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GENERALIZATION OF A VARIANCE-GAMMA-DRIVEN INTEREST RATE DERIVATIVE

A. M. Udoye [1,§], M. F. Salami [2]

[1] Department of Mathematics
Federal University Oye-Ekiti
Ekiti, P.M.B. 373, NIGERIA

[2] Department of Mathematics
University of Arkansas-Pulaski Technical College
North Little Rock, Arkansas, AR 72118, USA

Abstract

We derive a generalized Vasicek short rate model under a variance gamma Lévy process by applying Itô lemma, and use the derived model to obtain a generalized interest rate derivative motivated by the variance gamma process.

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Key Words and Phrases: interest rates, Lévy process, variance gamma process, zero-coupon bond

1. Introduction

The Lévy processes have contributed to better modelling of phenomenon in different fields (Wei [17], Udoye & Ekhaguere [13], Udoye et al [14]). A variance gamma (VG) process is a type of Lévy process that was launched by Madan and Seneta [7] in order to take care of unexpected occurrences which can lead to inadequate modelling of a given phenomenon. The VG process

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is acquired by changing the time of an arithmetic Brownian motion using a gamma process. Seneta [10] and Rathgeber [8] highlighted certain aspects of the process. Since its introduction, it has been applied in different fields which include mathematical finance (Bayazit & Nolder [3], Seneta [11], and Udoye et al [12], [15]), and engineering (Salem [9]), etc. Moreover, Hoyyi [6] discussed the process under Monte-Carlo simulation with closed form method for European call option price valuation. Aguilar [1] discussed different pricing tools of the process, while Azmoodeh et al. [2] emphasized its optimal approximation under second Wiener chaos. Furthermore, Bee et al. [4] highlighted likelihood risk estimates for models of the process. Moreover, Fischer [5] discussed update on distribution theory of the process. This work generalizes the work of Udoye and Ekhaguere [13] who derived an extended Vasicek model under a VG process and used the derived expression to obtain an interest rate derivative driven by the VG process.

In what follows, Section 2 considers important definitions and tools needed in deriving our result. Section 3 concerns the results, while Section 4 concludes the work.

2. **Mathematical Notion**

Definition 2.1. The dynamics of a Vasisek model [16] of an interest rate is given by

$$dr_t = \varpi(\beta - r_t)dt + \sigma dX_t, \tag{1}$$

where ϖ, β and σ denotes speediness of mean reversal, long-standing mean rate and volatility of the interest rate, while X_t denotes a Lévy process.

Definition 2.2. The dynamics of an interest rate derivative called *zero-coupon bond price* $P = P_t$ is given by

$$dP = r_t P dt + \sigma P dX_t, \tag{2}$$

where σ is the volatility of the interest rate while r_t is the interest rate at time t.

Lemma 2.1. (Itô formula for Lévy processes)
Let $X = X_t, t \geq 0$ be an n-dimensional Lévy process with characteristic triplet $(\mathbf{b}, \sigma^2, \nu)$ and a function $f \in C^{1,2}$ being a map $[0,T] \times \mathbb{R}^n \to \mathbb{R}$. Then,

$$f(t,X_t)=f(0,0)+\int_0^t \frac{\partial f}{\partial s}(s,X_s)ds+\int_0^t \sum_{1\le i\le n}\frac{\partial f}{\partial x_i}(s,X_{s-})\mathbf{b}_i(t)dX_s^i$$
$$+0.5\int_0^t \sum_{1\le i,j\le n}\sigma_{ij}^2\frac{\partial^2 f}{\partial x_i\partial x_j}(s,X_s)ds+\sum_{0\le s\le t}^{\Delta X_s\neq 0}\Big[f(s,X_{s-}$$
$$+\Delta X_s)-f(s,X_{s-})-\sum_{1\le i\le n}\Delta X_s^i\frac{\partial f}{\partial x_i}(s,X_{s-})\Big],$$

where $\Delta X_s = X_{s_+}-X_{s_-}$.

3. **Results**

To obtain our results, let an extended VG process be given by

$$X_t=\omega\lambda t+\theta[\lambda G_t+\rho t]+\widetilde{\sigma}\sqrt{\lambda G(t)+\rho t}Z, \tag{3}$$

where $\omega=\dfrac{1}{\kappa}\ln(1-\theta\kappa-\dfrac{1}{2}\widetilde{\sigma}^2\kappa)$, κ takes care of variance of the gamma process while θ and $\widetilde{\sigma}$ denote parameter for skewness and volatility, respectively, of the arithmetic Brownian motion used to obtain the VG process. $G=G(t)$ and $Z=Z(t)$ denote a gamma random variable and a Gaussian random variable, respectively. λ and ρ are deterministic parameters such that $0\le\lambda,\rho\le 1$.

