International Journal of Applied Mathematics

**Volume 37 No. 2 2024, 205–215**
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v37i2.6

# EXPLICIT SOLUTION OF TIME FRACTIONAL NONLOCAL HEAT EQUATION

**Yulian T. Tsankov**

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridsky"
"J. Bouchier" Bul., No 5, Sofia - 1164, BULGARIA

## Abstract

We study a one-dimensional fractional diffusion equation with the Caputo time-derivative of order $\mu \in (0,1]$. Applying spectral projectors, we find a series solution of the problem for a special choice of the initial function. Then, using operational calculus approach of Dimovski, we obtain an explicit representation of the solution in the general case. The expression obtained contains a non-classical convolution product of the particular solution and an arbitrary initial function. This result is an extension of the classical Duhamel principle, but for the space variable.

**MSC 2020**: 26A33, 34A08, 35R11, 34B10

**Key Words and Phrases**: time-fractional diffusion equation, Caputo fractional derivative, nonlocal spectral problem, spectral projector, non-classical convolution, operational calculus, Duhamel principle

## 1. Introduction

We consider the following initial-boundary value problem:

$$D_t^{\mu} u(x,t) - u_{xx}(x,t) = F(x,t), \quad 0 < x < 1, 0 < t, \tag{1}$$

$$u(x,0) = f(x), \quad 0 \le x \le 1, \tag{2}$$

$$u(0,t) = 0, \quad 0 \le t, \tag{3}$$

$$u_x(0,t) = u_x(1,t) + \alpha u(1,t), \quad 0 \le t. \tag{4}$$

Received: September 25, 2023

Here $D_t^\mu$ denotes the Caputo fractional derivative of order $\mu \in (0,1]$, (4) is a a nonlocal boundary condition involving a real parameter $\alpha \neq 0$. Problem (1-4) with $\mu = 1$ is introduced by Mokin in [9] and [10]. In [10] a representation of the solution in series is found. In Ali and Malik [8] it is obtained a series solution of problem (1-4) with fractional derivative of order $0 < \mu < 1$. Bazhlekova and Dimovski obtained explicit solution of Thornley's problem for one-dimensional fractional diffusion equation with the Caputo time-derivative of order $\mu \in (0,1]$ and nonlocal boundary condition in [2]. Here we use the operational calculus approach of Dimovski, we obtain Duhamel representation of the solution.

## 2. **Preliminaries**

Let

$$\phi_\mu = \frac{t^{\mu-1}}{\Gamma(\mu)}, \quad t > 0, \quad \mu > 0,$$

where $\Gamma(\mu)$ is the Gamma function.

The operation

$$(f \overset{t}{*} g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau, \quad f, g \in C[0,\infty), \tag{5}$$

bears the name of Duhamel convolution, but sometimes it is called either Borel, or Laplace convolution. The Riemann-Liouville fractional integral $l_t^\mu$ is defined on $C_{-1}(\bar{I})$ by

$$l_t^\mu f(t) = (\phi_\mu \overset{t}{*} f)(t). \tag{6}$$

The Caputo fractional derivative of order $\mu > 0$ is defined by

$$D_t^\mu g(t) = \begin{cases} (\phi_{m-\mu} \overset{t}{*} g^{(m)})(t), & m-1 < \mu < m, m \in \mathbb{N} \\ g^{(m)}(t), & \mu = m. \end{cases}$$

The Caputo derivative $D_t^\mu$ is a left inverse of $l_t^\mu$ : $D_t^\mu l_t^\mu f(t) = f(t)$, but in general, it is not a right inverse, since

$$l_t^\mu D_t^\mu f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0)\phi_{k+1}(t), \quad m-1 < \mu < m. \tag{7}$$

For more details on fractional calculus operators, see e.g. [3], [1], [6], [7]

The Mittag-Leffler function is the entire function defined by

$$E_\mu(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + 1)}, \quad \mu > 0, \ z \in \mathbb{C}.$$

