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NON-INSTANTANEOUS PERIODIC BVP OF FRACTIONAL VOLTERRA-FREDHOLM MODELS

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Abstract

In this manuscript, we study the sufficient conditions for existence results of PC-mild solutions of non-instantaneous impulses Caputo fractional Volterra-Fredholm integro-differential equations with periodic boundary conditions. Fractional calculus, semigroup theory, Krasnoselskii and Sadovskii fixed point theorems are used to prove the existence results. Moreover, an example is finally presented.

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Key Words and Phrases: Volterra-Fredholm model, Caputo fractional derivative, fixed point technique

1. Introduction

Due to its capacity to generalise classical calculus to non-integer orders, fractional calculus (FC) has drawn a lot of interest. This mathematical framework is effective for simulating complicated events with non-local and memory-dependent behaviours. The theory of FC was crucial in the development of differential equations as a powerful tool for simulating several real-world issues

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in a variety of scientific domains, including physics, control systems, engineering fields, etc [1, 14, 20]. Integro-differential equations (IDEs) with instantaneous impulse are unable to fully describe several dynamical difficulties in the evolution process. For instance, the transit of drugs through the circulation occurs gradually and continuously. On the other hand, non-instantaneous impulse models help clarify these problems, [8, 9, 10, 13]. Therefore, by using fixed point theory, various researchers have looked into some existence and uniqueness conclusions for nonlinear IDEs of fractional order, [7, 8, 11].

The authors of [2, 22] investigated if non-instantaneous impulsive differential equations have solutions. The presence of mild solutions was determined by [3] through the study of non-autonomous parabolic evolution equations with non-instantaneous impulses. In their study of periodic boundary value issues for integer differential equations with non-instantaneous memory impulses. The authors of [26] showed the existence of PC-mild solutions based on semigroup theory.

The authors of [12, 21] first looked at the existence of PC-mild solutions by a fixed point theorem of the Cauchy DEs with impulses non-instantaneous as follows:

$$\begin{cases} \Phi'(\nu) = \Xi\Phi(\nu) + \psi(\nu, \Phi(\nu)), & \nu \in (\delta_i, \nu_{i+1}], i = 0, 1, 2, \ldots, \zeta, \\ \Phi(\nu) = \omega_i(\nu, \Phi(\nu)), & \nu \in (\nu_i, \delta_i], i = 1, 2, \ldots, \zeta, \\ \Phi(0) = \Phi_0 \in \Upsilon, \end{cases}$$

where $\omega_i : (\nu_i, \delta_i] \times \Upsilon \to \Upsilon, i = 1, 2, \ldots, \zeta$, be continuous functions, $\Xi : \nabla(\Xi) \subset \Upsilon \to \Upsilon$ is the generator of a C_0-semigroup $\{S(\nu), \nu \geq 0\}$ on a Banach space Υ, the prefixed numbers δ_i, ν_i satisfy $0 = \delta_0 < \nu_1 \leq \delta_1 < \nu_2 \leq \delta_2 < \cdots < \nu_\zeta \leq \delta_\zeta < \nu_{\zeta+1} = E$, $\psi : [0, E] \times \Upsilon \to \Upsilon$.

In [23] the fixed point theorem is used to investigated periodic BVP for DEs with non-instantaneous impulses:

$$\begin{cases} \Phi'(\nu) = \psi(\nu, \Phi(\nu)), & \nu \in (\delta_i, \nu_{i+1}], i = 0, 1, 2, \ldots, \zeta, \\ \Phi(\nu) = \omega_i(\nu, \Phi(\nu)), & \nu \in (\nu_i, \delta_i], i = 1, 2, \ldots, \zeta, \\ \Phi(0) = \Phi(E). \end{cases}$$

Using non-instantaneous impulses, this work explores the presence of mild and PC-mild solutions for the semi-linear Caputo fractional Volterra-Fredholm model

$$\begin{cases} {}^c\nabla_\nu^\beta \Psi(\nu) = \Xi(\nu)\Psi(\nu) \\ \qquad\qquad + f\left(\nu, \Psi(\nu), \int_0^\nu \theta(\nu, \delta, \Psi(\delta))d\delta, \int_0^T \theta_1(\nu, \delta, \Psi(\delta))d\delta\right), \\ \qquad\qquad\qquad\qquad \nu \in (\delta_i, \nu_{i+1}], i = 0, 1, 2, \ldots, \wp, \\ \Psi(\nu) = h_i + \Omega_\beta(\nu, \nu_i) \int_{\nu_i}^\nu g_i(\delta, \Psi(\delta))d\delta, \\ \Psi(0) = \Psi(T),\ \nu \in (\nu_i, \delta_i], i = 1, 2, \ldots, \wp, \end{cases} \tag{1}$$

where $\beta \in (0,1], J=[0,T], \Xi(\nu)$ is a closed linear operator on a Banach space $\Upsilon, f, \theta, \theta_1$, with domain $\nabla(\Xi)$ defined and Ω_β are to be specified later, the prefixed numbers δ_i and $\nu_i(i=1,2,\ldots,\wp)$ satisfy $0=\delta_0<\nu_1\leq\delta_1<\nu_2\leq \cdots<\nu_\wp\leq\delta_\wp<t_{\wp+1}=T, g_i:(\nu_i,\delta_i]\times\Upsilon\to\Upsilon, i=1,2,\ldots,\wp$, be continuous functions, $h_i\in\Upsilon, i=1,2,\ldots,\wp$.

2. Preliminaries

Let $(\Upsilon,\|\cdot\|)$ be a Banach space, $J=[0,T]$ and $0<T<+\infty$. $C(J,\Upsilon)$ is the collection of all continuous functions from J into Υ equipped with the norm $\|\Psi\|_C=\max\{\|\Psi(\nu)\|, \nu\in J\}$.

Let $\mathrm{PC}(J,\Upsilon)=\{\Psi \mid \Psi: J\to\Upsilon: \Psi\in C\left(\left(\nu_k,\nu_{k+1}\right],\Upsilon\right)$, and there exist $\Psi\left(\nu_k^-\right)$ and $\Psi\left(\nu_k^+\right)$ with $\Psi\left(\nu_k\right)=\Psi\left(\nu_k^-\right), k=1,\ldots,\wp\}$ endowed with the PC-norm $\|\Psi\|_{\mathrm{PC}}=\sup\{\|\Psi(\nu)\|, \nu\in J\}$.

Definition 2.1. ([4]) Let $\Xi(\nu)$ be a closed and linear operator with domain $\nabla(\Xi)$ defined on a Banach space Υ and $\beta>0$. Let $\rho[\Xi(\nu)]$ be the resolvent set of $\Xi(\nu), \Xi(\nu)$ is called the generator of a β-resolvent family if there exist $\omega\geq 0$ and a strongly continuous function $\Omega_\beta: \mathbb{C}_+^2\to B(\Upsilon)$ such that $\left\{\lambda^\beta: \operatorname{Re}\lambda>\omega\right\}\subset\rho(\Xi)$ and

$$\left(\lambda^\beta I-\Xi(\delta)\right)^{-1}\Psi=\int_0^\infty e^{-\lambda(\nu-\delta)}\Omega_\beta(\nu,\delta)\Psi d\nu, \quad \operatorname{Re}(\lambda)>\omega, \Psi\in\Upsilon.$$

In this case, $\Omega_\beta(\nu,\delta)$ is called the β-resolvent family generated by $\Xi(\nu)$, denote $\Im=\max_{0\leq\delta<\nu\leq T}\|\Omega_\beta(\nu,\delta)\|$.

Definition 2.2. ([20]) The Caputo fractional derivative of order β of a function $f:(0,\infty)\to\mathbb{C}$ is defined as

$${}^c\nabla_\nu^\beta f(\nu)=\frac{1}{\Gamma(n-\beta)}\int_0^\nu(\nu-\delta)^{n-\beta-1}f^{(n)}(\delta)d\delta,$$

where $n-1<\beta<n, n\in N, \Gamma(\cdot)$ denotes the Gamma function. The Laplace transform of the Caputo fractional derivative of order β is given as

$$\mathcal{L}\left({}^c\nabla_\nu^\beta f(\nu)\right)(\delta)=\delta^\beta(\mathcal{L}f)(\delta)-\sum_{j=1}^{n-1}\delta^{\beta-j-1}\Psi^{(j)}(0), \quad n-1<\beta\leq n,$$

where $(\mathcal{L}f)(\Psi)=\int_0^\infty e^{-\Psi\nu}f(\nu)d\nu$ is the Laplace transform of the function $f(\nu)$.

Lemma 2.1. ([4]) *$\Omega_\beta(\nu,\delta)$ satisfies the following properties:*

(i) $\Omega_\beta(\delta,\delta)=I, \Omega_\beta(\nu,\delta)=\Omega_\beta(\nu,r)\Omega_\beta(r,\delta)$ for $0\leq\delta\leq r\leq\nu\leq a$;

(ii) $(\nu,\delta)\to\Omega_\beta(\nu,\delta)$ is strongly continuous for $0\leq\delta\leq\nu\leq a$;

(iii) If $\Omega_\beta(\nu,\delta)$ is compact for $\nu,\delta>0$, then $\Omega_\beta(\nu,\delta)$ is continuous in the uniform operator topology.

