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ON THE GEOMETRIC PROPERTIES OF THE MINKOWSKI OPERATOR

Mashrabjon Mamatov [1] , Jalolxon Nuritdinov [2,§]

[1] Department of Geometry and Topology
of National University of Uzbekistan
Tashkent - 100174, UZBEKISTAN

[2] Department of Digital Technologies and
Mathematics of Kokand University
Kokand - 150700, UZBEKISTAN

Abstract

This article is about Minkowski difference of sets, which is one of the Minkowski operators. The necessary and sufficient conditions for the existence of the Minkowski difference of given regular polygons in the plane are derived. The method of finding the Minkowski difference of given regular tetrahedrons in the Euclidean space $\mathbb{R}^3$ is explained. Results for finding the Minkowski difference of given n-dimensional cubes in space $\mathbb{R}^n$ are also presented.

At the end of the article, the obtained results are summarized and a geometric method for finding the Minkowski difference of the convex set M and compact set N given in $\mathbb{R}^n$ is shown. The theory of foliations is applied to find the Minkowski difference of sets. New geometric concepts such as "dense embedding" and "completely dense embedding" are introduced. An important geometric property of the Minkowski operator is introduced and proved as a theorem.

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1. **Introduction**

Not all operations on sets may have a geometric meaning. On sets whose elements are arbitrary in nature, we can perform union of sets, intersection of sets, and difference of sets. For example, let's take sets $A = \{4, 5, 6, 7\}$ and $B = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -5 \\ 9 & 2 \end{pmatrix} \right\}$. Set A is a part of the set of natural numbers, and set B is a part of the set of matrices of the second order, that is, they consist of elements of different natures. Their union is in form $A \cup B = \left\{ 4, 5, 6, 7, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -5 \\ 9 & 2 \end{pmatrix} \right\}$.

So, the above operations do not necessarily mean geometrically in some cases. The Minkowski sum and difference on the sets were introduced precisely for the purpose of solving geometric problems, and these operations depend on the nature of the elements that make up the sets. That is why Minkowski operations are not performed for the sets given in the above example.

Definitions and some properties of Minkowski operators are presented in works [1, 2]. Among the known scientific works, the Minkowski difference was first used in [3] to solve the problem of pursuit in differential games under the name "geometric difference". Later, in other works such as [4, 5], various properties of this "geometric difference" were studied, and with their help, the conditions for solving the problem of chasing were eased. Also, many geometric properties of Minkowski's difference and sum are presented in [6, 7]. To date, several scientific researches have been conducted to find algorithms for calculating the Minkowski sum. A. Kaul, M.A. O'Connor , V. Srinivasan, M.S. Kim, K. Sugihara, D. Leven, M. Sharir, J. Mikusinski, D. Mount, R. Silverman, E.R. Oks, P.K. Agarwal, E. Flato, D. Halperin, G.D. Ramkumar and other scientists obtained fundamental results on the calculation of the Minkowski sum of polygons in the plane [8, 9, 10, 11].

Finding the Minkowski difference of sets is more complicated than finding their Minkowski sum. There is also not much work on finding the Minkowski difference of given sets[12, 13, 14]. Several properties and calculation methods of the Minkowski difference are presented in the works of specialists such as K. Sugihara, Y.T. Feng, Y. Tan, S. Tomiskova, Y. Martinez-Maure, V.I. Danilov, G.A. Koshevoy, S.N. Avvakumov, Z.R. Gabudillina [15, 16, 17]. However, so far, the conditions for the Minkowski difference of an arbitrary given set to be empty or non-empty have not been obtained.

The theory of foliation is one of the developing branches of modern geometry, and it has applications to many areas of geometry, [18, 19, 20, 21]. In summarizing the obtained results in this article, the foliation theory was also used. Through new geometrical concepts, an efficient method for finding the Minkowski difference of given compact sets in $\mathbb{R}^n$ has been created.

This article presents important geometric properties of the Minkowski operator and geometric ways to find the Minkowski difference of some sets using these properties. In this article, we solved the following problems:

(1) A new geometric method and exact formula for finding the Minkowski difference of given regular polygons in the plane $\mathbb{R}^2$;
(2) Finding the Minkowski difference of two given regular tetrahedrons in the Euclidean space $\mathbb{R}^3$;
(3) The condition that the Minkowski difference of two n-dimensional cubes given in space $\mathbb{R}^n$ is non-empty;
(4) A new geometric property for finding the Minkowski difference of arbitrary sets;
(5) Applying foliation theory to finding the Minkowski difference.

2. **Research methodology**

As we know, arbitrary points x and y in the Euclidean space $\mathbb{R}^n$ can be represented by coordinates consisting of n numbers as follows: $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$. In addition, these points can be determined by their position vectors, that is, the position vector of point x is $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + ... + x_n\vec{e}_n$ and the position vector of point y is $\vec{y} = y_1\vec{e}_1 + y_2\vec{e}_2 + ... + y_n\vec{e}_n$. Here $\vec{e}_1, \vec{e}_2, ..., \vec{e}_n$ are basis vectors in Euclidean space.

In that case, the sum of these points means the point determined by the vector formed as a result of the sum of their position vectors:

$$\vec{x} + \vec{y} = x_1\vec{e}_1 + x_2\vec{e}_2 + ... + x_n\vec{e}_n + y_1\vec{e}_1 + y_2\vec{e}_2 + ... + y_n\vec{e}_n =$$

$$= (x_1 + y_1)\vec{e}_1 + (x_2 + y_2)\vec{e}_2 + ... + (x_n + y_n)\vec{e}_n.$$

So, as a result of adding points $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$, a point of form $(x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$ is formed.

Definition 2.1. Let the sets A and B be non-empty sets of the n dimensional Euclidean space $\mathbb{R}^n$. Their Minkowski sum is the set of points formed by adding each point of set A to each point of set B, i.e.

$$A + B = \{c \in \mathbb{R}^n : c = a + b,\ a \in A,\ b \in B\}. \tag{1}$$

Using this introduced operation, the Minkowski difference of two sets is defined as follows.

Definition 2.2. Let the sets A and B be non-empty sets of the n dimensional Euclidean space $\mathbb{R}^n$. The following set is called their Minkowski difference:

$$D = A \overset{*}{-} B = \{d \in \mathbb{R}^n : d + B \subset A\}. \tag{2}$$

Definition 2.3. The Minkowski operators of a multi-valued mapping $G : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ are the operators $A_G : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ and $B_G : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ given by the formulas

$$A_G S = \bigcup_{x \in S} (x + G(x)),$$
$$B_G S = \mathbb{R}^n \backslash (A_G(\mathbb{R}^n \backslash S)),$$

for any set S.

If, in particular, we take the multi-valued mapping G to be constant $G(x) = G_0$ for all $x \in S$, the Minkowski operators correspond to Minkowski sum and difference, respectively:

$$A_G S = S + G_0, \qquad B_G S = S \overset{*}{-} (-G_0). \tag{3}$$

The Minkowski sum and Minkowski difference have been used to obtain sufficient conditions for ending the game in differential games [3]-[5]. Today, the approximate calculation of Minkowski sum and difference takes an important place in solving practical problems with the help of differential games. At the same time, it is one of the most important issues to evaluate the Minkowski difference from below and above in theoretical studies.