The Cauchy problem for the ordinary fractional differential equation

$$D_t^\mu y(t) = \lambda y(t), \quad y(0) = y_0, \quad \mu \in (0,1], \ t > 0. \tag{8}$$

has a unique solution given by

$$y(t) = y_0 \, E_\mu(\lambda t^\mu). \tag{9}$$

## 3. **One-dimensional spectral problem, connected with BVP (1) - (4)**

In order to find the a series solution of (1) - (4), we are to consider the following non-local eigenvalue problem in $C^2[0,1]$:

$$\begin{aligned} &\frac{d^2}{dx^2}y(x) + \lambda^2 y(x) = 0, \quad 0 < x < 1, \\ &y(0) = 0, \qquad \Phi_\xi\{y(\xi)\} = 0, \end{aligned} \tag{10}$$

where

$$\Phi_\xi\{y(\xi)\} = \frac{1}{\alpha}(\alpha y(1) + y'(1) - y'(0)) = 0. \tag{11}$$

The sine indicatrix of the functional $\Phi$ is

$$E(\lambda) = \Phi_\xi\left\{\frac{\sin\lambda\xi}{\lambda}\right\} = \frac{1}{\alpha}\left(\alpha\frac{\sin\lambda}{\lambda} + \cos\lambda - 1\right). \tag{12}$$

We obtain

$$\begin{aligned} E(\lambda) &= \frac{2}{\alpha}\sin\frac{\lambda}{2}\left(\frac{\alpha}{\lambda}\cos\frac{\lambda}{2} - \sin\frac{\lambda}{2}\right), \\ E'(\lambda) &= \frac{\alpha\lambda\cos\lambda - (\alpha+\lambda^2)\sin\lambda}{\alpha\lambda^2}. \end{aligned}$$

The zeros of $\sin\frac{1}{2}\lambda = 0$ are $\lambda_k = 2k\pi$, $k = 1,2,3,....$ The zeros $\gamma_n$ of $\tan\frac{1}{2}\lambda = \frac{\alpha}{\lambda}$ satisfy $2\pi n < \gamma_n < 2\pi n + \pi$, $n = 0,1,2,....$ (see [9] Lemma 1). All the zeros are simple and coresponding eigenfunctions are $\sin\lambda_k x$, $k = 1,2,3,...$ and $\sin\gamma_n x$, $n = 0,1,2,....$ (Remark: 0 is not an eigenvalue.)

3.1. **The spectral projectors.** Let us consider the eigenvalues $-\lambda_k^2$, $k = 1,2,....$ Then the spectral Riesz' projectors $P_{\lambda_k} : C[0,1] \to \mathcal{E}_{\lambda_k} = \operatorname{span}\{\sin\lambda_k x\}$ are

$$\begin{aligned} P_{\lambda_k}\{f\} &= \frac{1}{\pi i}\int_{\Gamma_{\lambda_k}} R_{-\mu^2} f(x)\mu d\mu \\ &= -\frac{2}{\alpha\ \lambda_k\ E'(\lambda_k)}\left(\int_0^1 \lambda_k\cos((1-\eta)\lambda_k)f(\eta)d\eta\right. \\ &\qquad \left. + \alpha\int_0^1 \sin((1-\eta)\lambda_k)f(\eta)d\eta\right)\sin\lambda_k x, \end{aligned} \tag{13}$$

where $\Gamma_{\lambda_k}$ is a simple contour containing the zero $\lambda_k$ only.

We have the same representation for $P_{\gamma_n}\{f\}$, but in (13) we have replaced $\lambda_k$ with $\gamma_n$, and $n = 0, 1, 2, ....$

Let $u_k(x,t) = P_{\lambda_k}\{u(x,t)\} = A_k(t)\sin\lambda_k x,\ k = 1, 2, 3, ...,$ where $P_{\lambda_k}$ acts with respect to $x$. Analogically $u_n(x,t) = P_{\gamma_n}\{u(x,t)\} = B_n(t)\sin\gamma_n x$, $n = 0, 1, 2, ...,$ where $P_{\gamma_n}$ acts with respect to $x$.