Definition 2.3. A function $\Psi \in \mathrm{PC}(J, \Upsilon)$ is said to be a PC-mild solution of model (1), if $\Psi(\nu)$ satisfies the integral equation

$$\Psi(\nu) = \begin{cases} \Omega_\beta(\nu,0)\Big[\Omega_\beta(T,\delta_\wp)h_\wp + \Omega_\beta(T,\nu_\wp)\int_{\nu_\wp}^{\delta_\wp} g_\wp(\delta,\Psi(\delta))d\delta \\ \quad + \int_{\delta_\wp}^{T}\Omega_\beta(T,\delta) \\ \quad \times f\Big(\delta,\Psi(\delta),\int_0^\delta \theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T \theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\Big)d\delta\Big] \\ \quad\quad + \int_0^\nu \Omega_\beta(\nu,\delta) \\ \quad \times f\left(\delta,\Psi(\delta),\int_0^\delta \theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T \theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)d\delta, \\ \quad\quad \nu\in[0,\nu_1] \\ h_i+\Omega_\beta(\nu,\nu_i)\int_{\nu_i}^{\nu} g_i(\delta,\Psi(\delta))d\delta, \quad \nu\in(\nu_i,\delta_i], i=1,2,\ldots,\wp, \\ \quad \Omega_\beta(\nu,\delta_i)h_i + \Omega_\beta(\nu,\nu_i)\int_{\nu_i}^{\delta_i} g_i(\delta,\Psi(\delta))d\delta \\ \quad + \int_{\delta_i}^{\nu}\Omega_\beta(\nu,\delta) \\ \quad \times f\left(\delta,\Psi(\delta),\int_0^\delta \theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T \theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)d\delta \\ \quad \nu\in(\delta_i,\nu_{i+1}], i=1,2,\ldots,\wp. \end{cases} \tag{2}$$

Lemma 2.2. (Lemma 2.3, [16]) *Let $B \subset C(J,\Upsilon)$ be equicontinuous and bounded, then $\overline{Co}B \subset C(J,\Upsilon)$ is also equicontinuous and bounded. Let Υ be a Banach space and $\nabla \subset \Upsilon$ be bounded, then there exists a countable set $\nabla_0 \subset \nabla$ such that $\alpha(\nabla) \le 2\alpha(\nabla_0)$, where α denotes the measure of noncompactness.*

Lemma 2.3. ([17]) *Let $B \subset C(J,\Upsilon)$ be equicontinuous and bounded, then $\alpha(B(\nu))$ is continuous on J and*

$$\alpha\left(\int_J B(\delta)d\delta\right) \le \int_J \alpha(B(\delta))d\delta, \quad \alpha(B) = \max_{\nu\in J}\alpha(B(\nu)).$$

3. **Results**

First, we demonstrate the existence of PC-mild solutions for model (1) based on the measure of noncompactness and fixed point theorem.

Let us introduce the following assumptions:

(A1). The functions $\theta, \theta_1 : \nabla \times \Upsilon \to \Upsilon$ are continuous, $\nabla = \{(\nu, \delta) \mid 0 \leq \delta \leq \nu \leq T\}$, there exist $\Lambda(\nu, \cdot), \mathit{\Lambda}(\nu, \cdot) \in L^1\left(J, \complement_+\right)$ with

$$\Lambda_0 = \max_{\nu \in [0,T]} \int_0^\nu \Lambda(\nu, \delta) d\delta, \mathit{\Lambda}_0 = \max_{\nu \in [0,T]} \int_0^T \mathit{\Lambda}(\nu, \delta) d\delta$$

for $(\nu, \delta) \in \nabla, \Psi \in \Upsilon$ such that

$$\|\theta(\nu, \delta, \Psi)\| \leq \Lambda(\nu, \delta)\|\Psi\|, \ \|\theta_1(\nu, \delta, \Psi)\| \leq \mathit{\Lambda}(\nu, \delta)\|\Psi\|.$$

(A2). The functions $f : J \times T_\complement \times T_\complement \times T_\complement \to \Upsilon$ are continuous and bounded for every $\complement > 0$ such that

$$\lim_{\complement \to \infty} \sup \frac{\Im(\complement)}{\complement} < \frac{1}{\Delta},$$

where $\Im(\complement) = \max\left\{\Im_1(\complement), \Im_2(\complement)\right\}$,
$\Im_1(\complement) = \sup\left\{\|f\left(\nu, \Psi_1, \Psi_2, \Psi_3\right)\| : \left(\nu, \Psi_1, \Psi_2, \Psi_3\right) \in J \times T_\complement \times T_\complement \times \ T_\complement\right\}$,
$\Im_2(\complement) = \sup\left\{\|g_i(\nu, \Psi)\|, (\nu, \Psi) \in J \times T_\complement, i = 1, 2, \ldots, \wp\right\}$,
$T_\complement = \{\Psi \in \Upsilon : \|\Psi\| \leq \complement\}$,
$\Delta = \max\left\{\Im^2 a_0\left(T - t_\wp\right) + \Im\nu_1 a_0, \Im a_0\left(\nu_{i+1} - \nu_i\right), i = 1, 2, \ldots, \wp\right\}$,
$a_0 = \max\{1, h_0\}$.

(A3). $\forall\ \complement > 0$, there exist Lebesgue integrable nonnegative functions $L'_\theta, L'_{\theta_1}, L'_{g_i}, L'_1, L'_2 \in L^1\left(J, \complement_+\right)$ $(i = 1, 2, \ldots, \wp)$ for all equicontinuous and countable sets $\nabla, \nabla_i \subset T_\complement\, (i = 1, 2, 3)$ such that

$$\begin{aligned}
&\alpha(\theta(\nu, \delta, \nabla)) \leq L'_\theta(\nu)\alpha(\nabla), \\
&\alpha(\theta_1(\nu, \delta, \nabla)) \leq L'_{\theta_1}(\nu)\alpha(\nabla), \\
&\alpha\left(g_i(\nu, \nabla)\right) \leq L'_{g_i}(\nu)\alpha(\nabla),
\end{aligned}$$

and

$$\alpha\left(f\left(\nu, \nabla_1, \nabla_2, \nabla_3\right)\right) \leq L'_1(\nu)\alpha\left(\nabla_1\right) + L'_2(\nu)\alpha\left(\nabla_2\right) + L'_3(\nu)\alpha\left(\nabla_3\right).$$

Theorem 3.1. *Let the hypotheses* ***(A1)-(A3)*** *hold. Then model* (1) *has at least one PC-mild solution on* $\mathrm{PC}(J, \Upsilon)$ *provided that the resolvent* *operator $\Omega_\beta(\nu,\delta)$ is compact for $\nu,\delta>0$ and*

$$\begin{aligned}
&\rho=\max\Big\{2\Im^2\int_{\nu_\wp}^{\delta_\wp}L'_{g_\wp}(\delta)d\delta+2\Im^2\\
&\times\int_{\delta_\wp}^{T}\Big(L'_1(\delta)+L'_2(\delta)\int_0^T L'_\theta(\sigma)d\sigma+L'_3(\delta)\int_0^T L'_{\theta_1}(\sigma)d\sigma\Big)d\delta\\
&+2\Im\int_0^{\nu_1}\left(L'_1(\delta)+L'_2(\delta)\int_0^T L'_\theta(\sigma)d\sigma+L'_3(\delta)\int_0^T L'_{\theta_1}(\sigma)d\sigma\right)d\delta,\\
&2\Im\int_{\nu_i}^{\delta_i}L'_{g_i}(\delta)d\delta+2\Im\int_{\delta_i}^{\nu_{i+1}}\\
&\times\left(L'_1(\delta)+L'_2(\delta)\int_0^T L'_\theta(\sigma)d\sigma+L'_3(\delta)\int_0^T L'_{\theta_1}(\sigma)d\sigma\right)d\delta,\\
&i=1,2,\dots,\wp\Big\}<1.
\end{aligned}$$

Proof. Let $\mathcal{F}:\mathrm{PC}(J,\Upsilon)\to\mathrm{PC}(J,\Upsilon)$ defined by $(\mathcal{F}\Psi)(\nu)=$

$$\begin{cases}
\Omega_\beta(\nu,0)\Big[\Omega_\beta(T,\delta_\wp)h_\wp+\Omega_\beta(T,\nu_\wp)\int_{\nu_\wp}^{\delta_\wp}g_\wp(\delta,\Psi(\delta))d\delta\\
\quad+\int_{\delta_\wp}^{T}\Omega_\beta(T,s)f\left(\delta,\Psi(\delta),\int_0^\delta\theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)d\delta\Big]\\
\quad+\int_0^{\nu}\Omega_\beta(\nu,\delta)f\left(\delta,\Psi(\delta),\int_0^\delta\theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)d\delta,\\
\quad\nu\in[0,\nu_1]\\
\quad h_i+\Omega_\beta(\nu,\nu_i)\int_{\nu_i}^{\nu}g_i(\delta,\Psi(\delta))d\delta,\quad \nu\in(\nu_i,\delta_i],\,i=1,2,\dots,\wp,\\
\Omega_\beta(\nu,\delta_i)h_i+\Omega_\beta(\nu,\nu_i)\int_{\nu_i}^{\delta_i}g_i(\delta,\Psi(\delta))d\delta\\
\quad+\int_{\delta_i}^{\nu}\Omega_\beta(\nu,\delta)f\left(\delta,\Psi(\delta),\int_0^\delta\theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)d\delta\\
\quad\nu\in(\delta_i,\nu_{i+1}],\,i=1,2,\dots,\wp.
\end{cases}$$

Then, the operator $\mathcal{F}$ is well defined in $\mathrm{PC}(J,\Upsilon)$.

From **(A2)**, there exist $0<r<\frac{1}{\Delta}$ and $\mathsf{C}_0>0$, $\forall\ \mathsf{C}\ge a_0\mathsf{C}_0$ such that $\Im(\mathsf{C})<r\mathsf{C}$.