The Minkowski operator was first applied to the study of differential games in the works of L.S. Pontryagin [3], [4]. He called this operator geometric difference and marked it as ($\overset{*}{-}$). Also, the application of the Minkowski operator to differential games was made by N.Yu. Satimov[5], G.E. Ivanov, P.E. Dvurechensky [7]. In [12], the results on the calculation of the Minkowski difference of arbitrary triangles in the plane are described. In [13], a necessary and sufficient condition for the Minkowski difference of two squares to be non-empty was obtained. Formulas for calculating Minkowski differences are also presented in these works. Summarizing the methods in works [12] and [13], we derive a method for finding the Minkowski difference of two arbitrary regular n-sided polygons in the plane $\mathbb{R}^2$.

3. Minkowski difference of regular polygons

On the Euclidean plane $\mathbb{R}^2$, let regular n-sided polygons P^A and P^B be given by vertices $A_1, A_2, ..., A_n$ and $B_1, B_2, ..., B_n$, respectively. Using these points, we can express vectors corresponding to the sides of regular polygons P^A and P^B:

$$\overrightarrow{A_1A_2} = \vec{a}_1, \overrightarrow{A_2A_3} = \vec{a}_2, ..., \overrightarrow{A_nA_1} = \vec{a}_n, \tag{4}$$
$$\overrightarrow{B_1B_2} = \vec{b}_1, \overrightarrow{B_2B_3} = \vec{b}_2, ..., \overrightarrow{B_nB_1} = \vec{b}_n. \tag{5}$$

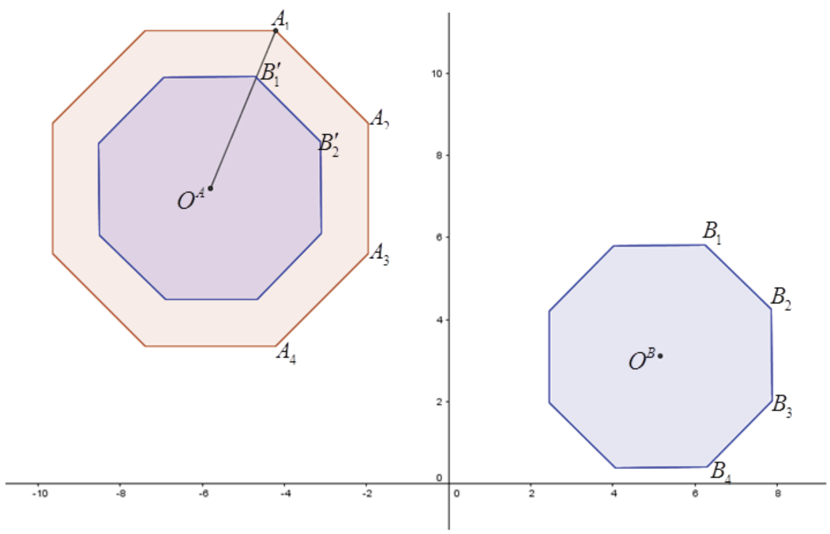

FIGURE 1. The Minkowski difference of regular polygons with parallel sides

THEOREM 3.1. *In order for the Minkowski difference $P^A \overset{*}{-} P^B$ of regular polygons P^A and P^B given on the Euclidean plane $\mathbb{R}^2$ to be non-empty, the following relation is necessary and sufficient:*

$$\frac{|\vec{a}_1|}{2\tan\left(\frac{\pi}{n}\right)} \geq \frac{\left|\vec{b}_1\right|}{2\sin\left(\frac{\pi}{n}\right)} \cdot \cos\left(\frac{\pi}{n} - \alpha_i\right). \tag{6}$$

Here $\alpha_i = \min\limits_{i=\overline{1,n}} \left\{ \arccos\left(\frac{\langle \vec{a}_1, \vec{b}_i \rangle}{|\vec{a}_1||\vec{b}_i|} \right) \right\}$ *is the smallest angle between vectors* $\vec{a}_1$ *and* $\vec{b}_i$, $i = \overline{1,n}$.

P r o o f. Since P^A is a regular polygon, the centers of the circumcircle and incircles of this polygon are at the same point. Let's denote this point as O^A. In the same way, we mark the center of circumcircle and incircles of the polygon P^B as O^B. $P^A \overset{*}{-} P^B \neq \emptyset$ means that the set P^B can be nested inside the set P^A. For this, we move the set P^B parallel until the point O^B falls on the point O^A, that is, we move the set P^B parallel along the vector $\overrightarrow{O^B O^A}$. There can be two cases.

In the first case, it can be $\vec{a}_1 \uparrow\uparrow \vec{b}_1$, $\vec{a}_2 \uparrow\uparrow \vec{b}_2, ..., \vec{a}_n \uparrow\uparrow \vec{b}_n$ (see Figure 1). In such a situation, the images of points $B'_1, B'_2, ..., B'_n$ formed by parallel displacement of points $B_1, B_2, ..., B_n$ along vector $\overrightarrow{O^B O^A}$ will be located on straight lines $O^A A_i$, $i = \overline{1,n}$. In order for the points $B'_1, B'_2, ..., B'_n$ to belong to the regular polygon P^A (here, the points inside the polygon are also considered to belong to the polygon), it is necessary and sufficient to satisfy the relation

$$\left|O^A A_i\right| \geq \left|O^A B'_i\right|, \, i = \overline{1,n}. \tag{7}$$

The length of the segments $O^A B'_i$, $i = \overline{1,n}$ is equal to the radius of the circumcircle of the P^B polygon, i.e

$$\left|O^A B'_i\right| = \frac{\left|\vec{b}_1\right|}{2\sin\left(\frac{\pi}{n}\right)}, \, i = \overline{1,n}. \tag{8}$$

The length of the segment $O^A A_i$, $i = \overline{1,n}$ is equal to the radius of the circumcircle of polygon P^A, but if we express it by the radius of the incircle of the polygon P^A, it be in the form of

$$\left|O^A A'_i\right| = \frac{|\vec{a}_1|}{2\tan\left(\frac{\pi}{n}\right)} \cdot \frac{1}{\cos\left(\frac{\pi}{n}\right)}, \, i = \overline{1,n}. \tag{9}$$

Since $\vec{a}_1 \uparrow\uparrow \vec{b}_1$, follows that $\alpha_i = \min\limits_{i=\overline{1,n}} \left\{ \arccos\left(\frac{\langle \vec{a}_1, \vec{b}_i \rangle}{|\vec{a}_1||\vec{b}_i|}\right)\right\} = 0$. From this we can write equation(9) as

$$\left|O^A A'_i\right| = \frac{|\vec{a}_1|}{2\tan\left(\frac{\pi}{n}\right)} \cdot \frac{1}{\cos\left(\frac{\pi}{n} - \alpha_i\right)}, \, i = \overline{1,n}. \tag{10}$$

If we put equations (10) and (8) to relation (7), condition (6) is obtained.

