

**RANDOM CORRELATION MATRICES
AND ONE DECOMPOSITION OF THE
WISHART DISTRIBUTION**

Anna Nikolova ^{1,§}, Tzvetan Ignatov ², Bozhidar Dyakov ³

¹ Technical University of Varna

Department of Mathematics and Physics

Varna-9010, BULGARIA

² Sofia University “St. Kliment Ohridski”

Faculty of Economics and Business Administration

Sofia-1113, BULGARIA

³ Technical University of Varna

Department of Navigation

Varna-9010, BULGARIA

Abstract

In the proposed article, random matrices whose distribution coincides with the Wishart distribution are considered. Their elements, which are dependent, are represented as algebraic functions of independent random variables. The densities of the independent random variables are also indicated. A Wishart matrix construction is thus obtained. The present paper shows the relationship between the proposed Wishart matrix construction and correlations and partial correlations. The considered construction was also used to obtain a factorization of the determinant of a correlation matrix.

Math. Subject Classification: 62H10, 62H20

Key Words and Phrases: correlation, correlation matrix, covariance matrix, partial correlation, Wishart distribution

1. Introduction

The covariance matrix plays a central role in almost all classical multivariate statistics (see [1]), time series analysis (see [5]), spatial data analysis (see [8]), machine learning (see [7]).

As Pourahmadi (see [18]) points out, finding a constraint-free and statistically interpretable parameterization of a covariance matrix is an important problem in statistics. His solution has implications for covariance matrix estimation, especially for high-dimensional data in recent years, for which complex constraints guaranteeing positive definiteness would cost a lot of computer time.

A suitable parameterization would allow the representation of each element of the sample covariance matrix as an algebraic function of a new set of independent random variables. In this way, new properties of the distribution of covariance matrices, known as the Wishart distribution, can be established.

Given the complex nature of the covariance matrix and the positive definiteness restriction, it is convenient to decompose Σ into two components, capturing the “variance” by a diagonal matrix and the “dependence” by a square matrix of order p . A decomposition is ideal if its “dependence” component is unbounded and represents a statistically interpretable matrix. The most commonly used decompositions are the variance-correlation, spectral, and Cholesky decompositions, where their “dependence” components are correlation, orthogonal, and lower triangular matrices, respectively. While the entries of the first two matrices are always constrained, those of the latter are unconstrained.

In the proposed article, random matrices whose distribution coincides with the Wishart distribution are considered. Their elements, which are dependent, are represented as algebraic functions of independent random variables. The densities of the independent random variables are also indicated. A Wishart matrix construction is thus obtained. The present paper shows the relationship between the proposed Wishart matrix construction and correlations and partial correlations. The considered construction was also used to obtain a factorization of the determinant of a correlation matrix.

2. Wishart distribution

Consider the sequence of independently distributed Gaussian random vectors (columns) $\xi^i = (\xi_{1i}, \xi_{2i}, \dots, \xi_{pi})^\top$, $\xi^i \in \mathbb{R}^p$, $i = 1, 2, \dots, n$, with mean vectors $\mu^i = (\mu_{1i}, \mu_{2i}, \dots, \mu_{pi})^\top$ and the same covariance matrix Σ , s.t.

$\xi^i \in \mathcal{N}_p(\mu^i, \Sigma)$. We usually assume that n is greater than or equal to p . The joint distribution of the elements of the $p \times p$ random matrix

$$\mathbf{W} = \sum_{i=1}^n \xi^i (\xi^i)^\top,$$

is known as a Wishart non-central distribution, $\mathbf{W} \in \mathcal{W}_p(n, \Sigma, \mathbf{M})$, where $\mathbf{M} = (\mu^1, \mu^2, \dots, \mu^n)$ is a matrix of dimension $p \times n$. In the particular case, when $\mu^i = (0, 0, \dots, 0)^\top$, $i = 1, 2, \dots, n$, the distribution of the random matrix $\mathbf{W}$ is known as a central Wishart distribution, $\mathbf{W} \in \mathcal{W}_p(n, \Sigma)$. If the random column vectors ξ^i , $i = 1, 2, \dots, n$ form a matrix $\mathbf{L} = (\xi^1, \xi^2, \dots, \xi^n)$ of dimension $p \times n$, then the square matrix $\mathbf{W}$ with size p can be written as

$$\mathbf{W} = \mathbf{L}\mathbf{L}^\top,$$

and it is easy to deduce that with probability 1 the realizations of the random matrix $\mathbf{W}$ are positive definite matrices. If $n \geq p$ then the Wishart distribution $\mathcal{W}_p(n, \Sigma)$ has a density with respect to the Lebesgue measure in $\mathbb{R}^{\frac{p(p+1)}{2}}$. The joint density of $\mathbf{W} \in \mathcal{W}_p(n, \Sigma)$ has the form

$$f_{\mathbf{W}}(\mathbf{X}) \propto [\det(\mathbf{X})]^{\frac{n-p-1}{2}} e^{-\frac{1}{2} \text{tr}(\Sigma^{-1}\mathbf{X})} I_A(\mathbf{X}),$$

where $\mathbf{X}$ is a symmetric positive definite $p \times p$ matrix; $\det(\cdot)$ is a determinant; $\text{tr}(\cdot)$ is a trace functional; A is the set of all symmetric positive definite matrices; $I_A(\cdot)$ is the indicator function.

3. Decomposition of the Wishart distribution

It should be noted that the Wishart distribution $\mathcal{W}_p(n, \Sigma)$ is a distribution of a dependent random variables. However, the joint distribution of these dependent random variables can be represented through the joint distribution of certain functions of independent random variables (see [10],[11]). There the authors show that a random matrix $\mathbf{S}$ with a Wishart distribution $\mathcal{W}_p(n, \mathbf{I})$ coincides in distribution with a matrix $\mathbf{V}$ of the type (see [10])

$$\mathbf{V} = \begin{pmatrix} \tau_1 & \nu_{12}\sqrt{\tau_1\tau_2} & \dots & \nu_{1p}\sqrt{\tau_1\tau_p} \\ \nu_{12}\sqrt{\tau_1\tau_2} & \tau_2 & \dots & \nu_{2p}\sqrt{\tau_2\tau_p} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{1p}\sqrt{\tau_1\tau_p} & \nu_{2p}\sqrt{\tau_2\tau_p} & \dots & \tau_p \end{pmatrix}, \quad (1)$$

where $\tau_1, \tau_2, \dots, \tau_p$ are independent and identically χ^2 -distributed random variables with n degrees of freedom ($n \geq p$). The set of random variables $\{\nu_{ij}, 1 \leq i < j \leq p\}$ and the set $\{\tau_1, \tau_2, \dots, \tau_p\}$ are also independent. The random variables $\{\nu_{ij}, 1 \leq i < j \leq p\}$ have a joint density

$$f_{\{\nu_{ij}, 1 \leq i < j \leq p\}}(\mathbf{X}) = C_{n,p} (\det \mathbf{X})^{\frac{n-p-1}{2}} I_A(\mathbf{X}), \quad (2)$$

where:

- $C_{n,p}$ is a constant (see [19])

$$C_{n,p} = \frac{[\Gamma(\frac{n}{2})]^p}{\pi^{\frac{p(p-1)}{4}} \prod_{i=1}^p \Gamma(\frac{n-i+1}{2})}, \quad (3)$$

- the matrix

$$\mathbf{X} = \begin{pmatrix} 1 & x_{12} & \dots & x_{1p} \\ x_{12} & 1 & \dots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1p} & x_{2p} & \dots & 1 \end{pmatrix}$$

is real symmetric,

- I_A is the indicator function of the set A of all symmetric positive definite matrices whose elements in the principle diagonal are all unity.

In fact, $f_{\{\nu_{ij}, 1 \leq i < j \leq p\}}(\mathbf{X})$, which is a joint density of $\frac{p(p+1)}{2}$ random variables, can be considered defined not in $\mathbb{R}^{\frac{p(p+1)}{2}}$ but in the set of symmetric matrices of size p and denoted by

$$f_{\mathbf{M}}(\mathbf{X}) = C_{n,p} (\det \mathbf{X})^{\frac{n-p-1}{2}} I_A(\mathbf{X}), \quad (4)$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & \nu_{12} & \dots & \nu_{1p} \\ \nu_{12} & 1 & \dots & \nu_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{1p} & \nu_{2p} & \dots & 1 \end{pmatrix}. \quad (5)$$

3.1. Decomposition of the random matrix with Wishart distribution.

If a random matrix $\mathbf{S}$ has a Wishart distribution $\mathcal{W}_p(n, \mathbf{I})$, then it coincides in distribution with a matrix $\mathbf{V}$ of (1), but the matrix $\mathbf{V}$ in turn has the following decomposition

$$\mathbf{V} = \text{diag}(\sqrt{\tau_1}, \sqrt{\tau_2}, \dots, \sqrt{\tau_p}) \mathbf{M} \text{diag}(\sqrt{\tau_1}, \sqrt{\tau_2}, \dots, \sqrt{\tau_p}). \quad (6)$$

The distribution of the random symmetric matrix $\mathbf{M}$ with density $f_{\mathbf{M}}(\mathbf{X})$ has been denoted as $\Psi(p, n)$ (see [10]).

If we interpret the Wishart distribution $\mathcal{W}_p(n, \mathbf{\Sigma})$ as the distribution of the sample covariance matrix when the sample is taken from Gaussian multivariate distribution $\mathcal{N}_p(\mathbf{0}, \mathbf{\Sigma})$, then we can consider the distribution of $\mathbf{M}$ as the distribution of the sample correlation matrix of the same Gaussian distribution.

The distribution of the random symmetric matrix $\mathbf{M}$ with density $f_{\mathbf{M}}(\mathbf{X})$ has been denoted as $\Psi(p, n)$ (see [10]).

We can consider the representation (6) as a decomposition of the matrix $\mathbf{V}$. Like the decomposition (6), there are also several methods based on well-known matrix decomposition. For instance, Banfield & Raftery, Celeux & Govaert, Bensmail et al. use the spectral decomposition of the matrix (see [2], [6], [4]). Another approach is to use the Cholesky decomposition of the covariance matrix (see [15], [16], [17]). Following Barnard et al. (see [3]) we will refer to the decomposition in (6) as a separation strategy, as we separate out the standard deviations and the correlations.

Further, the sets of random variables $\{\nu_{ij}, 1 \leq i < j \leq p\}$ will be denoted as $\{\nu_{ij}(p), 1 \leq i < j \leq p\}$ and represented as symmetric matrices $\mathbf{M}(p)$, with units on the main diagonal, for $p = 2, 3, \dots$

3.2. Recurrence relation for the matrices $\mathbf{M}(p-1)$ and $\mathbf{M}(p)$.

A recurrence relation is found (see [10],[11]) for the matrices $\mathbf{M}(p-1) \in \Psi(p-1, n-1)$ and $\mathbf{M}(p) \in \Psi(p, n)$. There the authors proved the following theorem:

THEOREM 3.1. *Let $\eta_1, \eta_2, \dots, \eta_{p-1}$ be independent and identically distributed random variables with density:*

$$f_{\eta_i}(x) = C_n (1 - x^2)^{\frac{1}{2}(n-3)} I_{(-1,1)}(x), \quad (7)$$

where

$$C_n = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right)} \quad (8)$$

and $I_{(-1,1)}(x)$ is the indicator function of the interval $(-1, 1)$, $n \in \mathbb{Z}$, $n \geq p$. Let the random matrix $\mathbf{M}(p-1) \in \Psi(p-1, n-1)$ and the both sets $\eta_1, \eta_2, \dots, \eta_{p-1}$ and $\mathbf{M}(p-1)$ be independent. Then the random matrix $\mathbf{M}(p)$ with entries, obtained by the formulas

$$\begin{aligned} \nu_{1j}(p) &= \eta_{j-1}, & 2 \leq j \leq p \\ \nu_{ij}(p) &= \eta_{i-1} \eta_{j-1} \\ &+ \nu_{i-1, j-1}(p-1) \sqrt{1 - \eta_{i-1}^2} \sqrt{1 - \eta_{j-1}^2}, & 2 \leq i < j \leq p \end{aligned} \quad (9)$$

has a joint distribution $\Psi(p, n)$.

The statement in Theorem 3.1 gives a construction for obtaining random matrices $\mathbf{M}(p)$ with a distribution $\Psi(p, n)$.