Let $\eta = \max\left\{\complement_0, \frac{\Im^2\|h_\wp\|}{1-\Im^2 ra_0(T-\nu_\wp)-\Im ra_0\nu_1}, \frac{\|h_i\|}{1-\Im ra_0(\delta_i-\nu_i)}, \frac{\Im\|h_i\|}{1-\Im ra_0(\nu_{i+1}-\nu_i)},\right\}$. For all $\Psi \in B_\eta = \{\Psi \in \mathrm{PC}(J,\Upsilon) : \|\Psi\|_{\mathrm{PC}} \leq \eta\}, \nu \in (\delta_i, \nu_{i+1}], i = 0, 1, \ldots, \wp$, then

$$\|\Psi\|_{\mathrm{PC}} \leq \eta \leq a_0\eta,$$

hence

$$\left\|\int_0^\nu \theta(\nu,\delta,\Psi)d\delta\right\| \leq \int_0^\nu \Lambda(\nu,\delta)\|\Psi\|_{\mathrm{PC}}d\delta \leq h_0\eta \leq a_0\eta.$$

$$\left\|\int_0^T \theta_1(\nu,\delta,\Psi)d\delta\right\| \leq \int_0^T \mathit{\Lambda}(\nu,\delta)\|\Psi\|_{\mathrm{PC}}d\delta \leq h_0\eta \leq a_0\eta.$$

Now, we prove that $\mathcal{F}\Psi \in B_\eta$.

For $\nu \in [0, \nu_1]$,

$$\begin{aligned}
&\|(\mathcal{F}\Psi)(\nu)\| \\
\leq&\Im^2\|h_\wp\| + \Im^2\int_{\nu_\wp}^{\delta_\wp}\|g_\wp(\delta,\Psi(\delta))\|\,d\delta \\
&+ \Im^2\int_{\delta_\wp}^T\left\|f\left(\delta,\Psi(\delta),\int_0^\delta\theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)\right\|d\delta \\
&+ \Im\int_0^\nu\left\|f\left(\delta,\Psi(\delta),\int_0^\delta\theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)\right\|d\delta \\
\leq&\Im^2\|h_\wp\| + \Im^2 ra_0\eta\,(\delta_\wp-\nu_\wp) + \Im^2 ra_0\eta\,(T-\delta_\wp) + \Im ra_0\eta\nu_1 \\
\leq&\Im^2\|h_\wp\| + \left(\Im^2 ra_0\,(T-\nu_\wp) + \Im ra_0\nu_1\right)\eta \leq \eta.
\end{aligned}$$

For $\nu \in (\nu_i, \delta_i], i = 1, 2, \ldots, \wp$,

$$\|(\mathcal{F}\Psi)(\nu)\| \leq \|h_i\| + \Im\int_{\nu_i}^\nu\|g_i(\delta,\Psi(\delta))\|\,d\delta \leq \|h_i\| + \Im ra_0\eta\,(\delta_i-\nu_i) \leq \eta.$$

For $\delta \in (\delta_i, \nu_{i+1}], i = 1, 2, \ldots, \wp$,

$$\begin{aligned}
&\|(\mathcal{F}\Psi)(\nu)\| \\
\leq&\Im\|h_i\| + \Im\int_{\nu_i}^{\delta_i}\|g_i(\delta,\Psi(\delta))\|\,d\delta \\
&+ \Im\int_{\delta_i}^\nu\left\|f\left(\delta,\Psi(\delta),\int_0^\delta\theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)\right\|d\delta \\
\leq&\Im\|h_i\| + \Im ra_0\eta\,(\delta_i-\nu_i) + \Im ra_0\eta\,(\nu_{i+1}-\delta_i) \\
\leq&\Im\|h_i\| + \Im \mathrm{ra}_0\,\eta\,(\nu_{i+1}-\nu_i) \leq \eta.
\end{aligned}$$

Then $\mathcal{F} : B_\eta \to B_\eta$.

Also, we show that $\mathcal{F} : B_\eta \to B_\eta$ is continuous. Let $\{\Psi_n\}_0^\infty$ with $\Psi_n \to \Psi$ in B_η. $\forall\ \nu \in [0, \nu_1]$, we get

$$
\begin{aligned}
&\left\|\left(\mathcal{F}\Psi_n\right)(\nu)-(\mathcal{F}\Psi)(\nu)\right\| \\
&\leq \Im^2 \int_{\nu_\wp}^{\delta_\wp}\left\|g_\wp\left(\delta, \delta_n(\delta)\right)-g_\wp(\delta, \Psi(\delta))\right\| d\delta \\
&\quad+\Im^2 \int_{\delta_\wp}^{T}\left\|f\left(\delta, \delta_n(\delta), \int_0^\delta \theta\left(\delta, \sigma, \Psi_n(\sigma)\right) d\sigma, \int_0^T \theta_1\left(\delta, \sigma, \Psi_n(\sigma)\right) d\sigma\right)\right. \\
&\quad\left.-f\left(\delta, \Psi(\delta), \int_0^\delta \theta(\delta, \sigma, \Psi(\sigma)) d\sigma, \int_0^T \theta_1(\delta, \sigma, \Psi(\sigma)) d\sigma\right)\right\| d\delta \\
&\quad+\Im \int_0^{\nu_1}\left\|f\left(\delta, \Psi_n(\delta), \int_0^\delta \theta\left(\delta, \sigma, \Psi_n(\sigma)\right) d\sigma, \int_0^T \theta_1\left(\delta, \sigma, \Psi_n(\sigma)\right) d\sigma\right)\right. \\
&\quad\left.-f\left(\delta, \Psi(\delta), \int_0^\delta \theta(\delta, \sigma, \Psi(\sigma)) d\sigma, \int_0^T \theta_1(\delta, \sigma, \Psi(\sigma)) d\sigma\right)\right\| d\delta \\
&\leq \Im^2\left(\delta_\wp-\nu_\wp\right) \sup_{\delta \in J}\left\|g_\wp\left(\delta, \Psi_n(\delta)\right)-g_\wp(\delta, \Psi(\delta))\right\|+\Im^2\left(T-\delta_\wp\right) \\
&\times \sup_{\delta \in J}\| f\left(\delta, \Psi_n(\delta), \int_0^\delta \theta\left(\delta, \sigma, \Psi_n(\sigma)\right) d\sigma, \int_0^T \theta_1\left(\delta, \sigma, \Psi_n(\sigma)\right) d\sigma\right) \\
&\quad-f\left(\delta, \Psi(\delta), \int_0^\delta \theta(\delta, \sigma, \Psi(\sigma)) d\sigma, \int_0^T \theta_1(\delta, \sigma, \Psi(\sigma)) d\sigma\right) \| \\
&\quad+\Im \nu_1 \sup_{\delta \in J}\left\|f\left(\delta, \Psi_n(\delta), \int_0^\delta \theta\left(\delta, \sigma, \Psi_n(\sigma)\right) d\sigma, \int_0^T \theta_1\left(\delta, \sigma, \Psi_n(\sigma)\right) d\sigma\right)\right. \\
&\quad\left.-f\left(\delta, \Psi(\delta), \int_0^\delta \theta(\delta, \sigma, \Psi(\sigma)) d\sigma, \int_0^T \theta_1(\delta, \sigma, \Psi(\sigma)) d\sigma\right)\right\| .
\end{aligned}
$$

For each $\nu \in (\nu_i, \delta_i]$, $i=1,2,\ldots,\wp$, we get

$$
\begin{aligned}
&\left\|\left(\mathcal{F}\Psi_n\right)(\nu)-(\mathcal{F}\Psi)(\nu)\right\| \\
&\leq \Im \int_{\nu_i}^{\delta_i}\left\|g_i\left(\delta, \Psi_n(\delta)\right)-g_i(\delta, \Psi(\delta))\right\| d\delta \\
&\leq \Im\left(\delta_i-\nu_i\right) \sup_{\delta \in J}\left\|g_i\left(\delta, \Psi_n(\delta)\right)-g_i(\delta, \Psi(\delta))\right\| .
\end{aligned}
$$

For each $\nu \in (\delta_i, \nu_{i+1}]$, $i=1,2,\ldots,\wp$, we get

$$
\begin{aligned}
&\left\|\left(\mathcal{F}\Psi_n\right)(\nu)-(\mathcal{F}\Psi)(\nu)\right\| \\
&\leq \Im \int_{\nu_i}^{\delta_i}\left\|g_i\left(\delta, \Psi_n(\delta)\right)-g_i(\delta, \Psi(\delta))\right\| d\delta+\Im \int_{\delta_i}^{\nu_{i+1}} \\
&\times\left\|f\left(\delta, \Psi_n(\delta), \int_0^\delta \theta\left(\delta, \sigma, \Psi_n(\sigma)\right) d\sigma, \int_0^T \theta_1\left(\delta, \sigma, \Psi_n(\sigma)\right) d\sigma\right)\right.
\end{aligned}
$$

$$- f\left(\delta, \Psi(\delta), \int_0^{\delta} \theta(\delta,\sigma,\Psi(\sigma))d\sigma, \int_0^{T} \theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)\Big\|d\delta$$
$$\leq \Im\left(\delta_i - \nu_i\right)\sup_{\delta\in J}\left\|g_i\left(\delta, \Psi_n(\delta)\right) - g_i(\delta, \Psi(\delta))\right\| + \Im\left(\nu_{i+1} - \delta_i\right)$$
$$\times \sup_{\delta\in J}\Big\|f\left(\delta, \Psi_n(\delta), \int_0^{\delta} \theta\left(\delta,\sigma,\Psi_n(\sigma)\right)d\sigma, \int_0^{T} \theta_1\left(\delta,\sigma,\Psi_n(\sigma)\right)d\sigma\right)$$
$$- f\left(\delta, \Psi(\delta), \int_0^{\delta} \theta(\delta,\sigma,\Psi(\sigma))d\sigma, \int_0^{T} \theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)\Big\|.$$