In the second case, relations $\vec{a}_i \uparrow\uparrow \vec{b}_j;\; i,\, j = \overline{1,n}$ are appropriate, that is, none of the sides of the polygons P^A and P^B are parallel to each other (see Figure 2). In studying this situation, we must first determine the smallest angle between the vectors $\vec{a}_1$ and $\vec{b}_i$, $i = \overline{1,4}$ and we denote this angle as α_i and calculate it as follows

$$\alpha_i = \min_{i=\overline{1,n}} \left\{ \arccos\left(\frac{\left\langle \vec{a}_1, \vec{b}_i \right\rangle}{|\vec{a}_1| \left|\vec{b}_i\right|} \right) \right\}. \tag{11}$$

Suppose this angle is the angle between the vector $\overrightarrow{A_1 A_2}$ and the vector $\overrightarrow{B_k B_{k+1}}, k = \overline{1,n}$ $(B_{n+1} = B_1)$. In that case, we construct the vector $\overrightarrow{O^A A}$, whose beginning is at the point O^A, and whose end is at the point A, the middle of the segment $A_1 A_2$. This vector forms an angle $\frac{\pi}{n} - \alpha_i$ with the vector $\overrightarrow{O^A B'_k}$, whose beginning is at point O^A and whose end is at point B'_k. In order for the points to belong to the regular polygon P^A, it is necessary and sufficient that the length of the orthogonal projection of the vector $\overrightarrow{O^A B'_k}$

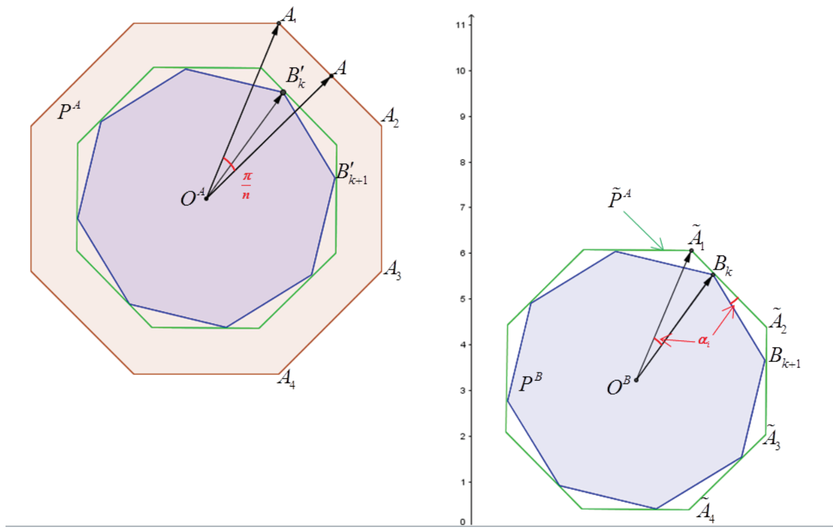

FIGURE 2. The Minkowski difference of regular polygons with corresponding sides not parallel.

onto the vector $\overrightarrow{O^A A}$ is not greater than the length of the vector $\overrightarrow{O^A A}$(see Figure 2), i.e.

$$\left|\overrightarrow{O^A A}\right| \geq \left|\overrightarrow{O^A B'_k}\right| \cdot \cos\left(\frac{\pi}{n} - \alpha_i\right). \tag{12}$$

The length of the vector $\overrightarrow{O^A A}$ is equal to the radius of the incircle of the regular polygon P^A,

$$\left|\overrightarrow{O^A A}\right| = \frac{|\vec{a}_1|}{2\tan\left(\frac{\pi}{n}\right)}. \tag{13}$$

The length of the vector $\overrightarrow{O^A B'_k}$ is equal to the radius of the circumcircle of the regular polygon P^B,

$$\left|\overrightarrow{O^A B'_k}\right| = \frac{\left|\vec{b}_1\right|}{2\sin\left(\frac{\pi}{n}\right)}. \tag{14}$$

If we put equations (14) and (13) to relation (12), condition (6) is obtained. This completes the proof. □

During the proof of the theorem, we created an algorithm (way) for finding the Minkowski difference of two regular n-sided polygons given by their vertices in a plane. Accordingly, to find the difference of polygons, the following is done sequentially:

1) First, we determine the coordinates of the vectors corresponding to the sides of the polygons P^A and P^B through the given vertices. If the points $A_1 = \{\alpha_1^1, \alpha_1^2,\}$, $A_2 = \{\alpha_2^1, \alpha_2^2\}$,..., $A_n = \{\alpha_n^1, \alpha_n^2\}$ are the vertices of the polygon P^A and the points $B_1 = \{\beta_1^1, \beta_1^2\}$, $B_2 = \{\beta_2^1, \beta_2^2\}$, ..., $B_n = \{\beta_n^1, \beta_n^2\}$ are the vertices of the polygon P^B, then we can find the vectors

$$\vec{a}_1 = \overrightarrow{A_1A_2} = \{\alpha_2^1 - \alpha_1^1, \alpha_2^2 - \alpha_1^2\},$$
$$\vec{a}_2 = \overrightarrow{A_2A_3} = \{\alpha_3^1 - \alpha_2^1, \alpha_3^2 - \alpha_2^2\},$$
$$...$$
$$\vec{a}_n = \overrightarrow{A_nA_1} = \{\alpha_1^1 - \alpha_n^1, \alpha_1^2 - \alpha_n^2\};$$

and

$$\vec{b}_1 = \overrightarrow{B_1B_2} = \{\beta_2^1 - \beta_1^1, \beta_2^2 - \beta_1^2\},$$
$$\vec{b}_2 = \overrightarrow{B_2B_3} = \{\beta_3^1 - \beta_2^1, \beta_3^2 - \beta_2^2\},$$
$$...$$
$$\vec{b}_n = \overrightarrow{B_nB_1} = \{\beta_1^1 - \beta_n^1, \beta_1^2 - \beta_n^2\};$$

2) By calculating the angles between the vector $\vec{a}_1$ and the vectors $\vec{b}_1, \vec{b}_2, ..., \vec{b}_n$, we find the smallest of these angles and designate it as α_i,

$$\alpha_i = \min_{i=\overline{1,n}} \left\{ \arccos \left(\frac{\left\langle \vec{a}_1, \vec{b}_i \right\rangle}{|\vec{a}_1| \left|\vec{b}_i\right|} \right) \right\};$$

3) We check that the Minkowski difference of polygons P^A and P^B is not empty. For this, we calculate the expression (6) and check that if the relation (6) is fulfilled, the difference $P^A \overset{*}{-} P^B$ is not empty, otherwise, the difference consists of an empty set;