Definition 3.1. Let $f \in C[0,1]$. The formal spectral expansion of $f(x)$ for eigenvalue problem (10) is the correspondence

$$f(x) \sim \sum_{k=1}^{\infty} P_{\lambda_k} + \sum_{n=0}^{\infty} P_{\gamma_n}. \tag{14}$$

This formal spectral expansion, in fact, is not completely formal, since it has the uniqueness property: if $P_{\lambda_k}\{f\} = 0$ for $k = 1, 2, ...$ and $P_{\gamma_n}\{f\} = 0$ for $n = 0, 1, 2, ...,$ then $f \equiv 0$. This follows immediately from a theorem of Bozhinov (see [11]). In general, it is not supposed the series in (14) to be convergent. If additionally it happens this series to be uniformly convergent on $[0,1]$, then $f(x) = \sum\limits_{k=1}^{\infty} P_{\lambda_k} + \sum\limits_{n=0}^{\infty} P_{\gamma_n}$. A sufficient condition for absolute uniform convergence of the series is: $f \in C^2[0,1]$ with $f(0) = 0$ and $\Phi_\xi\{f(\xi)\} = \dfrac{1}{\alpha}(\alpha f(1) + f'(1) - f'(0)) = 0$. Indeed, then, after integrating by parts in (13) one obtains: $|P_{\lambda_k}\{f\}| \le \dfrac{M_1}{k^2}$, $k >> 0$, $|P_{\gamma_n}\{f\}| \le \dfrac{M_2}{n^2}$, $n >> 0$, where $M_1$, $M_2$ are constants nondepending on $k$ and $n$. This ensures, absolute and uniform convergence of the spectral expansion (14) of $f$ and thus its sum to be a function from $C[0,1]$. Applying the uniqueness property, it follows that the sum of the series (14) is exactly $f(x)$, since their spectral projectors coincide.

## 4. **Nonclassical convolutions**

Theorem 4.1. (see Dimovski [4], p. 119) *Let $f, g \in C[0,1]$, then the operation*

$$(f * g)(x) = -\frac{1}{2}\Phi_\xi\left\{\int_0^\xi h(x,\zeta)d\zeta\right\} \tag{15}$$

*with*

$$h(x,\zeta)$$

$$= \int_x^\zeta f(\zeta + x - \eta)g(\eta)d\eta - \int_{-x}^\zeta f(|\zeta - x - \eta|)g(|\eta|)\mathrm{sgn}(\eta(\zeta - x - \eta))d\eta$$

*is a bilinear, commutative and associative operation on* $C[0,1]$ *such that*

$$R_{-\lambda^2} f = \left\{ \frac{\sin \lambda x}{\lambda E(\lambda)} \right\} * f \tag{16}$$

*and* $Lf(x) = \{x\} * f$.

In [4] the corresponding theorem is stated for an arbitrary linear functional $\Phi$ in $C^1[0,1]$. Next we will combine both the Duhamel convolution (5) and the Dimovski convolution (15) into a two-dimensional convolution in $C(G) = C([0,1] \times [0,\infty))$.

Theorem 4.2. *Let* $u, v \in C(G)$. *Then the operation*

$$(u \overset{x,t}{*} v)(x,t) = \int_0^t u(x, t-\tau) \overset{x}{*} v(x,\tau) d\tau,$$

*is a bilinear, commutative and associative operation in* $C(G)$, *such that*

$$l_t^{\mu} L_x u(x,t) = \{x\phi_\mu\} \overset{x,t}{*} u(x,t), \tag{17}$$

*where* $\phi_\mu = \dfrac{t^{\mu-1}}{\Gamma(\mu)}$, $t > 0$, $\mu > 0$.

For a proof, see [5] or [3].

## 5. **Ring of the multiplier fractions**

We consider the convolution algebra $(C(G), \overset{x,t}{*})$. Our direct operational calculus approach is based on the notion of a multiplier of the convolution algebra $(C(G), \overset{x,t}{*})$ (see Larsen [13]).

Definition 5.1. (Larsen [13]) An operator $M : C(G) \times C(G)$ is said to be a multiplier of the convolution algebra $(C(G), \overset{x,t}{*})$ iff $M(u \overset{x,t}{*} v) = (Mu) \overset{x,t}{*} v$ for all $u, v \in C(G)$.