By the functions $f : J \times \Upsilon \times \Upsilon \times \Upsilon \to \Upsilon$, $\theta, \theta_1 : \nabla \times \Upsilon \to \Upsilon$ and $g_i : (\nu_i, \delta_i] \times \Upsilon \to \Upsilon$ $(i = 1, 2, \ldots, \wp)$ that are continuous, we get

$$\lim_{n\to\infty}\sup_{\delta\in J}\Big\|f\left(\delta, \Psi_n(\delta), \int_0^{\delta} \theta\left(\delta,\sigma,\Psi_n(\sigma)\right)d\sigma, \int_0^{T} \theta_1\left(\delta,\sigma,\Psi_n(\sigma)\right)d\sigma\right)$$
$$-f\left(\delta, \Psi(\delta), \int_0^{\delta} \theta(\delta,\sigma,\Psi(\sigma))d\sigma, \int_0^{T} \theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)\Big\| = 0,$$

and

$$\lim_{n\to\infty}\sup_{\nu\in J}\left\|g_i\left(\delta, \Psi_n(\delta)\right) - g_i(\delta, \Psi(\delta))\right\| = 0 \quad (i = 1, 2, \ldots, \wp),$$

we get $\|\mathcal{F}\Psi_n - \mathcal{F}\Psi\|_{\mathrm{PC}} \to 0$ as $n \to \infty$. This proves that $\mathcal{F} : B_\eta \to B_\eta$ is continuous.

Now we prove that $\mathcal{F}\left(B_\eta\right)$ is equicontinuous.

For the interval $[0, \nu_1]$, $0 \leq e_1 < e_2 \leq \nu_1$, $\Psi \in B_\eta$, we get

$$\left\|\left(\mathcal{F}\Psi\right)(e_2) - \left(\mathcal{F}\Psi\right)(e_1)\right\|$$
$$\leq \left\|\Omega_\beta\left(e_2, 0\right) - \Omega_\beta\left(e_1, 0\right)\right\|\left(\Im\left\|h_\wp\right\| + \Im\left\|\int_{\nu_\wp}^{\delta_\wp} g_\wp(\delta, \Psi(\delta))d\delta\right\|\right.$$
$$\left.+\Im\left\|\int_{\delta_\wp}^{T} f\left(\delta, \Psi(\delta), \int_0^{\delta} \theta(\delta,\sigma,\Psi(\sigma))d\sigma, \int_0^{T} \theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)d\delta\right\|\right)$$
$$+\left\|\int_{e_1}^{e_2} \Omega_\beta\left(e_2, \delta\right) f\left(\delta, \Psi(\delta), \int_0^{\delta} \theta(\delta,\sigma,\Psi(\sigma))d\sigma, \int_0^{T} \theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)d\delta\right\|$$

$$+\left\| \int_0^{e_1} \left(\Omega_\beta (e_2,\delta) - \Omega_\beta (e_1,\delta)\right)\right.$$
$$\left.\times f\left(\delta, \Psi(\delta), \int_0^\delta \theta(\delta,\sigma,\Psi(\sigma))d\sigma, \int_0^T \theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right) d\delta\right\|$$
$$\leq \|\Omega_\beta (e_2,0) - \Omega_\beta (e_1,0)\| \Im \left(\|h_\wp\| + \left\|\int_{\nu_\wp}^{\delta_\wp} g_\wp(\delta,\Psi(\delta))d\delta\right\|\right.$$
$$\left. + \left\|\int_{\delta_\wp}^T f\left(\delta, \Psi(\delta), \int_0^\delta \theta(\delta,\sigma,\Psi(\sigma))d\sigma, \int_0^T \theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right) d\delta\right\|\right)$$
$$+ \Im_{ra}\eta (e_2 - e_1) + \sup_{\delta\in[0,\nu_1]} \|\Omega_\beta (e_2,\delta) - \Omega_\beta (e_1,\delta)\| ra_0\eta\nu_1.$$

For the interval $(\nu_i, \delta_i], i = 1, 2, \ldots, \wp, \nu_i < e_1 < e_2 \leq \delta_i, \Psi \in B_\eta$, we get

$$\|(\mathcal{F}\Psi)(e_2) - (\mathcal{F}\Psi)(e_1)\|$$
$$\leq \left\|\Omega_\beta (e_2,\nu_i) \int_{e_1}^{e_2} g_i(\delta,\Psi(\delta))d\delta\right\|$$
$$+ \left\|(\Omega_\beta (e_2,\nu_i) - \Omega_\beta (e_1,\nu_i)) \int_{\nu_i}^{e_1} g_i(\delta,\Psi(\delta))d\delta\right\|$$
$$\leq \Im ra_0\eta (e_2 - e_1) + \|\Omega_\beta (e_2,\nu_i) - \Omega_\beta (e_1,\nu_i)\| ra_0\eta (\delta_i - \nu_i).$$

For interval $(\delta_i, \nu_{i+1}], i = 1, 2, \ldots, \wp, \delta_i < e_1 < e_2 \leq \nu_{i+1}, \Psi \in B_\eta$, we get

$$\|(\mathcal{F}\Psi)(e_2) - (\mathcal{F}\Psi)(e_1)\|$$
$$\leq \|\Omega_\beta (e_2,\delta_i) - \Omega_\beta (e_1,\delta_i)\| \|h_i\|$$
$$+ \|\Omega_\beta (e_2,\nu_i) - \Omega_\beta (e_1,\nu_i)\| \left\|\int_{\nu_i}^{\delta_i} g_i(\delta,\Psi(\delta))d\delta\right\|$$
$$+ \left\| \int_{e_1}^{e_2} \Omega_\beta (e_2,s) f\left(\delta, \Psi(\delta), \int_0^\delta \theta(\delta,\sigma,\Psi(\sigma))d\sigma, \int_0^T \theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right) d\delta\right\|$$
$$+\left\| \int_{\delta_i}^{e_1} \left(\Omega_\beta (e_2,\delta) - \Omega_\beta (e_1,\delta)\right)\right.$$
$$\left.\times f\left(\delta, \Psi(\delta), \int_0^\delta \theta(\delta,\sigma,\Psi(\sigma))d\sigma, \int_0^T \theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right) d\delta\right\|$$
$$\leq \|\Omega_\beta (e_2,\delta_i) - \Omega_\beta (e_1,\delta_i)\| \|h_i\|$$
$$+ \|\Omega_\beta (e_2,\nu_i) - \Omega_\beta (e_1,\nu_i)\| \left\|\int_{\nu_i}^{\delta_i} g_i(\delta,\Psi(\delta))d\delta\right\|$$
$$+ \Im_{ra}\eta\eta (e_2 - e_1) + \sup_{\delta\in(\delta_i,\nu_{i+1}]} \|\Omega_\beta (e_2,\delta) - \Omega_\beta (e_1,\delta)\| ra_0\eta (t_{i+1} - \delta_i)$$
$$\to 0 \text{ as } e_2 \to e_1,$$

since the compactness of $\Omega_\beta(\nu,\delta)(\nu,\delta>0)$ implies the continuity in the uniform operator topology. This proves that $\mathcal{F}(B_\eta)$ is equicontinuous. From Lemma 2.2, $\overline{\mathrm{Co}}\mathcal{F}(B_\eta)\subset B_\eta$ is equicontinuous and bounded.

It remains to prove that $F:\overline{Co}\mathcal{F}(B_\eta)\to\overline{Co}\mathcal{F}(B_\eta)$ is a condensing operator. For any $\nabla\subset\overline{Co}\mathcal{F}(B_\eta)$, by Lemma 2.3, there exists a countable set $\nabla_0=\{\Psi_n\}\subset\nabla$ such that

$$\alpha(\mathcal{F}(\nabla))\leq 2\alpha\left(\mathcal{F}\left(\nabla_0\right)\right)$$

Using the fact that $\overline{Co}\mathcal{F}(B_\eta)$ is equicontinuous, $\nabla_0\subset\overline{Co}\mathcal{F}(B_\eta)$ is equicontinuous. By (H_3), for $\delta\in(\delta_i,\nu_{i+1}],i=0,1,\ldots,\wp$, then

$$\begin{aligned}
&\alpha\left(f\left(\delta,\nabla_0(\delta),\int_0^\delta\theta\left(\delta,\sigma,\nabla_0(\sigma)\right)d\sigma,\int_0^T\theta_1\left(\delta,\sigma,\nabla_0(\sigma)\right)d\sigma\right)\right)\\
&\leq L_1'(\delta)\alpha\left(\nabla_0(\delta)\right)+L_2'(\delta)\int_0^\delta L_g'(\sigma)\alpha\left(\nabla_0(\sigma)\right)d\sigma\\
&\leq\left(L_1'(\delta)+L_2'(\delta)\int_0^T L_g'(\sigma)d\sigma\right)\alpha(\nabla).
\end{aligned}$$