4) Suppose that in relation (6) the equality $\frac{|\vec{a}_1|}{2\tan\left(\frac{\pi}{n}\right)} = \frac{|\vec{b}_1|}{2\sin\left(\frac{\pi}{n}\right)} \cdot \cos\left(\frac{\pi}{n} - \alpha_i\right)$ is satisfied and the difference $P^A \overset{*}{-} P^B$ is not empty. Then this difference consists of a single point resulting from the subtraction of the centers of the circumcircles of the polygons P^A and P^B. Denoting these centers as O^A and O^B, respectively, we can define these points by vectors $\overrightarrow{A_2O^A}$ and $\overrightarrow{B_2O^B}$. The direction of the vector $\overrightarrow{A_2O^A}$ is the same as the direction of the vector $\overrightarrow{A_2A_3} - \overrightarrow{A_1A_2} = \vec{a}_2 - \vec{a}_1 = \left\{\alpha_1^1 + \alpha_3^1 - 2\alpha_2^1, \alpha_1^2 + \alpha_3^2 - 2\alpha_2^2\right\}$, and its length is equal to the length of the radius of the circumcircle of the polygon P^A, which we denote by r^A:

$$\left|\overrightarrow{A_2O^A}\right| = r^A = \frac{|\vec{a}_1|}{2\sin\left(\frac{\pi}{n}\right)}.$$

Then, the coordinate of the vector $\overrightarrow{A_2O^A}$ is of the form

$$\overrightarrow{A_2O^A} = \left\{ \frac{r^A}{|\vec{a}_2 - \vec{a}_1|} \left(\alpha_1^1 + \alpha_3^1 - 2\alpha_2^1\right), \frac{r^A}{|\vec{a}_2 - \vec{a}_1|} \left(\alpha_1^2 + \alpha_3^2 - 2\alpha_2^2\right) \right\}.$$

Using this vector, we determine the coordinate of point O^A:

$$O^A = \left\{ \frac{r^A}{|\vec{a}_2 - \vec{a}_1|} \left(\alpha_1^1 + \alpha_3^1 - 2\alpha_2^1\right) + \left(\alpha_3^1 - \alpha_2^1\right), \right.$$
$$\left. \frac{r^A}{|\vec{a}_2 - \vec{a}_1|} \left(\alpha_1^2 + \alpha_3^2 - 2\alpha_2^2\right) + \left(\alpha_3^2 - \alpha_2^2\right) \right\}.$$

Similarly, we find the coordinates of point O^B using the vector $\overrightarrow{B_2O^B}$:

$$O^B = \left\{ \frac{r^B}{\left|\vec{b}_2 - \vec{b}_1\right|} \left(\beta_1^1 + \beta_3^1 - 2\beta_2^1\right) + \left(\beta_3^1 - \beta_2^1\right), \right.$$
$$\left. \frac{r^A}{\left|\vec{b}_2 - \vec{b}_1\right|} \left(\beta_1^2 + \beta_3^2 - 2\beta_2^2\right) + \left(\beta_3^2 - \beta_2^2\right) \right\}.$$

Here, r^B is the radius of the circumcircle of the polygon P^B. So, for the case where $\frac{|\vec{a}_1|}{2\tan\left(\frac{\pi}{n}\right)} = \frac{\left|\vec{b}_1\right|}{2\sin\left(\frac{\pi}{n}\right)} \cdot \cos\left(\frac{\pi}{n} - \alpha_i\right)$is valid, the difference $P^A \overset{*}{-} P^B$ consists of points $O^A \overset{*}{-} O^B$;

5) Suppose that the inequality $\frac{|\vec{a}_1|}{2\tan\left(\frac{\pi}{n}\right)} \geq \frac{\left|\vec{b}_1\right|}{2\sin\left(\frac{\pi}{n}\right)} \cdot \cos\left(\frac{\pi}{n} - \alpha_i\right)$ holds. In order to calculate the difference $P^A \overset{*}{-} P^B$, we construct a regular n-sided polygon, the vectors on the corresponding sides of which are in the same direction as the vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$, and on the sides of which the points $B_1 = \{\beta_1^1, \beta_1^2\}$, $B_2 = \{\beta_2^1, \beta_2^2\}, ..., B_n = \{\beta_n^1, \beta_n^2\}$ lie. Let us mark the points at the ends of this polygon as $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n$ (see Figure 2). If we can find the coordinates of these points, we can determine the coordinates of the points corresponding to the vertices of the polygon $P^A \overset{*}{-} P^B$. To do this, it is enough to subtract the corresponding points $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n$ from the points $A_1 = \{\alpha_1^1, \alpha_1^2\}$, $A_2 = \{\alpha_2^1, \alpha_2^2\}$,...,$A_n = \{\alpha_n^1, \alpha_n^2\}$. It is known from the construction of polygon $\tilde{P}^A$ that the direction of vectors $\overrightarrow{O^B\tilde{A}_i}$, $i = \overrightarrow{1,n}$ is the same to the direction of vectors $\overrightarrow{O^AA_i}$, $i = \overrightarrow{1,n}$, and its length is equal to the length of the orthogonal projection of vector $\overrightarrow{O^AB'_k}$ to vector $\overrightarrow{O^AA}$ multiplied by the expression $\frac{1}{\cos\left(\frac{\pi}{n}\right)}$, that is

$$\left|\overrightarrow{O^B\tilde{A}_i}\right| = \frac{\left|\vec{b}_1\right|}{2\sin\left(\frac{\pi}{n}\right)} \cdot \frac{\cos\left(\frac{\pi}{n} - \alpha_i\right)}{\cos\left(\frac{\pi}{n}\right)} = \frac{\left|\vec{b}_1\right| \cdot \cos\left(\frac{\pi}{n} - \alpha_i\right)}{\sin\left(\frac{2\pi}{n}\right)}, \; i = \overrightarrow{1,n}.$$

Since the coordinates of the vectors $\overrightarrow{O^A A_i}$, $i = \overrightarrow{1,4}$ are in the form

$$\overrightarrow{O^A A_i} = \left\{ \alpha_i^1 - \frac{r^A}{|\vec{a}_2 - \vec{a}_1|} \left(\alpha_1^1 + \alpha_3^1 - 2\alpha_2^1\right) - \left(\alpha_3^1 - \alpha_2^1\right), \right.$$

$$\left. \alpha_i^2 - \frac{r^A}{|\vec{a}_2 - \vec{a}_1|} \left(\alpha_1^2 + \alpha_3^2 - 2\alpha_2^2\right) - \left(\alpha_3^2 - \alpha_2^2\right) \right\}, \ i = \overrightarrow{1,n},$$

the coordinates of the vectors $\overrightarrow{O^B \tilde{A}_i}$, $i = \overrightarrow{1,n}$ are found in the form

$$\overrightarrow{O^B \tilde{A}_i} = \left\{ \frac{\left|\vec{b}_1\right| \cdot \cos\left(\frac{\pi}{n} - \alpha_i\right)}{r^A \cdot \sin\left(\frac{2\pi}{n}\right)} \cdot \right.$$

$$\left(\alpha_i^1 - \frac{r^A}{|\vec{a}_2 - \vec{a}_1|} \left(\alpha_1^1 + \alpha_3^1 - 2\alpha_2^1\right) - \left(\alpha_3^1 - \alpha_2^1\right) \right),$$

$$\frac{\left|\vec{b}_1\right| \cdot \cos\left(\frac{\pi}{n} - \alpha_i\right)}{r^A \cdot \sin\left(\frac{2\pi}{n}\right)} \cdot$$

$$\left. \left(\alpha_i^2 - \frac{r^A}{|\vec{a}_2 - \vec{a}_1|} \left(\alpha_1^2 + \alpha_3^2 - 2\alpha_2^2\right) - \left(\alpha_3^2 - \alpha_2^2\right) \right) \right\}, \ i = \overrightarrow{1,n}.$$