THEOREM 3.1. *The generalized Vasicek model driven by a VG process is given by*

$$\begin{aligned}r_t=r_0e^{-\varpi t}+\beta(1-e^{-\varpi t})+\sigma\Bigg(&\frac{\omega\lambda}{\varpi}(1-e^{-\varpi t})\\ &+\frac{\theta\rho}{\varpi}(1-e^{-\varpi t})+\theta\lambda\sum_{0\le s\le t}\Delta G(s)e^{-\varpi(t-s)}\\ &+\widetilde{\sigma}\sum_{0\le s\le t}\Delta\sqrt{\lambda G(s)+\rho s}\,e^{-\varpi(t-s)}Z\Bigg),\end{aligned}\tag{4}$$

where $\Delta X_s=X_{s_+}-X_{s_-}$.*where* $\Delta G(s)=G(s_+)-G(s_-)$.

P r o o f. Applying Itó's lemma on equation (1) and evaluating, it follows that

$$r_t=r_0e^{-\varpi t}+\beta(1-e^{-\varpi t})+\sigma\int_0^t e^{-\varpi(t-s)}dX_s.$$

From equation (3),

$$dX_t = \omega\lambda dt + \theta\lambda\Delta G_t + \theta\rho dt + \widetilde{\sigma}\Delta\sqrt{\lambda G(t) + \rho t}Z. \tag{5}$$

Thus,

$$\int_0^t e^{-\varpi(t-s)}dX_s = \omega\lambda\int_0^t e^{-\varpi(t-s)}ds + \theta\lambda\sum_{0\leq s\leq t}\Delta G(s)e^{-\varpi(t-s)}$$

$$+\theta\rho\int_0^t e^{-\varpi(t-s)}ds + \widetilde{\sigma}\sum_{0\leq s\leq t}\Delta\sqrt{\lambda G(s)+\rho s}e^{-\varpi(t-s)}Z$$

$$= \frac{\omega\lambda}{\varpi}(1-e^{-\varpi t}) + \theta\rho\left[\frac{e^{-\varpi(t-s)}}{\varpi}\right]_0^t + \theta\lambda\sum_{0\leq s\leq t}\Delta G(s)$$

$$\times\ e^{-\varpi(t-s)} + \widetilde{\sigma}\sum_{0\leq s\leq t}\Delta\sqrt{\lambda G(s)+\rho s}e^{-\varpi(t-s)}Z.$$

Hence, the result follows. □

Theorem 3.2. *The generalized zero-coupon bond price driven by a VG process is given by*

$$P(t,T) = \exp\Bigg(-\Bigg(\frac{-r_0}{\varpi}(e^{-\varpi T} - e^{-\varpi t}) + \beta[T-t] + \frac{\beta}{\varpi}(e^{-\varpi T}$$

$$- e^{-\varpi t}) + \frac{\sigma\omega\lambda}{\varpi}[T-t] + \frac{\sigma\omega\lambda}{\varpi}\big(\frac{1}{\varpi}(e^{-\varpi T} - e^{-\varpi t})\big)$$

$$+ \frac{\sigma\theta\rho}{\varpi}[T-t] + \frac{\sigma\theta\rho}{\varpi}\big(\frac{1}{\varpi}(e^{-\varpi T} - e^{-\varpi t})\big) + \sigma\theta\lambda$$

$$\times\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta G(s)e^{-\varpi(u-s)} + \sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta \tag{6}$$

$$\times\ \sqrt{\lambda G(s)+\rho s}e^{-\varpi(u-s)}Z + \sigma\omega\lambda[T-t] + \sigma\theta\rho[T-t]$$

$$+\sigma\theta\lambda\sum_{t\leq u\leq T}\Delta G(u) + \sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\Delta\sqrt{\lambda G(u)+\rho u}Z$$

$$-\frac{\sigma^2}{2}\sum_{t\leq u\leq T}(\lambda\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{\lambda G(u)+\rho u}Z)^2\Bigg)\Bigg).$$