For each $\nu\in[0,\nu_1]$,

$$\begin{aligned}
&\alpha\left(\mathcal{F}\left(\nabla_0\right)(\nu)\right)\\
\leq&\Im^2\alpha\left(\int_{\nu_\wp}^{\delta_\wp}g_\wp\left(\delta,\nabla_0(\delta)\right)d\delta\right)\\
&+\Im^2\alpha\left(\int_{\delta_\wp}^T f\left(\delta,\nabla_0(\delta),\int_0^\delta\theta\left(\delta,\sigma,\nabla_0(\sigma)\right)d\sigma,\int_0^T\theta_1\left(\delta,\sigma,\nabla_0(\sigma)\right)d\sigma\right)d\delta\right)\\
&+\Im\alpha\left(\int_0^\nu f\left(\delta,\nabla_0(\delta),\int_0^\delta\theta\left(\delta,\sigma,\nabla_0(\sigma)\right)d\sigma,\int_0^T\theta_1\left(\delta,\sigma,\nabla_0(\sigma)\right)d\sigma\right)d\delta\right)\\
\leq&\Im^2\int_{\nu_\wp}^{\delta_\wp}L_{g_\wp}'(\delta)\alpha\left(\nabla_0(\delta)\right)d\delta\\
&+\Im^2\int_{\delta_\wp}^T\alpha\left(f\left(\delta,\nabla_0(\delta),\int_0^\delta\theta\left(\delta,\sigma,\nabla_0(\sigma)\right)d\sigma,\int_0^T\theta_1\left(\delta,\sigma,\nabla_0(\sigma)\right)d\sigma\right)\right)d\delta\\
&+\Im\int_0^\nu\alpha\left(f\left(\delta,\nabla_0(\delta),\int_0^\delta\theta\left(\delta,\sigma,\nabla_0(\sigma)\right)d\sigma,\int_0^T\theta_1\left(\delta,\sigma,\nabla_0(\sigma)\right)d\sigma\right)\right)d\delta
\end{aligned}$$

$$\leq\Big(\Im^2\int_{\nu_\wp}^{\delta_\wp} L'_{g_\wp}(\delta)d\delta$$
$$+\Im^2\int_{\delta_\wp}^{T}\left(L'_1(\delta)+L'_2(\delta)\int_0^T L'_\theta(\sigma)d\sigma+L'_3(s)\int_0^T L'_{\theta_1}(\sigma)d\sigma\right)d\delta$$
$$+\Im\int_0^{\nu_1}\left(L'_1(\delta)+L'_2(\delta)\int_0^T L'_\theta(\sigma)d\sigma+L'_3(s)\int_0^T L'_{\theta_1}(\sigma)d\sigma\right)d\delta\Big)\alpha(\nabla).$$

For each $\nu\in(\nu_i,\delta_i], i=1,\ldots,\wp$,

$$\alpha\left(\mathcal{F}\left(\nabla_0\right)(\nu)\right)\leq\Im\alpha\left(\int_{\nu_i}^{\nu} g_i\left(\delta,\nabla_0(\delta)\right)d\delta\right)\leq\Im\int_{\nu_i}^{\delta_i} L'_{g_i}(\delta)d\delta\alpha(\nabla).$$

For each $\nu\in(\delta_i,\nu_{i+1}], i=1,\ldots,\wp$,

$$\alpha\left(\mathcal{F}\left(\nabla_0\right)(\nu)\right)$$
$$\leq\Im\alpha\left(\int_{\nu_i}^{\delta_i} g_i\left(\delta,\nabla_0(\delta)\right)d\delta\right)$$
$$+\Im\alpha\left(\int_{\delta_i}^{\nu} f\left(\delta,\nabla_0(\delta),\int_0^\delta\theta\left(\delta,\sigma,\nabla_0(\sigma)\right)d\sigma,\int_0^T\theta_1\left(\delta,\sigma,\nabla_0(\sigma)\right)d\sigma\right)d\delta\right)$$
$$\leq\Big(\Im\int_{\nu_i}^{\delta_i} L'_{g_i}(\delta)d\delta+\Im\int_{\delta_i}^{\nu_{i+1}}$$
$$\times\left(L'_1(\delta)+L'_2(\delta)\int_0^T L'_\theta(\sigma)d\sigma+L'_3(\delta)\int_0^T L'_{\theta_1}(\sigma)d\sigma\right)d\delta\Big)\alpha(\nabla).$$

By Lemma (2.3),

$$\alpha\left(\mathcal{F}\left(\nabla_0\right)\right)=\max_{\nu\in J}\alpha\left(\mathcal{F}\left(\nabla_0\right)(\nu)\right).$$

Then

$$\alpha(\mathcal{F}(\nabla))\leq\rho\alpha(\nabla)<\alpha(\nabla).$$

These justifications allow us to deduce that $\mathcal{F}:\overline{Co}\mathcal{F}\left(B_\eta\right)\to\overline{Co}\mathcal{F}\left(B_\eta\right)$ is a condensing operator and by the Sadovskii's fixed point theorem, there exists one fixed point $\Psi^\star\in\overline{\mathrm{Co}}\mathcal{F}\left(B_\eta\right)\subset\mathrm{PC}(J,\Upsilon)$ for $\mathcal{F}$. In the final analysis, there is at least one PC-mild solution for model (1). □

First, using the fixed point theorem and the measure of noncompactness, we show that PC-mild solutions exist for model (1).

Let us introduce the following hypotheses:

(B1). The function $f:J\times\Upsilon\times\Upsilon\times\Upsilon\to\Upsilon$ is continuous, there exist nonnegative Lebesgue integrable functions $a,L_1,L_2,L_3\in L^1\left(J,\mathtt{C}_+\right)$ for $\nu\in$

$(\delta_i, \nu_{i+1}]\,(i = 0, 1, \dots, \wp)$ and $\Psi_1, \Psi_2, \Psi_3 \in \Upsilon$ such that

$$\|f(\nu, \Psi_1, \Psi_2, \Psi_3)\| \le a(\nu) + L_1(\nu)\|\Psi_1\| + L_2(\nu)\|\Psi_2\| + L_3(\nu)\|\Psi_3\|.$$

(B2). The functions $\theta, \theta_1 : \nabla \times \Upsilon \to \Upsilon$ are continuous, $\nabla = \{(\nu, \delta) \mid 0 \le \delta \le \nu \le T\}$, there exist Lebesgue integrable nonnegative functions $b, c, L_4, L_5 \in L^1(J, \mathtt{C}_+)$ for $(\nu, \delta) \in \nabla, \Psi \in \Upsilon$ such that

$$\|\theta(\nu, \delta, \Psi)\| \le b(\nu) + L_4(\nu)\|\Psi\|,$$
$$\|\theta_1(\nu, \delta, \Psi)\| \le c(\nu) + L_5(\nu)\|\Psi\|.$$

(B3). There exists a function $\omega_i(\nu)$ with $\varpi_i = \sup_{\nu\in[\nu_i,\delta_i]} \omega_i(\nu) < +\infty$ for $\nu \in (\nu_i, \delta_i]\,(i = 1, 2, \dots, \wp)$ and $\Psi \in \Upsilon$ such that

$$\|g_i(\nu, \Psi)\| \le \omega_i(\nu).$$

(B4). There exist constants $L_{g_i} > 0$ for $\nu \in (\nu_i, \delta_i]\,(i = 1, 2, \dots, \wp)$ and $\Psi, \Psi' \in \Upsilon$ such that

$$\left\|g_i(\nu, \Psi) - g_i(\nu, \Psi')\right\| \le L_{g_i}\left\|\Psi - \Psi'\right\|.$$

Theorem 3.2. *Suppose that the resolvent operator $\Omega_\beta(\nu, \delta)$ is compact for $\nu, \delta > 0$ and **(B1)-(B4)** satisfy. Then model* (1) *has at least one PC-mild solution on* PC(J, Υ) *provided that*

$$\vartheta = \max\Bigg\{\Im^2 \int_{\delta_\wp}^{T} b_1(\delta)d\delta + \Im \int_0^{\nu_1} b_1(\delta)d\delta,\ \Im \int_{\delta_i}^{\nu_{i+1}} b_1(\delta)d\delta, \Im^2 L_{g_\wp}(\delta_\wp - \nu_\wp),$$
$$\Im L_{g_i}(\delta_i - \nu_i), i = 1, \dots, \wp\Big\} < 1, \text{ where}$$

$$b_1(\delta) = L_1(\delta) + L_2(\delta)\int_0^T L_4(\sigma)d\sigma + L_3(\delta)\int_0^T L_5(\sigma)d\sigma.$$