With the help of these vectors, the exact coordinates of all points $\tilde{A}_i$, $i = \overrightarrow{1,n}$ can be determined:

$\tilde{A}_i =$

$$\left\{ \frac{\left|\vec{b}_1\right| \cdot \cos\left(\frac{\pi}{n} - \alpha_i\right)}{r^A \cdot \sin\left(\frac{2\pi}{n}\right)} \left(\alpha_i^1 - \frac{r^A}{|\vec{a}_2 - \vec{a}_1|} \left(\alpha_1^1 + \alpha_3^1 - 2\alpha_2^1\right) - \left(\alpha_3^1 - \alpha_2^1\right) \right) \right.$$

$$+ \frac{r^B}{\left|\vec{b}_2 - \vec{b}_1\right|} \left(\beta_1^1 + \beta_3^1 - 2\beta_2^1\right) + \left(\beta_3^1 - \beta_2^1\right),$$

$$\frac{\left|\vec{b}_1\right| \cdot \cos\left(\frac{\pi}{n} - \alpha_i\right)}{r^A \cdot \sin\left(\frac{2\pi}{n}\right)} \left(\alpha_i^2 - \frac{r^A}{|\vec{a}_2 - \vec{a}_1|} \left(\alpha_1^2 + \alpha_3^2 - 2\alpha_2^2\right) - \left(\alpha_3^2 - \alpha_2^2\right) \right)$$

$$\left. + \frac{r^A}{\left|\vec{b}_2 - \vec{b}_1\right|} \left(\beta_1^2 + \beta_3^2 - 2\beta_2^2\right) + \left(\beta_3^2 - \beta_2^2\right) \right\}, \ i = \overrightarrow{1,n}.$$

Using the determined points $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n$ and the previously given points $A_1, A_2, ..., A_n$, we can find the vertices $C_1, C_2, ..., C_n$ of the polygon resulting from the difference $P^A \overset{*}{-} P^B$:

$$C_i = A_i - \tilde{A}_i, \ i = \overline{1,n}.$$

Thus, the exact measures and geometric position of polygon $P^A \overset{*}{-} P^B$ were found.

4. Minkowski difference of regular tetrahedrons

We know that a polyhedron is called a regular polyhedron if all its faces are congruent regular polygons and all dihedral angles are also congruent. Since at least three edges of the polyhedron pass through each vertex, the sum of all plane angles at that end is less than 2π. A regular tetrahedron is a pyramid with all faces consisting of equilateral triangles, and it has 4 vertices, 4 faces and 6 edges. The spheres drawn inside and outside a regular tetrahedron have their centers at the same point. To define a tetrahedron in a three-dimensional Euclidean space, it is enough to give the coordinates of its vertices.

Let us say that the points corresponding to the vertices of the tetrahedron T^A are given by $A_1 = \left\{\alpha_1^1, \alpha_1^2, \alpha_1^3\right\}$, $A_2 = \left\{\alpha_2^1, \alpha_2^2, \alpha_2^3\right\}$, $A_3 = \left\{\alpha_3^1, \alpha_3^2, \alpha_3^3\right\}$, $A_4 = \left\{\alpha_4^1, \alpha_4^2, \alpha_4^3\right\}$ coordinates, and the points corresponding to the vertices of the tetrahedron T^B are given by $B_1 = \left\{\beta_1^1, \beta_1^2, \beta_1^3\right\}$, $B_2 = \left\{\beta_2^1, \beta_2^2, \beta_2^3\right\}$, $B_3 = \left\{\beta_3^1, \beta_3^2, \beta_3^3\right\}$, $B_4 = \left\{\beta_4^1, \beta_4^2, \beta_4^3\right\}$ coordinates. Then the coordinate of the center of the circumsphere and insphere of the tetrahedron T^A is in the form

$$O^A = \left\{\frac{\alpha_1^1 + \alpha_2^1 + \alpha_3^1 + \alpha_4^1}{4}, \frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2}{4}, \frac{\alpha_1^3 + \alpha_2^3 + \alpha_3^3 + \alpha_4^3}{4}\right\}.$$

Similarly, the coordinate of the center of the circumsphere and insphere of the tetrahedron T^B is also in the form

$$O^B = \left\{\frac{\beta_1^1 + \beta_2^1 + \beta_3^1 + \beta_4^1}{4}, \frac{\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2}{4}, \frac{\beta_1^3 + \beta_2^3 + \beta_3^3 + \beta_4^3}{4}\right\}.$$

We denote the vectors starting at point O^A and ending at the points where the medians of the faces of the tetrahedron T^A intersect as $\vec{r}_1^{\,A}$,$\vec{r}_2^{\,A}$, $\vec{r}_3^{\,A}$,$\vec{r}_4^{\,A}$ and the coordinates of these vectors are in the form

$$\vec{r}_1^{\,A} = \left\{\frac{\alpha_2^1 + \alpha_3^1 + \alpha_4^1 - 3\alpha_1^1}{12}, \frac{\alpha_2^2 + \alpha_3^2 + \alpha_4^2 - 3\alpha_1^2}{12}, \frac{\alpha_2^3 + \alpha_3^3 + \alpha_4^3 - 3\alpha_1^3}{12}\right\},$$

$$\vec{r}_2^{\,A} = \left\{\frac{\alpha_1^1 + \alpha_3^1 + \alpha_4^1 - 3\alpha_2^1}{12}, \frac{\alpha_1^2 + \alpha_3^2 + \alpha_4^2 - 3\alpha_2^2}{12}, \frac{\alpha_1^3 + \alpha_3^3 + \alpha_4^3 - 3\alpha_2^3}{12}\right\},$$

$$\vec{r}_3^{\,A} = \left\{\frac{\alpha_1^1 + \alpha_2^1 + \alpha_4^1 - 3\alpha_3^1}{12}, \frac{\alpha_1^2 + \alpha_2^2 + \alpha_4^2 - 3\alpha_3^2}{12}, \frac{\alpha_1^3 + \alpha_2^3 + \alpha_4^3 - 3\alpha_3^3}{12}\right\},$$

$$\vec{r}_4^{\,A} = \left\{\frac{\alpha_1^1 + \alpha_2^1 + \alpha_3^1 - 3\alpha_4^1}{12}, \frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 3\alpha_4^2}{12},\right.$$

$$\left.\frac{\alpha_1^3+\alpha_2^3+\alpha_3^3-3\alpha_4^3}{12}\right\}.$$

The lengths of these vectors are the same and equal to the radius of the insphere of the tetrahedron T^A, i.e.

$$\left|\vec{r}_i^{\,A}\right| = \frac{\sqrt{6}}{12}\left|\vec{a}_1\right|, \quad i=\overline{1,4}.$$

Where $\vec{a}_1=\overrightarrow{A_1A_2}$ and represents the vector corresponding to the edge of the tetrahedron T^A.