Here we will remind only some specific notations. The multipliers of the form $\{u(x,t)\} \overset{x,t}{*}$ will be denoted by $\{u\}$ or $u$ and the result of the application of the operator $u \overset{x,t}{*}$ to a function $F \in C(D)$ will be denoted simply by $\{u\}F$ or $uF$.

Lemma 5.1. *Let $f$ be a function from $C[0,1]$. The convolution operator $f\overset{x}{*}$, defined in $C(G)$ by $(f\overset{x}{*})u = f\overset{x}{*}u$ is a multiplier of the convolution algebra $(C(G),\overset{x,t}{*})$.*

Lemma 5.2. *Let $\varphi$ be a function from $C[0,\infty)$. The convolution operator $\varphi\overset{t}{*}$, defined in $C(G)$ by $(\varphi\overset{t}{*})u = \varphi\overset{t}{*}u$ is a multiplier of the convolution algebra $(C(G),\overset{x,t}{*})$.*

For proofs in more general situation, see [5].

Definition 5.2. Let $f = f(x) \in C[0,1]$ and $\varphi = \varphi(t) \in C[0,\infty)$, but both are considered as functions of $C(G)$. The operator $[f]_t$ defined by $[f]_t u = f\overset{x}{*}u$ is said to be a *partial numerical operator* with respect to $t$ and the operator $[\varphi]_x u = \varphi\overset{t}{*}u$ is said to be *partial numerical operator* with respect to $x$.

In these notations, we have $L_x = [x]_t$ and $l_t^\mu = [\phi_\mu(t)]_x$.

The notion of numerical operator for Duhamel convolution is introduced in [14].

Lemma 5.3. (Larsen [13]) *The set of all multipliers of the convolution algebra $(C(G),\overset{x,t}{*})$ is a commutative ring $\mathfrak{M}$.*

The multiplicative set $\mathfrak{N}$ of the non-zero non-divisors of 0 in $\mathfrak{M}$ is non-empty, since at least the operators $\{x\}\overset{x}{*} = [x]_t$ and $\{\phi_\mu\}\overset{t}{*} = [\phi_\mu]_x$ are non-divisors of 0.

Next we introduce the ring $\mathcal{M} = \mathfrak{N}^{-1}\mathfrak{M}$ of the multiplier fractions of the form $\dfrac{A}{B}$ where $A \in \mathfrak{M}$ and $B \in \mathfrak{N}$. The standard algebraic procedure of constructing of this ring, named "localization", is described, e.g. in Lang [12]. Basic for our construction are the algebraic inverses $S_x = \dfrac{1}{L_x}$ and $s_t^\mu = \dfrac{1}{l_t^\mu}$ in $\mathcal{M}$, of the multipliers $L_x$ and $l_t^\mu$ correspondingly. If $u \in C^2(G)$, then $u_{xx}$ and $D_t^\mu u$ are connected with $S_x u$ and $s_t^\mu u$ but they in general, are different from them.

LEMMA 5.4. *Let $u_{xx}$, $D_t^{\mu}u$ be continuous on $G$. Then*

$$u_{xx} = S_x u + S_x\{(x\Phi\{1\} - 1)u(0,t)\} - [\Phi_\xi\{u(\xi,t)\}]_x, \tag{18}$$

$$D_t^{\mu} u(x,t) = s_t^{\mu} - [u(x,0)]_t s_t^{\mu-1} = s_t^{\mu}(u(x,t) - u(x,0)). \tag{19}$$

P r o o f. The relation (19) is similar to a corresponding relation in Mikusinski [14] and its proof in [3]. Let us prove (18). It is easy to verify the identity

$$L_x\{u_{xx}\} = u(x,t) + (x\Phi\{1\} - 1)u(0,t) - x\Phi_\xi\{u(\xi,t)\}.$$

It remains to multiply it by $S_x$ and to use $S_x\{x\} = S_x L_x = 1$, in order to get (18). □