P r o o f. From the dynamics of a zero-coupon bond price given by equation (2),

$$dP = r_t P dt + \sigma P dX_t.$$

From Itô's lemma, $F(t,x)=\ln x$, $\dfrac{\partial F}{\partial t}=0$, $\dfrac{\partial F}{\partial x}=\dfrac{1}{x}$. Thus,

$$\begin{aligned} d\ln P &= (r_t dt+\sigma dX_t)-\frac{1}{2}\sigma^2(dX_t)^2 \\ &= r_t dt+\sigma dX_t-\frac{1}{2}\sigma^2\langle dX_t,dX_t\rangle, \end{aligned}$$

where dX_t is given by equation (5). Moreover,

$$(dX_t)^2=(\theta\lambda\Delta G_t+\widetilde{\sigma}\Delta\sqrt{\lambda G(t)+\rho t}Z)^2.$$

This implies that

$$\begin{aligned} d\ln P = r_t dt &+\sigma(\omega\lambda dt+\theta\lambda\Delta G_t+\theta\rho dt+\widetilde{\sigma}\Delta\sqrt{\lambda G(t)+\rho t}Z) \\ &-\frac{1}{2}\sigma^2(\theta\lambda\Delta G_t+\widetilde{\sigma}\Delta\sqrt{\lambda G(t)+\rho t}Z)^2. \end{aligned}$$

With $P(T,T)=1$, integrating gives

$$\begin{aligned} \ln P(t,T) = -\Big(&\int_t^T r_u du+\sigma\omega\lambda\int_t^T du+\sigma\theta\rho\int_t^T du \\ &+\sigma\theta\lambda\sum_{t\le u\le T}\Delta G(u)+\sigma\widetilde{\sigma}\sum_{t\le u\le T}\Delta\sqrt{\lambda G(u)+\rho u}Z \\ &-\frac{\sigma^2}{2}\sum_{t\le u\le T}(\lambda\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{\lambda G(u)+\rho u}Z)^2\Big) \\ = -\Big(&\frac{-r_0}{\varpi}(e^{-\varpi T}-e^{-\varpi t})+\beta[T-t]+\frac{\beta}{\varpi}(e^{-\varpi T}-e^{-\varpi t}) \\ &+\frac{\sigma\omega\lambda}{\varpi}[T-t]+\frac{\sigma\omega\lambda}{\varpi}\big(\frac{1}{\varpi}(e^{-\varpi T}-e^{-\varpi t})\big)+\frac{\sigma\theta\rho}{\varpi}[T-t] \\ &+\frac{\sigma\theta\rho}{\varpi}\big(\frac{1}{\varpi}(e^{-\varpi T}-e^{-\varpi t})\big)+\sigma\theta\lambda\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s) \\ &\times e^{-\varpi(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{\lambda G(s)+\rho s}e^{-\varpi(u-s)}Z \\ &+\sigma\omega\lambda[T-t]+\sigma\theta\rho[T-t]+\sigma\theta\lambda\sum_{t\le u\le T}\Delta G(u) \\ &+\sigma\widetilde{\sigma}\sum_{t\le u\le T}\Delta\sqrt{\lambda G(u)+\rho u}Z \\ &-\frac{\sigma^2}{2}\sum_{t\le u\le T}(\lambda\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{\lambda G(u)+\rho u}Z)^2\Big). \end{aligned}$$

Thus,

$$\ln P(t,T) = -\Big(\frac{-r_0}{\varpi}(e^{-\varpi T} - e^{-\varpi t}) + \beta[T-t] + \frac{\beta}{\varpi}(e^{-\varpi T} - e^{-\varpi t}) + \frac{\sigma\omega\lambda}{\varpi}[T-t] + \frac{\sigma\omega\lambda}{\varpi}\big(\frac{1}{\varpi}(e^{-\varpi T} - e^{-\varpi t})\big) + \frac{\sigma\theta\rho}{\varpi}[T-t] + \frac{\sigma\theta\rho}{\varpi}\big(\frac{1}{\varpi}(e^{-\varpi T} - e^{-\varpi t})\big) + \sigma\theta\lambda \times \sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-\varpi(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta \times \sqrt{\lambda G(s)+\rho s}e^{-\varpi(u-s)}Z + \sigma\omega\lambda[T-t] + \sigma\theta\rho[T-t] + \sigma\theta\lambda\sum_{t\le u\le T}\Delta G(u) + \sigma\widetilde{\sigma}\sum_{t\le u\le T}\Delta\sqrt{\lambda G(u)+\rho u}Z - \frac{\sigma^2}{2}\sum_{t\le u\le T}(\lambda\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{\lambda G(u)+\rho u}Z)^2\Big).$$

Hence, the result in equation (6) follows by taking exponential of the sides of the equation. □

4. Conclusion

The generalized version of Vasicek model driven by a variance gamma process and its corresponding interest rate derivative have been derived. These provide a wider atmosphere for different phenomenon to be captured in a financial instrument.

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