Let us denote the vectors starting at O^B and ending at points B_1, B_2, B_3, B_4 as $\vec{R}_1^B$,$\vec{R}_2^B$,$\vec{R}_3^B$,$\vec{R}_4^B$ respectively, and the coordinates of these vectors are in the form

$$\vec{R}_1^B=\left\{\frac{3\beta_1^1-\beta_2^1-\beta_3^1-\beta_4^1}{4},\frac{3\beta_1^2-\beta_2^2-\beta_3^2-\beta_4^2}{4},\right.$$

$$\left.\frac{3\beta_1^3-\beta_2^3-\beta_3^3-\beta_4^3}{4}\right\},$$

$$\vec{R}_2^B=\left\{\frac{3\beta_2^1-\beta_1^1-\beta_3^1-\beta_4^1}{4},\frac{3\beta_2^2-\beta_1^2-\beta_3^2-\beta_4^2}{4},\right.$$

$$\left.\frac{3\beta_2^3-\beta_1^3-\beta_3^3-\beta_4^3}{4}\right\},$$

$$\vec{R}_3^B=\left\{\frac{3\beta_3^1-\beta_1^1-\beta_2^1-\beta_4^1}{4},\frac{3\beta_3^2-\beta_1^2-\beta_2^2-\beta_4^2}{4},\right.$$

$$\left.\frac{3\beta_3^3-\beta_1^3-\beta_2^3-\beta_4^3}{4}\right\},$$

$$\vec{R}_4^B=\left\{\frac{3\beta_4^1-\beta_1^1-\beta_2^1-\beta_3^1}{4},\frac{3\beta_4^2-\beta_1^2-\beta_2^2-\beta_3^2}{4},\right.$$

$$\left.\frac{3\beta_4^3-\beta_1^3-\beta_2^3-\beta_3^3}{4}\right\}.$$

The lengths of these vectors are equal to the radius of the circumsphere of the tetrahedron T^B:

$$\left|\vec{R}_i^B\right| = \frac{\sqrt{6}}{4}\left|\vec{b}_1\right|, \quad i=\overline{1,4}.$$

Where $\vec{b}_1=\overrightarrow{B_1B_2}$ and represents the vector corresponding to the edge of the tetrahedron T^B. By α we denote the smallest angle between $\vec{r}_i^{\,A}, i=\overline{1,4}$ vectors and $\vec{R}_j^B, j=\overline{1,4}$ vectors.

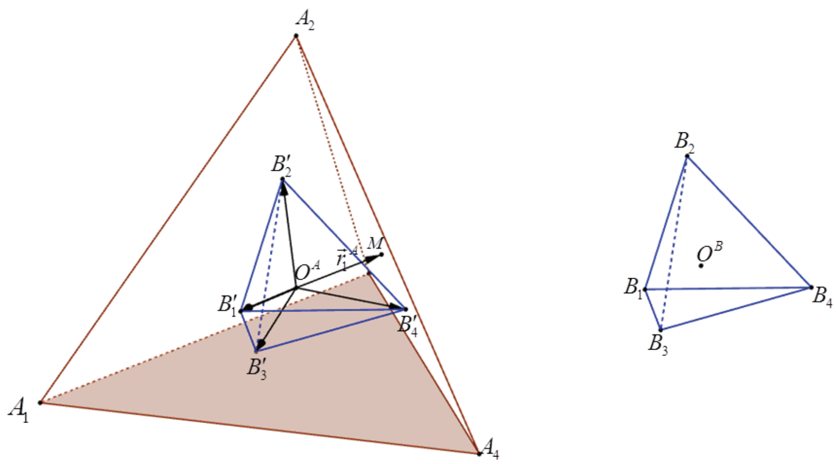

FIGURE 3. Minkowski difference of tetrahedrons

THEOREM 4.1. *In order for the Minkowski difference $T^A \overset{*}{-} T^B$ of regular tetrahedrons T^A and T^B given in Euclidean space $\mathbb{R}^3$ to be non-empty, the following relation is necessary and sufficient:*

$$\frac{\sqrt{6}}{12}\left|\vec{a}_1\right| \geq \frac{\sqrt{6}}{4}\left|\vec{b}_1\right| \cdot \cos\alpha. \tag{15}$$

P r o o f. To calculate the difference $T^A \overset{*}{-} T^B$, we move the tetrahedron T^B parallel to the vector $\overrightarrow{O^B O^A}$. Let us denote the images of points $B_1 = \left\{\beta_1^1, \beta_1^2, \beta_1^3\right\}$, $B_2 = \left\{\beta_2^1, \beta_2^2, \beta_2^3\right\}$, $B_3 = \left\{\beta_3^1, \beta_3^2, \beta_3^3\right\}$, $B_4 = \left\{\beta_4^1, \beta_4^2, \beta_4^3\right\}$ in this parallel displacement as B'_1, B'_2, B'_3, B'_4 respectively (see Figure 3). In order for the difference $T^A \overset{*}{-} T^B$ not to be empty, these points must lie inside the tetrahedron T^A or at most on its faces.

Let the points B'_1, B'_2, B'_3, B'_4 lie on the faces of the tetrahedron T^A. The radius of the insphere of the tetrahedron T^A drawn from the point O^A to the face formed by the vertices $A_2 = \left\{\alpha_2^1, \alpha_2^2, \alpha_2^3\right\}$, $A_3 = \left\{\alpha_3^1, \alpha_3^2, \alpha_3^3\right\}$, $A_4 = \left\{\alpha_4^1, \alpha_4^2, \alpha_4^3\right\}$ of the tetrahedron T^A falls on the point where the medians of the triangle $\triangle A_2 A_3 A_4$ intersect and is perpendicular to this face. Let us designate the vector corresponding to this radius as $\vec{r}_1^A$, its coordinate will be in the form

$$\vec{r}_1^A = \left\{\frac{\alpha_2^1 + \alpha_3^1 + \alpha_4^1 - 3\alpha_1^1}{12}, \frac{\alpha_2^2 + \alpha_3^2 + \alpha_4^2 - 3\alpha_1^2}{12}, \right.$$

$$\left. \frac{\alpha_2^3 + \alpha_3^3 + \alpha_4^3 - 3\alpha_1^3}{12}\right\}.$$

The length of the orthogonal projection of all vectors starting from O^A and ending at points lying on the face $A_2A_3A_4$ onto the vector $\vec{r}_1^A$ is equal to $\left|\vec{r}_1^A\right|$. Hence, if any point B'_1, B'_2, B'_3, B'_4 belongs to face $A_2A_3A_4$, equality

$$proj_{\vec{r}_1^A}\overrightarrow{O^A B'_i} = \left|\vec{r}_1^A\right|,\ i = \overline{1,4} \tag{16}$$

holds. Points B'_1, B'_2, B'_3, B'_4 can also be located inside the tetrahedron T^A, so we generalize equation (16) and write it in the form

$$proj_{\vec{r}_1^A}\overrightarrow{O^A B'_i} \le \left|\vec{r}_1^A\right|,\ i = \overline{1,4}. \tag{17}$$