Proof. Let $\mathcal{F}$ reduced as $\mathcal{F} = \mathcal{G} + \mathcal{H}$, where $(\mathcal{G}\Psi)(\nu) =$

$$\begin{cases} \Omega_\beta(\nu, 0)\left[\Omega_\beta(T, \delta_\wp)h_\wp + \Omega_\beta(T, \nu_\wp)\int_{\nu_\wp}^{\delta_\wp} g_\wp(\delta, \Psi(\delta))d\delta\right], & \nu \in [0, \nu_1], \\ h_i + \Omega_\beta(\nu, \nu_i)\int_{\nu_i}^{\nu} g_i(\delta, \Psi(\delta))d\delta, & \nu \in (\nu_i, \delta_i], i = 1, 2, \dots, \wp, \\ \Omega_\beta(\nu, \delta_i)h_i + \Omega_\beta(\nu, \nu_i)\int_{\nu_i}^{\delta_i} g_i(\delta, \Psi(\delta))d\delta, & \nu \in (\delta_i, \nu_{i+1}], i = 1, 2, \dots, \wp, \end{cases}$$

and $(\mathcal{H}\Psi)(\nu) =$

$$\begin{cases} \Omega_\beta(\nu, 0)\int_{\delta_\wp}^{T}\Omega_\beta(T, \delta)f\left(\delta, \Psi(\delta), \int_0^\delta \theta(\delta, \sigma, \Psi(\sigma))d\sigma, \int_0^T \theta_1(\delta, \sigma, \Psi(\sigma))d\sigma\right)d\delta \\ +\int_0^\nu \Omega_\beta(\nu, \delta)f\left(\delta, \Psi(\delta), \int_0^\delta \theta(\delta, \sigma, \Psi(\sigma))d\sigma, \int_0^T \theta_1(\delta, \sigma, \Psi(\sigma))d\sigma\right)d\delta, \nu \in [0, \nu_1] \\ 0,\ \nu \in (\nu_i, \delta_i], i = 1, 2, \dots, \wp \\ \int_{\delta_i}^{\nu} \Omega_\beta(\nu, \delta)f\left(\delta, \Psi(\delta), \int_0^\delta \theta(\delta, \sigma, \Psi(\sigma))d\sigma, \int_0^T \theta_1(\delta, \sigma, \Psi(\sigma))d\sigma\right)d\delta, \\ \nu \in (\delta_i, \nu_{i+1}], i = 1, 2, \dots, \wp. \end{cases}$$

Let us fix $\mathtt{C}^\star > 0$ such that

$$\mathtt{C}^{\star} \geq \max\left\{ \frac{\Im^2 \left\| h_{\wp} \right\| + \Im^2 \varpi_{\wp} \left(\delta_{\wp} - \nu_{\wp} \right) + \Im^2 \int_{\delta_{\wp}}^{T} a_1(\delta) d\delta + \Im \int_0^{\nu_1} a_1(\delta) d\delta}{1-\vartheta}, \right.$$

$$\left. \left\| h_i \right\| + \Im \varpi_i \left(\delta_i - \nu_i \right), \frac{\Im \left\| h_i \right\| + \Im \varpi_i \left(\delta_i - \nu_i \right) + \Im \int_{\delta_i}^{\nu_{i+1}} a_1(\delta) d\delta}{1-\vartheta} \right\},$$

where $a_1(\delta) = a(\delta) + L_2(\delta) \int_0^T b(\sigma) d\sigma + L_3(\delta) \int_0^T c(\sigma) d\sigma$.

Let the set $B_{\mathtt{C}^{\star}} = \left\{ \Psi \in \mathrm{PC}(J, \Upsilon) : \|\Psi\|_{\mathrm{PC}} \leq \mathtt{C}^{\star} \right\}$, $\forall\ \Psi \in B_{\mathtt{C}^{\star}}$. From conditions **(B1)** and **(B2)**, for all $\delta \in (\delta_i, \nu_{i+1}], i = 0, 1, \ldots, \wp$, one can find that

$$\begin{aligned}
&\left\| f\left(\delta, \Psi(\delta), \int_0^{\delta} \theta(\delta, \sigma, \Psi(\sigma)) d\sigma, \int_0^T \theta_1(\delta, \sigma, \Psi(\sigma)) d\sigma \right) \right\| \\
\leq & a(\delta) + L_1(\delta) \mathtt{C}^{\star} + L_2(\delta) \int_0^{\delta} \left(b(\sigma) + L_4(\sigma) \mathtt{C}^{\star} \right) d\sigma \\
& + L_3(\delta) \int_0^{\delta} \left(c(\sigma) + L_5(\sigma) \mathtt{C}^{\star} \right) d\sigma \\
\leq & a(\delta) + L_2(\delta) \int_0^T b(\sigma) d\sigma + L_3(\delta) \int_0^T c(\sigma) d\sigma \\
& + \left(L_1(\delta) + L_2(\delta) \int_0^T L_4(\sigma) d\sigma + L_3(\delta) \int_0^T L_5(\sigma) d\sigma \right) \mathtt{C}^{\star} \\
= & a_1(\delta) + b_1(\delta) \mathtt{C}^{\star}.
\end{aligned}$$

Obviously, $a_1(\delta)$ and $b_1(\delta)$ are Lebesgue integrable nonnegative functions.

Using condition (B3) and the aforementioned inequities, we can determine the following for any $\nu \in [0, \nu_1]$:

$$\begin{aligned}
& \|(\mathcal{F}\Psi)(\nu)\| \\
\leq & \left\| \Omega_{\beta}(\nu, 0) \right\| \left(\left\| \Omega_{\beta}\left(T, \delta_{\wp} \right) h_{\wp} \right\| + \left\| \Omega_{\beta}\left(T, \nu_{\wp} \right) \right\| \left\| \int_{\nu_{\wp}}^{\delta_{\wp}} g_{\wp}(\delta, \Psi(\delta)) d\delta \right\| \right. \\
& \left. + \left\| \int_{\delta_{\wp}}^T \Omega_{\beta}(T, \delta) f\left(\delta, \Psi(\delta), \int_0^{\delta} \theta(\delta, \sigma, \Psi(\sigma)) d\sigma, \int_0^T \theta_1(\delta, \sigma, \Psi(\sigma)) d\sigma \right) d\delta \right\| \right) \\
& + \left\| \int_0^{\nu} \Omega_{\beta}(\nu, \delta) f\left(\delta, \Psi(\delta), \int_0^{\delta} \theta(\delta, \sigma, \Psi(\sigma)) d\sigma, \int_0^T \theta_1(\delta, \sigma, \Psi(\sigma)) d\sigma \right) d\delta \right\|
\end{aligned}$$

$$\leq \Im^2 \|h_\wp\| + \Im^2 \varpi_\wp (\delta_\wp - \nu_\wp) + \Im^2 \int_{\delta_\wp}^T \left(a_1(\delta) + b_1(\delta)\mathtt{C}^\star\right) d\delta$$
$$+ \Im \int_0^\nu \left(a_1(\delta) + b_1(\delta)\mathtt{C}^\star\right) d\delta$$
$$\leq \Im^2 \|h_\wp\| + \Im^2 \varpi_\wp (\delta_\wp - \nu_\wp) + \Im^2 \int_{\delta_\wp}^T a_1(\delta) d\delta + \Im \int_0^{\nu_1} a_1(\delta) d\delta + \vartheta \mathtt{C}^\star$$
$$\leq \mathtt{C}^\star.$$

For any $\nu \in (\nu_i, \delta_i], i = 1, 2, \ldots, \wp$, we have

$$\begin{aligned}\|(\mathcal{F}\Psi)(\nu)\| &\leq \|h_i\| + \left\|\Omega_\beta(\nu, \nu_i) \int_{\nu_i}^\nu g_i(\delta, \Psi(\delta)) d\delta\right\| \\ &\leq \|h_i\| + \Im \int_{\nu_i}^\nu \|g_i(\delta, \Psi(\delta))\| d\delta \\ &\leq \|h_i\| + \Im \varpi_i (\delta_i - \nu_i) \leq \mathtt{C}^\star.\end{aligned}$$

For any $\nu \in (\delta_i, \nu_{i+1}], i = 1, 2, \ldots, \wp$, we have

$$\|(\mathcal{F}\Psi)(\nu)\|$$
$$\leq \|\Omega_\beta(\nu, \delta_i) h_i\| + \left\|\Omega_\beta(\nu, \nu_i) \int_{\nu_i}^{\delta_i} g_i(\delta, \Psi(\delta)) d\delta\right\|$$
$$+ \left\|\int_{\delta_i}^\nu \Omega_\beta(\nu, \delta) f\left(\delta, \Psi(\delta), \int_0^\delta \theta(\delta, \sigma, \Psi(\sigma)) d\sigma, \int_0^T \theta_1(\delta, \sigma, \Psi(\sigma)) d\sigma\right) d\delta\right\|$$
$$\leq \Im \|h_i\| + \Im \varpi_i (\delta_i - \nu_i) + \Im \int_{\delta_i}^\nu \left(a_1(\delta) + b_1(\delta)\mathtt{C}^\star\right) d\delta$$
$$\leq \Im \|h_i\| + \Im \varpi_i (\delta_i - t_i) + \Im \int_{\delta_i}^{\nu_{i+1}} a_1(\delta) d\delta + \vartheta \mathtt{C}^\star \leq \mathtt{C}^\star.$$

From the above inequities, we conclude $\mathcal{F}\Psi = \mathcal{G}\Psi + \mathcal{H}\Psi \in B_{\mathtt{C}^\star}$.