3.3. Representation of the entries of the matrix $\mathbf{M}(p)$.

The next theorem gives recurrent relations between the dependent random variables $\nu_{ij}(p)$ (see [10], [11]). Let

$$\eta_i(p-m), \text{ for } p=2, 3, \dots, 0 \leq m \leq p-2, 1 \leq i \leq p-m-1 \quad (10)$$

be independent random variables with densities in the interval $(-1, 1)$

$$f_{\eta_i(p-m)}(x) = C_{n,m}(1-x^2)^{\frac{1}{2}(n-3-m)} I_{(-1,1)}(x), \quad (11)$$

where

$$C_{n,m} = \frac{\Gamma\left(\frac{n-m}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-m-1}{2}\right)} = \frac{1}{B\left(\frac{1}{2}, \frac{n-m-1}{2}\right)} \quad (12)$$

and $I_{(-1,1)}(x)$ is the indicator function of the interval $(-1, 1)$, $n \in \mathbb{Z}$, $n \geq p$. We consequently introduce the random variables:

$$\begin{aligned} \nu_{1j}(s) &= \eta_{j-1}(s), \quad 2 \leq j \leq s \\ \nu_{ij}(s) &= \eta_{i-1}(s) \eta_{j-1}(s) \\ &\quad + \nu_{i-1,j-1}(s-1) \sqrt{1-\eta_{i-1}^2(s)} \sqrt{1-\eta_{j-1}^2(s)}, \quad 2 \leq i < j \leq s, \end{aligned} \quad (13)$$

for $s = 2, 3, \dots, p$.

THEOREM 3.2. *The random matrix*

$$\mathbf{M}(p) = \begin{pmatrix} 1 & \nu_{12}(p) & \dots & \nu_{1p}(p) \\ \nu_{12}(p) & 1 & \dots & \nu_{2p}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{1p}(p) & \nu_{2p}(p) & \dots & 1 \end{pmatrix}, \quad (14)$$

with entries from (13), has distribution $\Psi(p, n)$ for $p = 2, 3, \dots$.

The dependent random variables $\nu_{ij}(p)$, $1 \leq i < j \leq p$ can be represented through independent $\eta_i(p-m)$ from (10) as follows (see [14]).

THEOREM 3.3. *For the entries $\nu_{ij}(p)$ of a random matrix $\mathbf{M}(p) \in \Psi(p, n)$ hold relations:*

$$\begin{aligned} \nu_{ij}(p) &= \eta_{i-1}(p) \eta_{j-1}(p) \\ &\quad + \sum_{m=2}^i \left\{ \eta_{i-m}(p-m+1) \eta_{j-m}(p-m+1) \right. \\ &\quad \times \left. \prod_{k=1}^{m-1} \sqrt{1-\eta_{i-k}^2(p-k+1)} \sqrt{1-\eta_{j-k}^2(p-k+1)} \right\}, \end{aligned} \quad (15)$$

for $1 \leq i < j \leq p$, where $\eta_0(p) = 1$, $\nu_{0j}(p) = 1$, $1 \leq j \leq p$, $p = 1, 2, \dots$.

It can be shown that the independent random variables $\eta_i(p-m)$ of (10) have in fact a Beta distribution in the interval $(-1, 1)$

$$\eta_i(p-m) \in \text{Beta}(\alpha_m, \alpha_m)$$

where $\alpha_m = \frac{n-m-1}{2}$.

The family of distributions $\mathcal{W}_p(n, \Sigma)$, Σ positive definite, can be generated from $\mathcal{W}_p(n, \mathbf{I})$ and $p \times p$ matrices for which $\mathbf{A}\mathbf{A}^\top = \Sigma$ i.e. $\mathbf{W} = \mathbf{A}\mathbf{S}\mathbf{A}^\top \in \mathcal{W}_p(n, \Sigma)$. In the role of matrix $\mathbf{A}$ can be taken $\Sigma^{\frac{1}{2}}$.

4. One factorization of the determinant of a matrix $\mathbf{M}(p) \in \Psi(p, n)$

If we have one realization of the random variables $\eta_i(p-m)$ and this realization is denoted by $\bar{\eta}_i(p-m)$, $p=2, 3, \dots$, $0 \leq m \leq p-2$, $1 \leq i \leq p-m-1$, then with the real numbers $\bar{\eta}_i(p-m)$, $p=2, 3, \dots$, $0 \leq m \leq p-2$, $1 \leq i \leq p-m-1$ we can get one realization $\bar{\mathbf{M}}(p)$ of the random matrix $\mathbf{M}(p)$. Indeed, with the numbers

$$\begin{aligned} \bar{\nu}_{ij}(p) &= \bar{\eta}_{i-1}(p)\bar{\eta}_{j-1}(p) \\ &+ \bar{\nu}_{i-1,j-1}(p-1)\sqrt{1-\bar{\eta}_{i-1}^2(p)}\sqrt{1-\bar{\eta}_{j-1}^2(p)}, \end{aligned} \quad (16)$$

$1 \leq i < j \leq p$, we get the non-random matrix $\bar{\mathbf{M}}(p)$. This matrix is a symmetric positive definite with all diagonal elements equal to 1. The positive definiteness of $\bar{\mathbf{M}}(p)$ follows from Lemma 1 (see [11]). We will reformulate this Lemma 1 in terms of real numbers $\bar{\eta}_i(p)$, $1 \leq i \leq p-1$ and $\bar{\nu}_{ij}(p-1)$, $2 \leq i < j \leq p-1$.

LEMMA 4.1. *If $\bar{\mathbf{M}}(p)$ is a non-random matrix with entries given in (16), then we have:*

$$\det(\bar{\mathbf{M}}(p)) = (1 - \bar{\eta}_1^2(p))(1 - \bar{\eta}_2^2(p)) \dots (1 - \bar{\eta}_{p-1}^2(p)) \det(\bar{\mathbf{M}}(p-1)).$$

By consequently applying the last equality to $\bar{\mathbf{M}}(p)$, $\bar{\mathbf{M}}(p-1)$, $\dots$, $\bar{\mathbf{M}}(2)$ we get:

$$\begin{aligned} \det(\bar{\mathbf{M}}(p)) &= (1 - \bar{\eta}_1^2(p))(1 - \bar{\eta}_2^2(p)) \dots (1 - \bar{\eta}_{p-1}^2(p)) \\ &\times (1 - \bar{\eta}_1^2(p-1))(1 - \bar{\eta}_2^2(p-1)) \dots (1 - \bar{\eta}_{p-2}^2(p-1)) \\ &\dots \\ &\times (1 - \bar{\eta}_1^2(3))(1 - \bar{\eta}_2^2(3)) \\ &\times (1 - \bar{\eta}_1^2(2)). \end{aligned}$$