## 6. Algebraization of boundary value problem (1) - (4)

Relations (18) and (19) allow to reduce both the equation (1) and BVCs (2) - (4) to a single linear algebraic equation for $u$. Indeed, substituting $D_t^{\mu}u(x,t)$ and $u_{xx}$ from (18) and (19) in the equation $D_t^{\mu}u(x,t) - u_{xx}(x,t) = F(x,t)$, we get

$$s_t^{\mu} u - [u(x,0)]_t s_t^{\mu-1} - S_x u$$
$$-S_x\{(x\Phi\{1\} - 1)u(0,t)\} + \Phi_\xi\{u(\xi,t)\} = F(x,t).$$

Now using initial condition (2) and boundary value conditions (3) and (4), we obtain

$$s_t^{\mu} u - [f(x)]_t s_t^{\mu-1} - S_x u = F(x,t),$$

$$(s_t^{\mu} - S_x)u = [f(x)]_t s_t^{\mu-1} + \{F(x,t)\}. \tag{20}$$

Thus, we reduced BVP (1) - (4) to the single linear algebraic equation (20) for $u$ in $\mathcal{M}$. It is reasonable to introduce the notation of a *weak solution* of BVP (1) - (4).

DEFINITION 6.1. A function $u \in C(G)$ is said to be a weak solution of BVP (1) - (4), if it is a solution of (20).

Let us consider the problem of uniqueness of the solution of (1) - (4). Equation (20) reduces it to the algebraic question, whether $s_t - S_x$ is a divisor of zero in $\mathcal{M}$ or not.

THEOREM 6.1. *The element $s_t^{\mu} - S_x$ is a nondivisor of zero in $\mathcal{M}$.*

P r o o f. Assume the contrary, i.e. that there exists a non-zero multiplier fraction $\frac{A}{B} \neq 0$ with $(s_t^{\mu} - S_x)\frac{A}{B} = 0$. The last relation is equivalent to $(s_t^{\mu} - S_x)A = 0$. Since $A \neq 0$, then there exist a function $v \in C(D)$ such that $Av = u \neq 0$. Then $(s_t^{\mu} - S_x)A = 0$ implies $(s_t^{\mu} - S_x)u = 0$. We multiply the last equation by $l_t^\mu$ and obtain $u - S_x l_t^\mu u = 0$. If $t = 0$, then $u(x, 0) = 0$. Next we multiply $(s_t^\mu - S_x)u = 0$ by $\psi_k(x)$ and obtain $(s_t^\mu - S_x)v_k = 0$, where $v_k(x,t) = \alpha_k(t)\sin\mu_k x$, $k = 1, 2, \ldots$. We find

$$s_t^\mu\{\alpha_k(t)\sin\mu_k x\} - S_x\{\alpha_k(t)\sin\mu_k x\} = 0.$$

Using (18) and (19) we find

$$D_t^\mu \alpha_k(t)\sin\mu_k x + \mu_k^2\alpha_k(t)\sin\mu_k x + \alpha_k(t)\Phi_\xi\{\sin\mu_k\xi\} = 0.$$

But $\Phi_\xi\{\sin\mu_k\xi\} = 0$ and we get

$$D_t^\mu\alpha_k(t) + \mu_k^2\alpha_k(t) = 0, \quad \alpha_k(0) = 0.$$

This initial problem has unique solution $\alpha_k(t) = 0$. We find $u(x,t) = 0$. □

The solution of (20) in $\mathcal{M}$ is

$$u = \frac{s_t^{\mu-1}}{s_t^\mu - S_x}[f(x)]_t + \frac{1}{s_t^\mu - S_x}\{F(x,t)\}. \tag{21}$$

We may call (21) the formal (generalized) solution of problem (1)- (4).