We can write the same relation for other faces of the tetrahedron T^A:

$$\begin{aligned} proj_{\vec{r}_2^A}\overrightarrow{O^A B'_i} &\le \left|\vec{r}_2^A\right|,\ i = \overline{1,4}, \\ proj_{\vec{r}_3^A}\overrightarrow{O^A B'_i} &\le \left|\vec{r}_3^A\right|,\ i = \overline{1,4}, \\ proj_{\vec{r}_4^A}\overrightarrow{O^A B'_i} &\le \left|\vec{r}_4^A\right|,\ i = \overline{1,4}. \end{aligned} \tag{18}$$

Summarizing relations (17) and (18), we can write as follows

$$proj_{\vec{r}_j^A}\overrightarrow{O^A B'_i} \le \left|\vec{r}_j^A\right|,\ i = \overline{1,4},\ j = \overline{1,4}. \tag{19}$$

We know that the lengths of vectors $\overrightarrow{O^A B'_i}$ are the same and equal to the radius of the circumsphere of the tetrahedron T^B. $\vec{r}_j^A$ vectors have the same length and are equal to the radius of the insphere of the tetrahedron T. Based on these, we write relation (19) in form

$$\frac{\sqrt{6}}{12}\left|\vec{a}_1\right| \ge \frac{\sqrt{6}}{4}\left|\vec{b}_1\right| \cdot \cos\alpha.$$

Here α is the smallest of the angles between vectors $\vec{r}_j^A, j = \overline{1,4}$ and vectors $\overrightarrow{O^A B'_i},\ i = \overline{1,4}$. Because the cosine of a smaller angle is greater than the cosine of a larger angle. This means that if relation (15) holds for the smallest angle, it holds for the rest of the angles as well. Therefore, (15) is considered a necessary and sufficient condition for the relation $T^A \overset{*}{-} T^B$ not to be empty. □

5. The Minkowski difference of n-dimensional cubes

Continuing the considerations for finding the Minkowski difference of squares presented in [13] and generalizing the method presented in [14], we obtained sufficient and necessary conditions for the existence of the Minkowski difference of two n-dimensional cubes given in Euclidean space $\mathbb{R}^n$. In order for the definition of an n-dimensional cube in the Euclidean space $\mathbb{R}^n$ to be one-valued, it is enough to give its $n+1$ vertices that do not lie in the same hyperplane and are located on edges from one vertex. Let C^A and C^B be cubes given by vertices A_0,A_1,A_2,...,A_n and B_0,B_1,B_2,...,B_n respectively. We introduce the following notations:

$$\overrightarrow{A_0A_1} = \vec{a}_1,\ \overrightarrow{A_0A_2} = \vec{a}_2, ..., \overrightarrow{A_0A_n} = \vec{a}_n;$$
$$\overrightarrow{B_0B_1} = \vec{b}_1,\ \overrightarrow{B_0B_2} = \vec{b}_2, ..., \overrightarrow{B_0B_3} = \vec{b}_3.$$

The number of all diagonals of cube C^B is found by the expression 2^{n-1}, and we can express the vectors corresponding to these diagonals by all combinations of vectors $\vec{b}_1$, $\vec{b}_2, ...\vec{b}_n$. For example, when $n = 2$ is a two-dimensional cube, that is, a square, the number of diagonals is 2, but the number of the vectors corresponding to the diagonals is 4, and we express these vectors by all combinations of the two vectors $\vec{b}_1$, $\vec{b}_2$ that define the square and correspond to its sides:

$$\begin{aligned}
\vec{d}_1 &= \vec{b}_1 + \vec{b}_2,\\
\vec{d}_2 &= -\vec{b}_1 + \vec{b}_2,\\
\vec{d}_3 &= \vec{b}_1 - \vec{b}_2,\\
\vec{d}_4 &= -\vec{b}_1 - \vec{b}_2.
\end{aligned}$$

Since $\left|\vec{d}_1\right| = \left|\vec{d}_4\right|$ and $\left|\vec{d}_2\right| = \left|\vec{d}_3\right|$, the vectors $\vec{d}_1$ and $\vec{d}_4$ represent one diagonal and the vectors $\vec{d}_2$ and $\vec{d}_3$ represent another diagonal.

When $n = 3$ is a three-dimensional cube, the number of diagonals is 4, but the number of the vectors corresponding to the diagonals is 8, and we express these vectors by all combinations of the three vectors $\vec{b}_1$, $\vec{b}_2, \vec{b}_3$ that define the cube and correspond to its edges from one vertex:

$$\begin{aligned}
\vec{d}_1 &= \vec{b}_1 + \vec{b}_2 + \vec{b}_3, & \vec{d}_5 &= -\vec{b}_1 - \vec{b}_2 - \vec{b}_3,\\
\vec{d}_2 &= -\vec{b}_1 + \vec{b}_2 + \vec{b}_3, & \vec{d}_6 &= \vec{b}_1 - \vec{b}_2 - \vec{b}_3,\\
\vec{d}_3 &= \vec{b}_1 - \vec{b}_2 + \vec{b}_3, & \vec{d}_7 &= -\vec{b}_1 + \vec{b}_2 - \vec{b}_3,\\
\vec{d}_4 &= \vec{b}_1 + \vec{b}_2 - \vec{b}_3, & \vec{d}_8 &= -\vec{b}_1 - \vec{b}_2 + \vec{b}_3.
\end{aligned}$$

Here two vectors represent one diagonal.

When $n = 3$ is a four-dimensional cube, a tesseract the number of diagonals is 8, but the number of the vectors corresponding to the diagonals is 16, and we express these vectors by all combinations of the four vectors $\vec{b}_1$, $\vec{b}_2, \vec{b}_3, \vec{b}_4$ that define the cube and correspond to its edges from one end:

$$\begin{aligned}
\vec{d}_1 &= \vec{b}_1 + \vec{b}_2 + \vec{b}_3 + \vec{b}_4, & \vec{d}_9 &= -\vec{b}_1 - \vec{b}_2 - \vec{b}_3 - \vec{b}_4,\\
\vec{d}_2 &= -\vec{b}_1 + \vec{b}_2 + \vec{b}_3 + \vec{b}_4, & \vec{d}_{10} &= \vec{b}_1 - \vec{b}_2 - \vec{b}_3 - \vec{b}_4,\\
\vec{d}_3 &= \vec{b}_1 - \vec{b}_2 + \vec{b}_3 + \vec{b}_4, & \vec{d}_{11} &= -\vec{b}_1 + \vec{b}_2 - \vec{b}_3 - \vec{b}_4,\\
\vec{d}_4 &= \vec{b}_1 + \vec{b}_2 - \vec{b}_3 + \vec{b}_4, & \vec{d}_{12} &= -\vec{b}_1 - \vec{b}_2 + \vec{b}_3 - \vec{b}_4,\\
\vec{d}_5 &= \vec{b}_1 + \vec{b}_2 + \vec{b}_3 - \vec{b}_4, & \vec{d}_{13} &= -\vec{b}_1 - \vec{b}_2 - \vec{b}_3 + \vec{b}_4,\\
\vec{d}_6 &= -\vec{b}_1 - \vec{b}_2 + \vec{b}_3 + \vec{b}_4, & \vec{d}_{14} &= \vec{b}_1 + \vec{b}_2 - \vec{b}_3 - \vec{b}_4,
\end{aligned}$$

$$\vec{d_7} = -\vec{b}_1 + \vec{b}_2 - \vec{b}_3 + \vec{b}_4, \qquad \vec{d}_{15} = \vec{b}_1 - \vec{b}_2 + \vec{b}_3 - \vec{b}_4,$$

$$\vec{d_8} = -\vec{b}_1 + \vec{b}_2 + \vec{b}_3 - \vec{b}_4, \qquad \vec{d}_{16} = \vec{b}_1 - \vec{b}_2 - \vec{b}_3 + \vec{b}_4.$$