Thus for $\det(\bar{\mathbf{M}}(p))$ we have:

THEOREM 4.1. *For the determinant of the matrix $\overline{\mathbf{M}}(p)$ the following factorization holds:*

$$\det(\overline{\mathbf{M}}(p)) = \prod_{m=0}^{p-2} \prod_{i=1}^{p-m-1} (1 - \overline{\eta}_i^2(p-m)). \quad (17)$$

Consequently,

$$\det(\overline{\mathbf{M}}(p)) > 0, \quad (18)$$

taking into account the inequalities $-1 < \overline{\eta}_i(p-m) < 1$,

$$p = 2, 3, \dots, \quad 0 \leq m \leq p-2, \quad 1 \leq i \leq p-m-1.$$

The last inequalities easily follow from definition of the density function of $\eta_i(p-m)$ in (11). The positive definiteness of $\overline{\mathbf{M}}(p)$ we can deduce using the Sylvester's criterion, i.e.

$$\det(\overline{\mathbf{M}}(k)) > 0, \quad \text{for } k = 1, 2, 3, \dots, p$$

(here $\det(\mathbf{M}(1)) = 1$).

5. A connection between partial correlations and one construction of the Wishart matrix

Here the relationship between the partial correlations and the presented Wishart matrix decomposition will be shown. We will use the same notations as in the paper of Joe (see [12]). Let $\mathbf{R} = (\rho_{ij})$ be a d -dimensional positive definite correlation matrix. The diagonal elements are all 1. $\mathbf{R}$ can be parameterized following Joe (see [12]) in terms of $\rho_{i,i+1}$ for $i = 1, \dots, d-1$ and the partial correlations $\rho_{j,j+k|j+1\dots j+k-1}$ for $j = 1, \dots, d-k$, $k = 2, \dots, d-1$. We could also write $\rho_{j,j+1} = \rho_{j,j+k|j+1\dots j+k-1}$ with $k = 1$ since the indices $j+1, \dots, j+k-1$ form an empty set if $k = 1$. In Anderson (see [1]) it can be found the definition of partial correlations

$$\rho_{ij|kL} = \frac{\rho_{ij|L} - \rho_{ik|L}\rho_{jk|L}}{\sqrt{(1 - \rho_{ik|L}^2)(1 - \rho_{jk|L}^2)}}, \quad (19)$$

where i, j, k are distinct integers in $1, 2, \dots, d$ and L is a subset of $\{1, \dots, d\} \setminus \{i, j, k\}$. From (19) we get

$$\rho_{ij|L} = \rho_{ik|L}\rho_{jk|L} + \rho_{ij|kL}\sqrt{(1 - \rho_{ik|L}^2)(1 - \rho_{jk|L}^2)}. \quad (20)$$

Compare formula (20) and formula (16) we see that the values of $\overline{\eta}_{i-1}(p)$, $\overline{\eta}_{j-1}(p)$ and $\overline{\nu}_{i-1,j-1}(p-1)$ have the same relations as values of $\rho_{ik|L}$, $\rho_{jk|L}$ and $\rho_{ij|kL}$. More precisely let $p = d$. It is easily to see that the non-random

matrix $\overline{\mathbf{M}}(p)$, has the properties of some correlation matrix $\mathbf{R} = (\rho_{ij})$ i.e. there exists a suitable correlation matrix $\mathbf{R} = (\rho_{ij})$ such that $\overline{\mathbf{M}}(p) \equiv (\rho_{ij})$.

From the construction (16) of $\overline{\mathbf{M}}(p)$ we have:

$$\overline{\mathbf{M}}(p) = \begin{pmatrix} 1 & \overline{\nu}_{12}(p) \dots \overline{\nu}_{1p}(p) \\ \overline{\nu}_{12}(p) & & \\ \vdots & & \Delta(p-1) \\ \overline{\nu}_{1p}(p) & & \end{pmatrix}. \quad (21)$$

The symmetric $(p-1)$ -dimensional square matrix $\Delta(p-1)$ has the form $((\Delta(p-1))_{ij})$, where

$$(\Delta(p-1))_{ij} = \begin{cases} \overline{\nu}_{1i}(p)\overline{\nu}_{1j}(p) + \overline{\nu}_{i-1,j-1}(p-1)\sqrt{1-\overline{\nu}_{1i}^2(p)}\sqrt{1-\overline{\nu}_{1j}^2(p)}, & 2 \leq i < j \leq p, \\ 1, & i = j. \end{cases}$$

In terms of $\mathbf{R}$ using (20) with $k=1$ and L empty set we get:

$$\mathbf{R} = \begin{pmatrix} 1 & \rho_{12} \dots \rho_{1p} \\ \rho_{12} & & \\ \vdots & & \gamma(p-1) \\ \rho_{1p} & & \end{pmatrix},$$

$$\gamma(p-1) \equiv (\gamma(p-1)_{ij}),$$

$$\gamma(p-1)_{ij} = \begin{cases} \rho_{1i}\rho_{1j} + \rho_{ij|1}\sqrt{(1-\rho_{1i}^2)(1-\rho_{1j}^2)}, & 2 \leq i < j \leq p, \\ 1, & i = j. \end{cases}$$

Consequently, the matrix of partial correlations

$$\begin{pmatrix} 1 & \rho_{23|1} & \dots & \rho_{2p|1} \\ & 1 & \dots & \rho_{3p|1} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}$$

coincides with the matrix $\overline{\mathbf{M}}(p-1)$. The same is true for

$$\begin{pmatrix} 1 & \rho_{34|12} & \rho_{35|12} & \dots & \rho_{3p|12} \\ & 1 & \rho_{45|12} & \dots & \rho_{4p|12} \\ & & 1 & & \vdots \\ & & & \ddots & \rho_{p-1,p|12} \\ & & & & 1 \end{pmatrix} = \overline{\mathbf{M}}(p-2)$$

and so on. Finally we have that the matrix

$$\begin{pmatrix} 1 & \rho_{k,k+1|12\dots k-1} & \rho_{k,k+2|12\dots k-1} & \cdots & \rho_{k,p|12\dots k-1} \\ & 1 & \rho_{k+1,k+2|12\dots k-1} & \cdots & \rho_{k+1,p|12\dots k-1} \\ & & 1 & & \vdots \\ & & & \ddots & \rho_{p-1,p|12\dots k-1} \\ & & & & 1 \end{pmatrix} \quad (22)$$

coincides with the matrix $\overline{\mathbf{M}}(p-k+1)$, $k = 1, 2, \dots, p-1$. For $k = p-1$ from (22) we get

$$\begin{pmatrix} 1 & \rho_{p-1,p|12\dots p-2} \\ \rho_{p-1,p|12\dots p-2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \overline{\nu}_{12}(2) \\ \overline{\nu}_{12}(2) & 1 \end{pmatrix} = \overline{\mathbf{M}}(2).$$