## 7. Interpretation of the formal (generalized) solution of (1)- (4) as a function

Our next task is to interpret (if possible) (21) as a function of $C(D)$. To this end, we consider a special case of problem (1)- (4) for $F(x,t) \equiv 0$ and $f(x) = L_x\{x\} = L_x^2 = \dfrac{1}{S_x^2} = \dfrac{x^3}{6} - \dfrac{x(3+\alpha)}{6\alpha}$. We denote its solution, if it exists, by $\Omega = \Omega(x,t)$. Having in mind that $L_x\{x\} = L_x^2 = \dfrac{1}{S_x^2}$, we have the following algebraic representation of this solution:

$$\Omega = \frac{s_t^{\mu-1}}{s_t^\mu - S_x}\left[\frac{x^3}{6} - \frac{x(3+\alpha)}{6\alpha}\right]_t = \frac{s_t^{\mu-1}}{S_x^2(s_t^\mu - S_x)}. \tag{22}$$

As for the special solution $\Omega(x,t)$, it can be found in an explicit series form, using the spectral projectors (13). Thus we obtain the following result.

Lemma 7.1. *If $f(x) = \dfrac{x^3}{6} - \dfrac{x(3+\alpha)}{6\alpha}$ and $F(x,t) \equiv 0$, then the solution $\Omega(x,t)$ of the BVP (1)-(4) is:*

$$\Omega(x,t) = \sum_{k=1}^\infty U_k(x,t) + \sum_{n=0}^\infty V_n(x,t), \tag{23}$$

*where*

$$U_k(x,t) = E_\mu(-\lambda_k^2 t^\mu)\frac{2\sin\lambda_k x}{\lambda_k^4 E'(\lambda_k)},$$

$$V_n(x,t) = E_\mu(-\gamma_n^2 t^\mu)\frac{2\sin\gamma_n x}{\gamma_n^4 E'(\gamma_n)},$$

$k = 1, 2, ..., n = 0, 1, ....$

The proof may be accomplished by a direct check, too.

The generalized solution of problem (1) - (4) for arbitrary $f(x)$ and $F(x,t)$ can be written in the form:

$$\begin{aligned} u &= \frac{s_t^{\mu-1}}{s_t^\mu - S_x}[f(x)]_t + \frac{1}{s_t^\mu - S_x}\{F(x,t)\} \\ &= S_x^2 \frac{s_t^{\mu-1}}{S_x^2(s_t^\mu - S_x)}[f(x)]_t + \frac{S_x^2}{s_t^{\mu-1}}\frac{s_t^{\mu-1}}{S_x^2(s_t^\mu - S_x)}F(x,t). \end{aligned} \tag{24}$$

By virtue of (22) and (18), we obtain for the first term in (24)

$$S_x^2 \frac{s_t^{\mu-1}}{S_x^2(s_t^\mu - S_x)}[f(x)]_t = S_x^2\ \Omega\ [f(x)]_t = \frac{\partial^4}{\partial x^4}\left[\Omega \overset{x}{*} f(x)\right].$$

Now we can rewrite the second term in (24) as follows:

$$\begin{aligned} \frac{S_x^2}{s_t^{\mu-1}}\frac{s_t^{\mu-1}}{S_x^2(s_t^\mu - S_x)}F(x,t) &= S_x^2 \frac{1}{s_t^{\mu-1}}\Omega F(x,t) = S_x^2 \frac{s_t}{s_t^\mu}\Omega F(x,t) \\ &= l_t^\mu \frac{\partial^4}{\partial x^4}\left[\frac{\partial}{\partial t}(\Omega \overset{x,t}{*} F)(x,t) + (\Omega \overset{x,t}{*} F)(x,t)\big|_{t=0}\right] \\ &= l_t^\mu \frac{\partial^4}{\partial x^4}\frac{\partial}{\partial t}(\Omega \overset{x,t}{*} F)(x,t). \end{aligned}$$

Under the corresponding assumptions for smoothness of the functions $f(x)$ and $F(x,t)$, the solution can be written as a function of the form

$$u = \frac{\partial^4}{\partial x^4}\left[\Omega \overset{x}{*} f(x) + \frac{\partial}{\partial t}(\phi_\mu \overset{t}{*} \Omega \overset{x,t}{*} F(x,t))\right]. \tag{25}$$

Then (25) gives the following Duhamel-type representation of the solution of (1) - (4).