Here, we show that the operator $\mathcal{G}$ be a contraction on $B_{\mathtt{C}^\star}$. By **(B4)**, for $\Psi, \Psi' \in B_{\mathtt{C}^\star}$, for any $\nu \in [0, \nu_1]$, we get

$$\begin{aligned}\left\|(\mathcal{G}\Psi)(\nu) - \left(\mathcal{G}\Psi'\right)(\nu)\right\| &\leq \Im^2 \int_{\nu_\wp}^{\delta_\wp} \left\|g_\wp(\delta, \Psi(\delta)) - g_\wp\left(\delta, \Psi'(\delta)\right)\right\| d\delta \\ &\leq \Im^2 L_{g_\wp} \left\|\Psi - \Psi'\right\|_{\mathrm{PC}} (\delta_\wp - \nu_\wp).\end{aligned}$$

For any $\nu \in (\nu_i, \delta_i], i = 1, 2, \ldots, \wp$, we get

$$\begin{aligned}\left\|(\mathcal{G}\Psi)(\nu) - \left(\mathcal{G}\Psi'\right)(\nu)\right\| &\leq \Im \int_{\nu_i}^\nu \left\|g_i(\delta, \Psi(\delta)) - g_i\left(\delta, \Psi'(\delta)\right)\right\| d\delta \\ &\leq \Im L_{g_i} \left\|\Psi - \Psi'\right\|_{\mathrm{PC}} (\delta_i - \nu_i).\end{aligned}$$

For any $\nu \in (\delta_i, \nu_{i+1}], i = 1, 2, \ldots, \wp$, we get

$$\begin{aligned}\left\|(\mathcal{G}\Psi)(\nu)-\left(\mathcal{G}\Psi'\right)(\nu)\right\| &\leq \Im \int_{\nu_i}^{\delta_i}\left\|g_i(\delta,\Psi(\delta))-g_i\left(\delta,\Psi'(\delta)\right)\right\| d\delta \\ &\leq \Im L_{g_i}\left\|\Psi-\Psi'\right\|_{\mathrm{PC}}\left(\delta_i-\nu_i\right).\end{aligned}$$

From the above inequities with $\vartheta<1$, we have $\|\mathcal{G}\Psi-\mathcal{G}\Psi'\|_{\mathrm{PC}}<\|\Psi-\Psi'\|_{\mathrm{PC}}$. Then $\mathcal{G}$ be a contraction.

We will show that $\mathcal{H}$ is completely continuous on $B_{\mathtt{C}^\star}$.

Then, we claim that $\mathcal{H}$ is continuous applying the arguments employed from Theorem 3.1. In additional, $\mathcal{H}$ is uniformly bounded on $B_{\mathtt{C}^\star}$ since $\|\mathcal{H}\Psi\|_{\mathrm{PC}}\leq \mathtt{C}^\star$. Now, we prove that $\mathcal{H}\left(B_{\mathtt{C}^\star}\right)$ is equicontinuous. Thus, for $\Psi\in B_{\mathtt{C}^\star}, e_1,e_2\in[0,\nu_1]$ with $e_1<e_2$, we have

$$\begin{aligned}&\left\|\left(\mathcal{H}\Psi\right)(e_2)-\left(\mathcal{H}\Psi\right)(e_1)\right\| \\ &\leq \left\|\Omega_\beta\left(e_2,0\right)-\Omega_\beta\left(e_1,0\right)\right\| \Im\int_{\delta_\wp}^{T} \\ &\times\left\|f\left(\delta,\Psi(\delta),\int_0^\delta \theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)\right\| d\delta \\ &+\left\|\int_{e_1}^{e_2}\Omega_\beta\left(e_2,\delta\right)\right. \\ &\times f\left(\delta,\Psi(\delta),\int_0^\delta \theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right) d\delta \\ &+\left\|\int_0^{e_1}\left(\Omega_\beta\left(e_2,\delta\right)-\Omega_\beta\left(e_1,\delta\right)\right)\right. \\ &\times f\left(\delta,\Psi(\delta),\int_0^\delta \theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right) d\delta\Big\|\| \\ &\leq \left\|\Omega_\beta\left(e_2,0\right)-\Omega_\beta\left(e_1,0\right)\right\| \Im\int_{\delta_\wp}^{T}\left(a_1(\delta)+b_1(\delta)\mathtt{C}^\star\right)d\delta \\ &\quad+\Im\int_{e_1}^{e_2}\left(a_1(\delta)+b_1(\delta)\mathtt{C}^\star\right)d\delta \\ &\quad+\sup_{\delta\in[0,\nu_1]}\left\|\Omega_\beta\left(e_2,\delta\right)-\Omega_\beta\left(e_1,\delta\right)\right\|\int_0^{e_1}\left(a_1(\delta)+b_1(\delta)\mathtt{C}^\star\right)d\delta.\end{aligned}$$

For $e_1,e_2\in(\nu_i,\delta_i]$ with $e_1<e_2, i=1,2,\ldots,\wp$, we have

$$\left\|\left(\mathcal{H}\Psi\right)(e_2)-\left(\mathcal{H}\Psi\right)(e_1)\right\|=0.$$

For $e_1,e_2\in(\delta_i,\nu_{i+1}]$ with $e_1<e_2, i=1,2,\ldots,\wp$, we have

$$\begin{aligned}
&\left\|(\mathcal{H}\Psi)(e_2)-(\mathcal{H}\Psi)(e_1)\right\| \\
&\leq \int_{e_1}^{e_2}\left\|\Omega_\beta(e_2,\delta)\right\| \\
&\times\left\|f\left(\delta,\Psi(\delta),\int_0^\delta\theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)\right\|d\delta \\
&\quad+\int_{\delta_i}^{e_1}\left\|\Omega_\beta(e_2,\delta)-\Omega_\beta(e_1,\delta)\right\| \\
&\times\left\|f\left(\delta,\Psi(\delta),\int_0^\delta\theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)\right\|d\delta \\
&\leq \Im\int_{e_1}^{e_2}\left(a_1(\delta)+b_1(\delta)\mathtt{C}^\star\right)d\delta \\
&+\sup_{\delta\in(\delta_i,\nu_{i+1}]}\left\|\Omega_\beta(e_2,\delta)-\Omega_\beta(e_1,\delta)\right\|\int_{\delta_i}^{e_1}\left(a_1(\delta)+b_1(\delta)\mathtt{C}^\star\right)d\delta.
\end{aligned}$$

By Lemma 2.1, the compactness of the resolvent operator $\Omega_\beta(\nu,\delta)$ implies the continuity in the uniform operator topology and together with $a_1(\delta), b_1(\delta)\in L^1\left(J,\mathtt{C}_+\right)$, we infer that $\left\|(\mathcal{H}\Psi)(e_2)-(\mathcal{H}\Psi)(e_1)\right\|\to 0$ as $e_2\to e_1$. Consequently, $\mathcal{H}\left(B_{\mathtt{C}^*}\right)$ is equicontinuous.

Now, we show that $\mathcal{H}\left(B_{\mathtt{C}^*}\right)$ is precompact. For $\nu\in[0,\nu_1], 0<\epsilon<\nu, \Psi\in B_{\mathtt{C}^*}$, we have

$$\begin{aligned}
&(\mathcal{H}_\epsilon\Psi)(\nu) \\
=&\Omega_\beta(\nu,0)\int_{\delta_\wp}^T\Omega_\beta(T,\delta)f\left(\delta,\Psi(\delta),\int_0^\delta\theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)d\delta \\
&+\int_0^{\nu-\epsilon}\Omega_\beta(\nu,\delta)f\left(\delta,\Psi(\delta),\int_0^\delta\theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)d\delta.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\|(\mathcal{H}\Psi)(\nu)-(\mathcal{H}_\epsilon\Psi)(\nu)\right\| \\
&\leq\left\|\int_{\nu-\epsilon}^\nu\Omega_\beta(\nu,\delta)f\left(\delta,\Psi(\delta),\int_0^\delta\theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)d\delta\right\| \\
&\leq\Im\int_{\nu-\epsilon}^\nu\left(a_1(\delta)+b_1(\delta)\mathtt{C}^\star\right)d\delta.
\end{aligned}$$

For $\nu\in(\nu_i,\delta_i], 0<\epsilon<\nu, \Psi\in B_{\mathtt{C}^*}, i=1,2,\ldots,\wp$, define $(\mathcal{H}_\epsilon\Psi)(\nu)=0$. Obviously, $\left\|(\mathcal{H}\Psi)(\nu)-(\mathcal{H}_\epsilon\Psi)(\nu)\right\|=0$.

For $\nu\in(\delta_i,\nu_{i+1}], 0<\epsilon<\nu, \Psi\in B_{\mathtt{C}^*}, i=1,2,\ldots,\wp$, define

$$(\mathcal{H}_\epsilon\Psi)(\nu)=\int_{\delta_i}^{\nu-\epsilon}\Omega_\beta(\nu,\delta)f\left(\delta,\Psi(\delta),\int_0^\delta\theta(\delta,\sigma,\Psi(\sigma))d\sigma,\int_0^T\theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right)d\delta.$$

Thus

$$\begin{aligned}
&\|(\mathcal{H}\Psi)(\nu) - (\mathcal{H}_\epsilon\Psi)(\nu)\| \\
&\leq \left\| \int_{\nu-\epsilon}^{\nu} \Omega_\beta(\nu,\delta) f\left(\delta, \Psi(\delta), \int_0^\delta \theta(\delta,\sigma,\Psi(\sigma))d\sigma, \int_0^T \theta_1(\delta,\sigma,\Psi(\sigma))d\sigma\right) d\delta \right\| \\
&\leq \Im \int_{\nu-\epsilon}^{\nu} \left(a_1(\delta) + b_1(\delta)\complement^\star\right) d\delta.
\end{aligned}$$

Since $\Omega_\beta(\nu,\delta)$ is a resolvent compact operator, thus the set $Y_\epsilon(\nu) = \Big\{\Big(\mathcal{H}_\epsilon \Psi\Big)(\nu) : \Psi \in B_{\complement^*}\Big\}$ is compact relatively in Υ, $\forall\ 0 < \epsilon < t$. Thus $Y(\nu) = \{(\mathcal{H}\Psi)(\nu) : \Psi \in B_{\complement^*}\}$ is totally bounded. Thus, $Y(\nu)$ is compact relatively in Υ, and so, with the help of the Arzela-Ascoli theorem, $\mathcal{H}$ is completely continuous on $B_{\complement^*}$. Therefore, there exists a point for $\mathcal{F} = \mathcal{G} + \mathcal{H}$, which corresponds to a PC-mild solution of model (1) on PC(J,Υ). □