5.1. One factorization of the determinant of a correlation matrix.

The equality in Lemma 4.1 we can rewrite in terms of elements of $\overline{\mathbf{M}}(p)$ and $\mathbf{R} = (\rho_{ij})$:

$$\det(\overline{\mathbf{M}}(p)) = (1 - \overline{\eta}_1^2(p))(1 - \overline{\eta}_2^2(p)) \dots (1 - \overline{\eta}_{p-1}^2(p)) \det(\overline{\mathbf{M}}(p-1)),$$

$$\begin{aligned} \det(\mathbf{R}) &= (1 - \rho_{12}^2)(1 - \rho_{13}^2) \dots (1 - \rho_{1p}^2) \det \begin{pmatrix} 1 & \rho_{23|1} & \cdots & \rho_{2p|1} \\ & 1 & \cdots & \rho_{3p|1} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \\ &= (1 - \rho_{12}^2)(1 - \rho_{13}^2) \dots (1 - \rho_{1p}^2) (1 - \rho_{23|1}^2) \dots (1 - \rho_{2p|1}^2) \\ &\quad \times \det \begin{pmatrix} 1 & \rho_{34|12} & \rho_{35|12} & \cdots & \rho_{3p|12} \\ & 1 & \rho_{45|12} & \cdots & \rho_{4p|12} \\ & & 1 & & \vdots \\ & & & \ddots & \rho_{p-1,p|12} \\ & & & & 1 \end{pmatrix}. \end{aligned}$$

After finite number iteration we get:

$$\begin{aligned} \det(\mathbf{R}) &= (1 - \rho_{12}^2) (1 - \rho_{13}^2) \dots (1 - \rho_{1p}^2) \\ &\quad (1 - \rho_{23|1}^2) (1 - \rho_{24|1}^2) \dots (1 - \rho_{2p|1}^2) \\ &\quad (1 - \rho_{34|12}^2) (1 - \rho_{35|12}^2) \dots (1 - \rho_{3p|12}^2) \\ &\quad \dots \dots \dots \\ &\quad (1 - \rho_{p-1,p|12\dots p-2}^2). \end{aligned}$$

This result coincides with the result of Kurowicka and Cooke (see [13]) Theorem 3.2 in the case of canonical vine on p elements.

5.2. A joint density of the partial correlations.

A next application of our four theorems will be the proof of Theorem 5 of Joe (see [12]).

According to the consideration on page 2182 of Joe the joint distribution of $(\rho_{12}, \rho_{23}, \rho_{13|2})$ coincides with the distribution of independent not-identically Beta distributed respectively $B(\alpha_1, \alpha_1)$, $B(\alpha_1, \alpha_1)$, $B(\alpha_2, \alpha_2)$, where $\alpha_1 = \alpha_2 + \frac{1}{2}$. If we choose

$$\alpha_1 = \frac{n-1}{2}, \quad \alpha_2 = \frac{n-2}{2},$$

the joint distribution of $(\rho_{12}, \rho_{23}, \rho_{13|2})$ coincides with the distribution of $(\eta_1(3), \eta_2(3), \eta_1(2))$.

Indeed, from (11) the distribution of $\eta_i(p-m)$ is $Beta\left(\frac{n-m-1}{2}, \frac{n-m-1}{2}\right)$ and $\eta_i(p-m)$, $p = 2, 3, \dots$, $0 \leq m \leq p-2$, $1 \leq i \leq p-m-1$ are independent random variables. Consequently $(\eta_1(3), \eta_2(3), \eta_1(2))$ are independent beta distributed random variables respectively

$$Beta\left(\frac{n-1}{2}, \frac{n-1}{2}\right), Beta\left(\frac{n-1}{2}, \frac{n-1}{2}\right), Beta\left(\frac{n-2}{2}, \frac{n-2}{2}\right).$$

It is evident that

$$(\rho_{12}, \rho_{23}, \rho_{13|2}) \stackrel{d}{=} (\eta_1(3), \eta_2(3), \eta_1(2)).$$

On the other hand using (13) we have

$$(\nu_{12}(3), \nu_{13}(3), \nu_{12}(2)) \equiv (\eta_1(3), \eta_2(3), \eta_1(2)).$$

Finally, we get

$$\begin{aligned} (\nu_{12}(3), \nu_{13}(3), \nu_{12}(2)) &\stackrel{d}{=} (\eta_1(3), \eta_2(3), \eta_1(2)) \\ &\stackrel{d}{=} (\rho_{12}, \rho_{23}, \rho_{13|2}). \end{aligned} \quad (23)$$

Let us consider the joint distribution of $(\rho_{12}, \rho_{23}, \rho_{13})$. From (20) with $k = 2$ and L empty and (23) we get

$$\begin{aligned} (\rho_{12}, \rho_{23}, \rho_{13}) &\stackrel{d}{=} \left(\rho_{12}, \rho_{23}, \rho_{12}\rho_{23} + \rho_{13|2}\sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2)} \right) \\ &\stackrel{d}{=} \left(\eta_1(3), \eta_2(3), \eta_1(3)\eta_2(3) + \eta_1(2)\sqrt{(1-\eta_1^2(3))(1-\eta_2^2(3))} \right) \\ &\stackrel{d}{=} \left(\nu_{12}(3), \nu_{13}(3), \nu_{12}(3)\nu_{13}(3) + \nu_{12}(2)\sqrt{(1-\nu_{12}^2(3))(1-\nu_{13}^2(3))} \right) \\ &\stackrel{d}{=} (\nu_{12}(3), \nu_{13}(3), \nu_{23}(3)). \end{aligned}$$