Theorem 7.1. *I) If $f(x) \in C^2[0,1]$, $f(0) = 0$ and $\Phi_\xi\{f(\xi)\} = \frac{1}{\alpha}(\alpha f(1) + f'(1) - f'(0)) = 0$, then*

$$\begin{aligned} u &= \frac{\partial^4}{\partial x^4}(\Omega(x,t) \overset{x}{*} f(x)) \\ &= \frac{1}{2\alpha}\int_0^1 \Big(\alpha\big(\Omega_x(1-x-\eta,t) - \Omega_x(1+x-\eta,t)\big) \\ &\qquad + \Omega_{xx}(1-x-\eta,t) - \Omega_{xx}(1+x-\eta,t)\Big) f''(\eta)d\eta \end{aligned}$$

*is a weak solution of* (1) -(4) *for* $F(x,t) \equiv 0$.

*II) If* $F_{xx}(x,t) \in C(G)$, $F(0,t) = 0$ *and*

$$\Phi_\xi\{F(\xi,t)\} = \frac{1}{\alpha}(\alpha F(1,t) + F_x(1,t) - F_x(0,t)) = 0,$$

*then*

$$\begin{aligned} u &= \frac{\partial^4}{\partial x^4}\frac{\partial}{\partial t}(\phi_\mu \overset{t}{*} \Omega \overset{x,t}{*} F(x,t)) \\ &= \frac{1}{2\alpha}\frac{\partial}{\partial t}\int_0^1 \Big(\alpha\big(\Omega_x(1-x-\eta,t) - \Omega_x(1+x-\eta,t)\big) \\ &\quad + \Omega_{xx}(1-x-\eta,t) - \Omega_{xx}(1+x-\eta,t)\Big) \overset{t}{*} F_{xx}(\eta,t) \overset{t}{*} \phi_\mu(t)d\eta \end{aligned}$$

*is a weak solution of* (1) -(4) *for* $f(x) \equiv 0$.

The proof may be accomplished by a direct check, too.

## References


[1] E. Bajlekova, *Fractional Evolution Equations in Banach Spaces* [Ph.D. Thesis], Eindhoven University of Technology, The Netherlands (2001). Available from: http://alexandria.tue.nl/extra2/200113270.pdf.

[2] E. Bazhlekova, I.H. Dimovski, Time-fractional Thornley's problem, *Journal of Inequalities and Special Functions*, **4**, No 1 (2013), 21–35.

[3] E. Bazhlekova, I.H. Dimovski, Exact solution of two-term time-fractional Thornley's problem by operational method *Integral Transforms and Special Functions*, **25**, No 1 (2014), 61-74.

[4] I.H. Dimovski, *Convolutional Calculus*, Kluwer, Dordrecht (1990).

[5] I.H. Dimovski, Nonlocal boundary value problems. In: *Mathematics and Math. Education, Proc. 38th Spring Conf. UBM* (2009), 31-40.

[6] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier, Amsterdam (2006).

[7] Yu. Luchko, R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives, *Acta Math Vietnam*, **24** (1999), 207–233.

[8] M. Ali, S.A. Malik, An inverse problem for a family of time fractional diffusion equations, *Inverse Problems in Science and Engineering*, **25**, No 9 (2017), 1299-1322.

[9] A.Yu. Mokin, On a family of initial-boundary value problems for the heat equation, *Differential Equations*, **45** (2009), 126-141.

[10] A.Yu. Mokin, Application of nonclassical separation of variables to a nonlocal heat problem, *Differential Equations*, **49**, No 1 (2013), 59–67.

[11] N.S. Bozhinov, On theorems of uniqueness and completeness of expansion on eigen and associated eigenfunctions of the nonlocal Sturm-Liouville operator on a finite interval, *Diferenzial'nye Uravnenya*, **26**, No 5 (1990), 741-453 (in Russian).

[12] S. Lang, *Algebra*. Addison - Wesley, Reading, Mass. (1965).

[13] R. Larsen, *An Introduction to the Theory of Multipliers*, Springer, Berlin-New York-Heidelberg (1971).

[14] J. Mikusinski, *Operational Calculus*, Oxford-Warszawa (1959).