4. **An application**

We present an example of Caputo fractional V-FIDE to test our main findings:

$$\begin{cases}
{}^c\nabla_\nu^\beta \Psi(\hbar,\nu) = \nu\frac{\partial^2}{\partial\hbar^2}\Psi(\hbar,\nu) + \frac{\nu}{5\Im^2(1+\nu^2)}\Psi(\hbar,\nu) \\
\qquad + \int_0^\nu \frac{e^{\nu-\delta}|\Psi(\hbar,\delta)|}{10\Im^2 e^4} d\delta + \int_0^3 \frac{e^{\nu-\delta}|\Psi(\hbar,\delta)|}{5\Im^2(1+\nu^2)} d\delta, \\
\qquad \nu \in [0,1) \cup (2,3], \hbar \in (0,1), \\
\frac{\partial}{\partial\hbar}\Psi(0,\nu) = \frac{\partial}{\partial\hbar}\Psi(1,\nu) = 0, \quad \nu \in [0,1) \cup (2,3], \\
\Psi(\hbar,\nu) = y_1\hbar + \Omega_\beta(\nu,1)\int_1^\nu \frac{|\Psi(\hbar,\delta)|}{10\Im^2(1+\nu)} d\delta, \quad \nu \in (1,2], \hbar \in (0,1), \\
\Psi(0,\nu) = \Psi(3,\nu), \quad \nu \in (0,1),
\end{cases} \tag{3}$$

where $\beta, 0 < \beta < 1, \Upsilon = L^2[0,3], 0 = \nu_0 = \delta_0, \nu_1 = 1, \delta_1 = 2$. The operator $\Xi : \nabla(\Xi) \subset \Upsilon \to \Upsilon$ is given as $\Xi(\nu)(\hbar) = \nu\frac{\partial^2\Psi}{\partial\hbar^2}$, where $\nabla(\Xi) = \{\Psi \in \Upsilon : \Psi'' \in \Upsilon, \Psi(0) = \Psi(1) = 0\}$. It is well known that the operator $\Xi(\nu)$ generates a β-resolvent family $\Omega_\beta(\nu,\delta)$ and $\max_{0\leq\delta<\nu\leq T}\|\Omega_\beta(\nu,\delta)\| \leq \Im, (\Im > 1)$. By setting

$$\begin{aligned}
&\Psi(\nu)(\hbar) = \Psi(\hbar,\nu), \quad h_1\hbar = y_1\hbar, \quad g_1(\nu,\Psi(\nu))(\hbar) = \frac{|\Psi(\hbar,\delta)|}{10\Im^2(1+\nu)}, \\
&f\left(\nu, \Psi(\nu), \int_0^\nu \theta(\nu,\delta,\Psi(\delta))d\delta, \int_0^1 \theta_1(\nu,\delta,\Psi(\delta))d\delta\right)(\hbar) \\
&= \frac{\nu}{5\Im^2(1+\nu^2)}\Psi(\hbar,\nu) + \int_0^\nu \frac{e^{\nu-\delta}|\Psi(\hbar,\delta)|}{10\Im^2 e^4} d\delta + \int_0^3 \frac{e^{\nu-\delta}|\Psi(\hbar,\delta)|}{5\Im^2(1+\nu^2)} d\delta,
\end{aligned}$$

model (3) can be rewritten as the following abstract form:

$$\begin{cases} {}^{c}\nabla_{\nu}^{\beta}\Psi(\nu) = \Xi(\nu)\Psi(\nu) \\ \qquad\qquad +f\left(\nu, \Psi(\nu), \int_0^{\nu} \theta(\nu, \delta, \Psi(\delta))d\delta, \int_0^{3} \theta_1(\nu, \delta, \Psi(\delta))d\delta\right), \\ \Psi(\nu) = h_1 + \Omega_{\beta}(\nu, 1)\int_1^{\nu} g_1(\delta, \Psi(\delta))d\delta, \\ \Psi(0) = \Psi(3). \end{cases} \tag{4}$$

The function $f : J \times T_{\mathtt{C}} \times T_{\mathtt{C}} \times T_{\mathtt{C}} \to \Upsilon$ is bounded and continuous, for every $\mathtt{C} > 0$, such that

$$\lim_{\mathtt{C}\to\infty} \sup \frac{\Im(\mathtt{C})}{\mathtt{C}} < \frac{1}{\Im a_0(\Im + 1)},$$

where $\Im(\mathtt{C}) = \max\left\{\Im_1(\mathtt{C}), \Im_2(\mathtt{C})\right\}$,
$\Im_1(\mathtt{C}) = \sup\{\|f(\nu, \Psi_1, \Psi_2, \Psi_3)\| : (\nu, \Psi_1, \Psi_2, \Psi_3) \in J \times T_{\mathtt{C}} \times T_{\mathtt{C}} \times T_{\mathtt{C}}\}$,
$\Im_2(\mathtt{C}) = \sup\{\|g_1(\nu, \Psi)\|, (\nu, \Psi) \in J \times T_{\mathtt{C}}, \}, T_{\mathtt{C}} = \{\Psi \in \Upsilon : \|\Psi\| \leq \mathtt{C}\}$, $a_0 = \max\{1, h_0\}$. Let

$$\|\theta(\nu, \delta, \Psi)\| \leq \frac{e^{\nu-\delta}}{10\Im^2 e^4}\|\Psi\|,\ \|\theta_1(\nu, \delta, \Psi)\| \leq \frac{e^{\nu-\delta}}{5\Im^2(1+\delta^2)}\|\Psi\|,$$

$$\alpha(\theta(\nu, \delta, \nabla)) \leq e^{\nu}\alpha(\nabla),\ \alpha(\theta_1(\nu, \delta, \nabla)) \leq e^{\nu}\alpha(\nabla),$$

$$\alpha(g_1(\nu, \delta, \nabla)) \leq \frac{1}{10\Im^2(1+\nu)}\alpha(\nabla),$$

$$\alpha(f(\nu, \nabla_1, \nabla_2, \nabla_3)) \leq \frac{\nu}{5\Im^2(1+\nu^2)}\alpha(\nabla_1) + e^{\nu}\alpha(\nabla_2) + e^{\nu}\alpha(\nabla_3),$$

then

$$\begin{aligned} &2\Im^2\int_1^2 \frac{1}{10\Im^2(1+\delta)}d\delta + 2\Im^2\int_2^3\Big(\frac{\delta}{5\Im^2(1+\delta^2)} \\ &\qquad + e^{\delta}\int_0^3 \frac{e^{\delta-\sigma}}{10\Im^2 e^4}d\sigma + e^{\delta}\int_0^3 \frac{e^{\delta-\sigma}}{5\Im^2(1+\delta^2)}d\sigma\Big)d\delta \\ &+ 2\Im\int_0^1\Big(\frac{\delta}{5\Im^2(1+\delta^2)} + e^{\delta}\int_0^3 \frac{e^{\delta-\sigma}}{10\Im^2 e^4}d\sigma \\ &+ e^{\delta}\int_0^3 \frac{e^{\delta-\sigma}}{5\Im^2(1+\delta^2)}d\sigma\Big)d\delta \\ &< \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = 1, \end{aligned}$$

$$\begin{aligned} &2\Im\int_1^2 \frac{1}{10\Im^2(1+\delta)}d\delta + 2\Im\int_2^3 \\ &\times\left(\frac{\delta}{5\Im^2(1+\delta^2)} + e^{\delta}\int_0^3 \frac{e^{\delta-\sigma}}{10\Im^2 e^4}d\sigma + e^{\delta}\int_0^3 \frac{e^{\delta-\sigma}}{5\Im^2(1+\delta^2)}d\sigma\right)d\delta \\ &< \frac{1}{5\Im} + \frac{1}{5\Im} + \frac{1}{5\Im} + \frac{1}{5\Im} < 1. \end{aligned}$$

We have

$$
\begin{aligned}
\rho = \max\Big\{ & 2\Im^2 \int_1^2 \frac{1}{10\Im^2(1+\delta)} d\delta + 2\Im^2 \int_2^3 \Big(\frac{\delta}{5\Im^2\,(1+\delta^2)} \\
& + e^{\delta} \int_0^3 \frac{e^{\delta-\sigma}}{10\Im^2 e^4} d\sigma + e^{\delta} \int_0^3 \frac{e^{\delta-\sigma}}{5\Im^2\,(1+\delta^2)} d\sigma \Big) d\delta \\
& + 2\Im \int_0^1 \left(\frac{\delta}{5\Im^2\,(1+\delta^2)} + e^{\delta} \int_0^3 \frac{e^{\delta-\sigma}}{10\Im^2 e^4} d\sigma + e^{\delta} \int_0^3 \frac{e^{\delta-\sigma}}{5\Im^2\,(1+\delta^2)} d\sigma \right) d\delta, \\
& 2\Im \int_1^2 \frac{1}{10\Im^2(1+\delta)} d\delta + 2\Im \int_2^3 \Big(\frac{\delta}{5\Im^2\,(1+\delta^2)} \\
& + e^{\delta} \int_0^3 \frac{e^{\delta-\sigma}}{10\Im^2 e^4} d\sigma + e^{\delta} \int_0^3 \frac{e^{\delta-\sigma}}{5\Im^2\,(1+\delta^2)} d\sigma \Big) d\delta \Big\} < 1.
\end{aligned}
$$

Since issue (4) meets the requirements of Theorem 3.1, it has a PC-mild solution, which implies that model (3) also has a mild solution.

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