It is easy to see by induction in the general case that:

THEOREM 5.1. *For the joint distributions of the random vectors below hold:*

$$\begin{aligned} & (\rho_{12}, \rho_{23}, \dots, \rho_{d-1,d}, \rho_{13|2}, \rho_{24|3}, \dots, \rho_{d-2,d|d-1}, \\ & \rho_{14|23}, \rho_{25|34}, \dots, \rho_{d-3,d|d-2,d-1}, \dots, \rho_{1,d|2,3,\dots,d-1}) \\ \underline{\underline{d}} \quad & (\eta_1(d), \dots, \eta_{d-1}(d), \eta_1(d-1), \eta_2(d-1), \dots, \eta_{d-2}(d-1), \\ & \eta_1(d-2), \dots, \eta_{d-3}(d-2), \dots, \eta_1(2)) \end{aligned}$$

and

$$\begin{aligned} & (\rho_{12}, \rho_{23}, \dots, \rho_{d-1,d}, \rho_{13}, \rho_{24}, \dots, \rho_{d-2,d}, \dots, \rho_{1d}) \\ \underline{\underline{d}} \quad & (\nu_{12}(d), \dots, \nu_{1d}(d), \nu_{23}(d), \dots, \nu_{2d}(d), \dots, \nu_{d-1,d}(d)). \end{aligned} \quad (24)$$

Now we will formulate a result of Harry Joe (see [12]):

THEOREM 5.2. *If $\alpha_k = \alpha_{d-1} + \frac{1}{2}(d-1-k)$, $k = 1, \dots, d-1$ and $\rho_{i,i+k|i+1\dots i+k-1}$ is Beta(α_k, α_k) on $(-1, 1)$ for $1 \leq i < i+k \leq d$, then the joint density of (ρ_{ij}) becomes*

$$c_d^{-1} \left(\det(r_{ij})_{1 \leq i, j \leq d} \right)^{\alpha_{d-1}-1}$$

on the set of correlation matrices (r_{ij}) , where the normalizing constant c_d is

$$\begin{aligned} & 2^{\sum_{k=1}^{d-1} (2\alpha_{d-1}-2+d-k)(d-k)} \\ & \times \prod_{k=1}^{d-1} \left[B \left(\alpha_{d-1} + \frac{1}{2}(d-1-k), \alpha_{d-1} + \frac{1}{2}(d-1-k) \right) \right]^{d-k}. \end{aligned}$$

If $\alpha_{d-1} = 1$ and $\alpha_k = \frac{1}{2}(d+1-k)$ for $k = 1, \dots, d-2$ leading to uniform joint density for $\{\rho_{ij}, i < j\}$, then normalizing constant is

$$\begin{aligned} c_d &= 2^{\sum_{k=1}^{d-1} (d-k)^2} \times \prod_{k=1}^{d-1} \left[B \left(\frac{1}{2}(d-k+1), \frac{1}{2}(d-k+1) \right) \right]^{d-k} \\ &= 2^{\sum_{k=1}^{d-1} k^2} \times \prod_{k=1}^{d-1} \left[B \left(\frac{1}{2}(k+1), \frac{1}{2}(k+1) \right) \right]^k, \end{aligned}$$

and the recursion is

$$c_d = c_{d-1} \times 2^{(d-1)^2} \times \left[B \left(\frac{d}{2}, \frac{d}{2} \right) \right]^{d-1}.$$

P r o o f. (Sketch of proof) Using (3), (4) and (5) for the joint density function of random variables

$$(\nu_{12}(d), \dots, \nu_{1d}(d), \nu_{23}(d), \dots, \nu_{2d}(d), \dots, \nu_{d-1,d}(d)),$$

we get:

$$f_{\mathbf{M}}((r_{ij})) = \frac{\left(\Gamma\left(\frac{n}{2}\right)\right)^{d-1}}{\pi^{\frac{d(d-1)}{4}} \prod_{i=1}^{d-1} \Gamma\left(\frac{n-i}{2}\right)} (\det(r_{ij}))^{\frac{n-d-1}{2}} I_A((r_{ij})). \quad (25)$$

On the other hand, substituting

$$\alpha_k = \frac{n-k}{2}, \quad k = 1, \dots, d-1,$$

we get the same expression

$$\alpha_k = \alpha_{d-1} + \frac{1}{2}(d-1-k), \quad k = 1, \dots, d-1$$

as in Joe (see [12]). With new values α_k , $k = 1, \dots, d-1$ the joint density in (25) becomes

$$f_{\mathbf{M}}((r_{ij})) = \frac{\left(\Gamma\left(\alpha_{d-1} + \frac{d-1}{2}\right)\right)^{d-1}}{\pi^{\frac{d(d-1)}{4}} \prod_{k=1}^{d-1} \Gamma(\alpha_k)} (\det(r_{ij}))^{\alpha_{d-1}-1} I_A((r_{ij})). \quad (26)$$

If we have randomized value $\rho_{i,i+k|i+1\dots i+k-1}$ as *Beta* (α_k, α_k) on $(-1, 1)$ for $1 \leq i < i+k \leq d$, and suppose independence of $\rho_{i,i+k|i+1\dots i+k-1}$ for $1 \leq i < i+k \leq d$, then taking into account (24) we get that the joint density of

$$(\rho_{12}, \rho_{23}, \dots, \rho_{d-1,d}, \rho_{13}, \rho_{24}, \dots, \rho_{d-2,d}, \dots, \rho_{1,d})$$

coincides with the joint density of

$$(\nu_{12}(d), \dots, \nu_{1d}(d), \nu_{23}(d), \dots, \nu_{2d}(d), \dots, \nu_{d-1,d}(d))$$

given in (25), i.e. the joint density of

$$(\rho_{12}, \rho_{23}, \dots, \rho_{d-1,d}, \rho_{13}, \rho_{24}, \dots, \rho_{d-2,d}, \dots, \rho_{1,d})$$

is

$$c_d^{-1} \left(\det(r_{ij})_{1 \leq i,j \leq d} \right)^{\alpha_{d-1}-1}$$

where

$$c_d^{-1} = \frac{\left(\Gamma\left(\alpha_{d-1} + \frac{d-1}{2}\right)\right)^{d-1}}{\pi^{\frac{d(d-1)}{4}} \prod_{k=1}^{d-1} \Gamma(\alpha_k)}.$$

