \tag{12}$$

The Fourier cosine transform of $\frac{a+k}{(a+k)^2+z^2}$ is $\sqrt{\frac{\pi}{2}} e^{-(a+k)t}$ and applying convolution theorem of Fourier cosine transform [1] in (12), we obtain

$$A(a; U, 1) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sin 2a\pi} \sum_{k=0}^{\infty} \int_0^\infty \left(e^{-(a+k)t} - e^{-(1-a+k)t} \right) \mathcal{F}_{\mathcal{C}}(t, U) dt. \tag{13}$$

After simplification, yields that

$$A(a; U, 1) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sin 2a\pi} \int_0^\infty \frac{e^{-at} - e^{-(1-a)t}}{1 - e^{-t}} \mathcal{F}_{\mathcal{C}}(t, U) dt. \tag{14}$$

Using the definition of Laplace transform, (4) can be easily found for $a \neq \frac{1}{2}$. If $a = \frac{1}{2}$, then

$$\lim_{a \to \frac{1}{2}} A(a; U, 1) = \lim_{a \to \frac{1}{2}} \frac{1}{\sqrt{2\pi}} \frac{1}{\sin 2a\pi} \int_0^\infty \frac{e^{-at} - e^{-(1-a)t}}{1 - e^{-t}} \mathcal{F}_{\mathcal{C}}(t, U) dt. \tag{15}$$

Now, applying L'hôpital's rule [5] in (15), gives

$$A(1/2, U, 1) = \frac{1}{\pi}\frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{te^{-t/2}}{1-e^{-t}}\mathcal{F}_{\mathcal{C}}(t, U)dt. \tag{16}$$

Similarly, if $V(z) = \sinh 2z\pi$, then for $0 < a < 1$

$$A(a; U, \sinh 2z\pi) = \frac{1}{2i}\int_0^\infty U(z)\left(\cot\pi(a - zi) - \cot\pi(a + zi)\right)dz. \tag{17}$$

Using the cotangent series in (8), we arrive

$$\begin{aligned} &A(a; U, \sinh 2z\pi) \\ &= \frac{1}{\pi}\sum_{k=0}^\infty \int_0^\infty \left(\frac{z}{(a+k)^2 + z^2} + \frac{z}{(1-a+k)^2 + z^2}\right)U(z)dz. \end{aligned} \tag{18}$$

Using the convolution theorem of Fourier sine transform [1] and definition of Laplace transform, entry (5) can be easily obtained. This completes proof of the theorem.

4.2. **Proof of Theorem 2.2.** If $V(z) = \sinh z\pi$ and $0 < a < 1$, then the integral $A(a; U, \sinh z\pi)$ can be written as

$$\begin{aligned} &A(a; U, \sinh z\pi) \\ &= \frac{1}{4i\cos a\pi}\int_0^\infty U(z)\left(\csc\pi(a - zi) - \csc\pi(a + zi)\right)dz. \end{aligned} \tag{19}$$

Applying the cosecant series found in (9), gives

$$\begin{aligned} A(a; U, \sinh z\pi) &= \frac{1}{2\pi}\frac{1}{\cos a\pi} \\ &\times \sum_{k=0}^\infty \int_0^\infty (-1)^k\left(\frac{z}{(a+k)^2 + z^2} - \frac{z}{(1-a+k)^2 + z^2}\right)U(z)dz. \end{aligned}$$

Applying the convolution theorem of Fourier sine transform [1], we find

$$A(a; U, \sinh z\pi) = \frac{1}{2\sqrt{2\pi}}\frac{1}{\cos a\pi}\int_0^\infty \frac{e^{-at} - e^{-(1-a)t}}{1+e^{-t}}\mathcal{F}_{\mathcal{S}}(t, U)dt. \tag{20}$$

Now, using definition of Laplace transform, entry (6) can be easily found for $a \neq \frac{1}{2}$. If $a = \frac{1}{2}$, then using L'hôpital's rule in (20), this gives

$$\begin{aligned} \lim_{a\to\frac{1}{2}} A(a; U, \sinh z\pi) =& \frac{1}{2\sqrt{2\pi}}\lim_{a\to\frac{1}{2}}\frac{1}{\cos a\pi}\int_0^\infty \frac{e^{-at} - e^{-(1-a)t}}{1+e^{-t}}\mathcal{F}_{\mathcal{S}}(t, U)dt \\ =& \frac{1}{\pi}\frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{te^{-t/2}}{1+e^{-t}}\mathcal{F}_{\mathcal{S}}(t, U)dt. \end{aligned} \tag{21}$$

Similarly, if $V(z) = \cosh z\pi$ and $0 < a < 1$, then

$$A(a; U, \cosh z\pi) = \frac{1}{4 \sin a\pi} \int_0^\infty U(z) \left(\csc(a - zi)\pi + \csc(a + iz)\pi \right) dz. \quad (22)$$

Using cosecant series found in (9), gives

$$A(a; U, \cosh z\pi) = \frac{1}{2\pi \sin a\pi}$$
$$\times \int_0^\infty \sum_{k=0}^\infty (-1)^k \left(\frac{a+k}{(a+k)^2 + z^2} + \frac{1-a+k}{(1-a+k)^2 + z^2} \right) U(z) dz.$$

Using convolution theorem of Fourier cosine transform and definition of Laplace transform, entry (7) can be easily found. This completes proof of the theorem.

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