To prove Theorem 5.2 we need to establish the identity

$$\frac{\left(\Gamma\left(\alpha_{d-1} + \frac{d-1}{2}\right)\right)^{d-1}}{\pi^{\frac{d(d-1)}{4}} \prod_{k=1}^{d-1} \Gamma(\alpha_k)} \quad (27)$$

$$= \left\{ 2^{\sum_{k=1}^{d-1} (2\alpha_{d-1}-2+d-k)(d-k)} \times \prod_{k=1}^{d-1} \left[B \left(\alpha_{d-1} + \frac{1}{2} (d-1-k), \alpha_{d-1} + \frac{1}{2} (d-1-k) \right) \right]^{d-k} \right\}^{-1}.$$

Using the properties of the Beta function

$$B(p, q) = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)}, \quad p > 0, q > 0$$

and $\Gamma(p) = (p-1) \Gamma(p-1)$ for $p > 1$, it is easy to verify (27). $\square$

COROLLARY 5.1. *Let $\mathbf{R} = (\rho_{ij})$ be a d -dimensional correlation matrix generated in Theorem 5.2. Let $\tau_1, \tau_2, \dots, \tau_d$ be independent identically distributed random variables with n degrees of freedom. Suppose the sets of random variables $\{\rho_{ij}, 1 \leq i < j \leq d\}$ and $\{\tau_1, \tau_2, \dots, \tau_d\}$ are independent. The random matrix*

$$\mathbf{A} = \text{diag}(\sqrt{\tau_1}, \sqrt{\tau_2}, \dots, \sqrt{\tau_d}) \mathbf{R} \text{diag}(\sqrt{\tau_1}, \sqrt{\tau_2}, \dots, \sqrt{\tau_d})$$

has the Wishart distribution $W_d(n, \mathbf{I})$ under the conditions of Theorem 5.2 and assuming $\alpha_{d-1} = \frac{s}{2}$, where s is positive integer. Here $n = s + d - 1$.

P r o o f. According to Theorem 5.1 we have that the joint density of

$$(\rho_{12}, \rho_{23}, \dots, \rho_{d-1,d}, \rho_{13}, \rho_{24}, \dots, \rho_{d-2,d}, \dots, \rho_{1d})$$

coincides with the joint density of

$$(\nu_{12}(d), \dots, \nu_{1d}(d), \nu_{23}(d), \dots, \nu_{2d}(d), \dots, \nu_{d-1,d}(d))$$

given in (25). From Theorem 3.2 follows that the sequence

$$(\rho_{12}, \rho_{23}, \dots, \rho_{d-1,d}, \rho_{13}, \rho_{24}, \dots, \rho_{d-2,d}, \dots, \rho_{1,d})$$

has a distribution $\Psi(d, n)$. Using equality

$$\mathbf{V} = \text{diag}(\sqrt{\tau_1}, \sqrt{\tau_2}, \dots, \sqrt{\tau_d}) \mathbf{M} \text{diag}(\sqrt{\tau_1}, \sqrt{\tau_2}, \dots, \sqrt{\tau_d})$$

and (1), (6) we can conclude that the matrix $\mathbf{A}$ has a Wishart distribution $W_d(n, \mathbf{I})$. $\square$

From the above considerations it is easy to obtain the Joe's result (a) on page 2183 (see [12]) about the volume of the set of the d -dimensional positive definite correlation matrices in $\binom{d}{2}$ dimensional space.

COROLLARY 5.2. *The Lesbegue volume of the set of $\binom{d}{2}$ correlation matrices in $\binom{d}{2}$ -dimensional space is:*

$$\left\{ \begin{array}{ll} \pi^{\frac{d^2-1}{4}} \frac{\prod_{m=1}^{(d-1)/2} \Gamma(2m)}{2^{\frac{(d-1)^2}{4}} \Gamma^{d-1}\left(\frac{d+1}{2}\right)}, & \text{if } d \text{ is odd,} \\ \pi^{\frac{d(d-2)}{4}} \frac{2^{\frac{3d^2-4d}{4}} \Gamma^d\left(\frac{d}{2}\right) \prod_{m=1}^{(d-2)/2} \Gamma(2m)}{\Gamma^{d-1}(d)}, & \text{if } d \text{ is even.} \end{array} \right. \quad (28)$$

P r o o f. Let $\alpha_{d-1} = 1$, then the joint density in (26) becomes

$$\frac{(\Gamma(1 + \frac{d-1}{2}))^{d-1}}{\pi^{\frac{d(d-1)}{4}} \prod_{k=1}^{d-1} \Gamma(1 + \frac{1}{2}(d-1-k))} (\det(r_{ij}))^0 I_A((r_{ij})). \quad (29)$$

Thus we have a uniform distribution over the $d \times d$ correlation matrices depending on $\binom{d}{2}$ values r_{ij} , $1 \leq i < j \leq d$. Consequently the volume V of all correlation matrices in $\binom{d}{2}$ Euclidian space coincides with the reciprocal value of the coefficient in (29), i.e.

$$V = \frac{\pi^{\frac{d(d-1)}{4}} \prod_{k=1}^{d-1} \Gamma\left(\frac{d-k+1}{2}\right)}{(\Gamma(\frac{d+1}{2}))^{d-1}}.$$

It is easy to find that V is equal to (28). $\square$

Example. When $d = 3$ the volume of correlation matrix

$$\mathbf{R} = \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix},$$

where

$$-1 < x < 1, \quad -1 < y < 1, \quad -1 < z < 1, \quad (30)$$

and

$$\det(\mathbf{R}) > 0, \quad \text{i.e.} \quad 1 - x^2 - y^2 - z^2 + 2xyz > 0$$

is equal to $\frac{\pi^2}{2}$. If we compare the volume of the figure depicted by the points (x, y, z) satisfying (30), and the volume of a ball inscribed in a cube $(-1, 1) \times (-1, 1) \times (-1, 1)$, we get $\frac{3\pi}{8}$, that is greater than 1, i.e. the volume of our figure is bigger then the volume of the inscribed ball.

Let us consider the Joe's construction of random correlation matrices (see [12], Theorem 5.2). On the constant α_{d-1} we have only the constraint $\alpha_{d-1} > 0$. If we put the condition $\alpha_{d-1} = \frac{l}{2}$ where l is a positive integer we get:

COROLLARY 5.3. Let $\mathbf{R} = (\rho_{ij})$ be a d -dimensional correlation matrix. Under the conditions of Theorem 5.2 and supposing α_{d-1} of the form

$$\alpha_{d-1} = \frac{l}{2}, \quad (31)$$

where l is a positive integer then $\mathbf{R} \stackrel{d}{=} \mathbf{M}$, where $\mathbf{M}$ is the matrix in (5).

P r o o f. Using Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 5.1, Theorem 5.2 it is easy to conclude that conditions (31) and

$$\alpha_k = \alpha_{d-1} + \frac{1}{2}(d-1-k), \quad k = 1, \dots, d-1$$

are sufficient to get the coincidence of the joint distribution of (ρ_{ij}) and the joint distribution of $\mathbf{M}$ from (5). $\square$

Finally, we will give an example connected with the above consideration.

Example. Let $\xi_{11}, \xi_{21}, \xi_{12}, \xi_{22}, \xi_{13}, \xi_{23}$ be independent $\mathcal{N}(0, 1)$. Consider the correlation matrix of the random vectors $\zeta_1 = (\xi_{11}, \xi_{12}, \xi_{13})$, $\zeta_2 = (\xi_{21}, \xi_{22}, \xi_{23})$. Then the correlation matrix of ζ_1 and ζ_2 is

$$\begin{aligned} \mathbf{R} &= \begin{pmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\xi_{11}\xi_{21} + \xi_{12}\xi_{22} + \xi_{13}\xi_{23}}{\sqrt{\xi_{11}^2 + \xi_{12}^2 + \xi_{13}^2} \sqrt{\xi_{21}^2 + \xi_{22}^2 + \xi_{23}^2}} \\ \cdot & 1 \end{pmatrix} \\ &\equiv \mathbf{M} = \begin{pmatrix} 1 & \nu_{12} \\ \nu_{12} & 1 \end{pmatrix}. \end{aligned}$$

According to equality (4) we get

$$\begin{aligned} &f_{\mathbf{M}} \left(\begin{pmatrix} 1 & x_{12} \\ x_{12} & 1 \end{pmatrix} \right) \\ &= \begin{cases} C_{3,2} \left(\det \begin{pmatrix} 1 & x_{12} \\ x_{12} & 1 \end{pmatrix} \right)^{\frac{3-2-1}{2}}, & -1 \leq x_{12} \leq 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (32)$$

In this case the set of all positive definite matrices coincide with the set of all random variables with values in $(-1, 1)$. On the other hand the random variable ρ_{12} has uniform distribution, consequently ρ_{12} is uniformly distributed on the interval $(-1, 1)$.

6. Conclusions

In the article, a decomposition of a matrix with a Wishart distribution $\mathcal{W}_p(n, \mathbf{\Sigma})$ by independent random variables on which the distributions are found is considered. The relation of this decomposition to correlations and

partial correlations is shown. A similar decomposition can also be applied to a generalized Wishart distribution (with non-integer degrees of freedom).

The proposed recurrence relation between matrices with a Wishart distribution can be used to generate matrices with this distribution.

Acknowledgments. We would like to acknowledge Prof. Tzvetan Ignatov's collaboration, who unfortunately passed away shortly after the completion of the work on this paper, which is also related to previous research results with him.

References

- [1] T.W. Anderson, *An Introduction to Multivariate Statistical Analysis*, 3rd ed., Wiley, New York (2003).
- [2] J. Banfield, A. Raftery, Model-based Gaussian and non-Gaussian clustering, *Biometrics*, **49** (1993), 803-821.
- [3] J. Barnard, R. McCulloch, Xiao-Li Meng, Modeling covariance matrices in terms of standard deviations and correlations, with application to shrinkage, *Statistica Sinica*, **10** (2000), 1281-1311.
- [4] H. Bensmail, G. Celeux, A. Raftery, C. Robert, Inference in model-based cluster analysis, *Statist. Comput.*, **7** (1997), 1-10.
- [5] G.E.P. Box, G.M. Jenkins and G. Reinsel, *Time Series Analysis-Forecasting and Control*, Revised 3rd ed., Prentice Hall, NJ (1994).
- [6] G. Celeux, G. Govaert, Gaussian parsimonious clustering models, *J. Pattern Recognition Soc.*, **28** (1995), 781-793.
- [7] R. Couillet, Z. Liao, *Random Matrix Methods for Machine Learning*, Cambridge University Press, (2022).
- [8] N.A.C. Cressie, *Statistics for Spatial Data*, Revised ed., Wiley, New York (1993).
- [9] N. Giri, *Multivariate Statistical Analysis*, Marcel Dekker, New York (2004).
- [10] Tz. Ignatov, A. Nikolova, About Wishart's distribution, *Annuaire de l'Universite de Sofia "St. Kliment Ohridski", Faculty of Economics and Business Administration*, **3** (2004), 79-94.
- [11] Tz. Ignatov, A. Nikolova, A decomposition of the Wishart distribution, *Annuaire de l'Universite de Sofia "St. Kliment Ohridski", Faculty of Economics and Business Administration*, **4** (2005), 131-147.
- [12] H. Joe, Generating random correlation matrices based on partial correlations, *J. Multivariate Anal.*, **97** (2006), 2177-2189.
- [13] D. Kurowicka, R. Cooke, Completion problem with partial correlation vines, *Linear Algebra Appl.*, **418**, No 1 (2006), 188-200.
- [14] A. Nikolova, Evaluation the Moments of the Wishart Distribution, *Adv. Stud. Contemp. Math.*, **15**, No 2 (2007), 171-186.

- [15] J. Pinheiro, D. Bates, Unconstrained parametrizations for variance-covariance matrices, *Statist. Comput.*, **6** (1996), 289-296.
- [16] M. Pourahmadi, Joint mean-covariance models with applications to longitudinal data: Unconstrained parameterization, *Biometrika*, **86** (1999), 677-690.
- [17] M. Pourahmadi, Maximum likelihood estimation of generalized linear models for multivariate normal covariance matrix, *Biometrika*, **87** (1999), 425-435.
- [18] M. Pourahmadi, Covariance estimation: The GLM and regularization perspectives. *Statist. Sci.*, **26**, No 3 (2011), 369-387.
- [19] E. Veleva, Joint densities of correlation coefficients for samples from multivariate standard normal distribution, *Pliska Stud. Math. Bulgar.*, **18** (2007), 379-386.