Here, too, one diagonal is represented by two vectors. Similarly, the number of diagonals of an n-dimensional cube is 2^{n-1}, but the number of the vectors corresponding to the diagonals is 2^n, and we can express these vectors by all combinations of vectors $\vec{b}_1$, $\vec{b}_2$,...,$\vec{b}_n$:

$$\vec{d_1} = \vec{b}_1 + \vec{b}_2 + ... + \vec{b}_n,$$

$$\vec{d_2} = -\vec{b}_1 + \vec{b}_2 + ... + \vec{b}_n,$$

$$\vec{d_3} = \vec{b}_1 - \vec{b}_2 + ... + \vec{b}_n,$$

$$\cdot \quad \cdot \quad \cdot$$

$$\vec{d}_{2^n} = -\vec{b}_1 - \vec{b}_2 - ... - \vec{b}_n.$$

THEOREM 5.1. *In order for the Minkowski difference $C^A \overset{*}{-} C^B$ not to be empty, it is necessary and sufficient that the length of the orthogonal projection of the vectors $\vec{d_i}, i = \overline{1, 2^{n-1}}$ corresponding to all diagonals of the cube C^B to the vectors $\vec{a}_1$, $\vec{a}_2$,...,$\vec{a}_n$, should not be greater than the length of the vector $\overrightarrow{a}_1$.*

P r o o f. There can be two cases when calculating the difference $C^A \overset{*}{-} C^B$.

In the first case, all $\vec{a}_1$, $\vec{a}_2$,...,$\vec{a}_n$ vectors are parallel to all $\vec{b}_1$, $\vec{b}_2$,...,$\vec{b}_n$ vectors, respectively, then the orthogonal projections of vectors $\vec{d_i}, i = \overline{1, 2^n}$ to vectors $\vec{a}_1$, $\vec{a}_2$,...,$\vec{a}_n$ are equal to the vector length $\vec{b}_1$, that is,

$$\left|proj_{\vec{a}_j} \vec{d_i}\right| = \left|\vec{b}_1\right|, \ i = \overline{1, 2^n}, \ j = \overline{1, n}\,.$$

According to the determination of the Minkowski difference, in this case, in order to be able to place the cube C^B inside the cube C^A, that is, so that the difference $C^A \overset{*}{-} C^B$ is not empty, the relation $|\vec{a}_1| \geq \left|\vec{b}_1\right|$ is necessary and sufficient. This means that $\left|proj_{\vec{a}_j} \vec{d_i}\right| \leq |\vec{a}_1|\,, \ i = \overline{1, 2^n}, \ j = \overline{1, n}\,.$

In the second case, at least one of the vectors $\vec{a}_1$, $\vec{a}_2$,...,$\vec{a}_n$ is not parallel to one of the corresponding vectors $\vec{b}_1$, $\vec{b}_2$,...,$\vec{b}_n$. We assume that none of these vectors are parallel to each other in the general case. Then the lengths of orthogonal projections of vectors $\vec{d_i}, i = \overline{1, 2^n}$ to vectors $\vec{a}_1$, $\vec{a}_2$,...,$\vec{a}_n$ are found using the formula

$$\left|proj_{\vec{a}_j} \vec{d_i}\right| = \frac{\left|\left\langle \vec{a}_j, \vec{d_i} \right\rangle\right|}{|\vec{a}_j|}, \ i = \overline{1, 2^n}, \ j = \overline{1, n}\,.$$

Here $\left\langle \vec{a}_j, \vec{d}_i \right\rangle$ is the scalar product of vectors $\vec{a}_j$ and $\vec{d}_i$. We designate the vectors whose length is the longest among the orthogonal projections of vectors $\vec{d}_i, i = \overline{1, 2^n}$ onto vectors $\vec{a}_j$ and whose direction is the same as the direction of the vectors $\vec{a}_j$, respectively, as $\vec{b}_1'', \vec{b}_2'', ..., \vec{b}_n''$. We construct an n-dimensional rectangular parallelepiped P' whose edges consist of vectors $\vec{b}_j'', j = \overline{1, n}$ and which contains the cube C^B. According to the construction of this rectangular parallelepiped, $P' \overset{*}{-} C^B \neq \emptyset$ is valid. As in the first case, so that the parallelepiped P' can be placed inside the cube C^A by parallel displacement. It is necessary and sufficient that the edges of P' are not greater than the corresponding edges of C^A, i.e.

$$|\vec{a}_1| \geq \left|\vec{b}_1''\right|, \ |\vec{a}_2| \geq \left|\vec{b}_2''\right|, ..., \ |\vec{a}_n| \geq \left|\vec{b}_n''\right|, \tag{20}$$

from the relations(20),

$$|\vec{a}_1| \geq \left|proj_{\vec{a}_j} \vec{d}_i\right| \ , \ i = \overline{1, 2^n} \ , \ j = \overline{1, n}.$$

The theorem has been proved. □

6. Generalization of the results

In this section, we summarize the results obtained above [18, 19, 20, 21]. Let, we are given convex set M and compact set N in $\mathbb{R}^n$. We denote by $\partial M_0 = L_0$ the boundary of a convex compact set $M = M_0$. M_α, $\partial M_\alpha = L_\alpha$, $\alpha \in A$ are chosen in such a way that: 1) $\bigcup\limits_{\alpha \in A} L_\alpha = M$; 2) $M_\alpha \overset{*}{-} M_\beta \neq \emptyset$ for arbitrary $\alpha, \beta \in A$ and $\alpha \leq \beta$. Based on I. Tamura [18], we call $F = \{L_\alpha : L_\alpha = \partial M_\alpha, \ \alpha \in A\}$ a foliation and L_α, $\alpha \in A$ a leaves of the foliation. Let $\partial \left(M_\alpha \overset{*}{-} M_\beta\right) \in F$ be for arbitrary $\alpha, \beta \in A$.

THEOREM 6.1. *If the condition $N \subset M_\alpha$ is satisfied for the convex compact sets M,M_α and compact set N given in $\mathbb{R}^n$, the equality*

$$M \overset{*}{-} N = \left(M \overset{*}{-} M_\alpha\right) + \left(M_\alpha \overset{*}{-} N\right) \tag{21}$$

holds.

P r o o f. Let be $z \in M \overset{*}{-} N$, then we show that there are elements $z_1 \in M \overset{*}{-} M_\alpha$ and $z_2 \in M_\alpha \overset{*}{-} N$ such that $z = z_1 + z_2$. We can write the relation $z + N \subset M$ using the definition of the Minkowski difference for the element $z \in M \overset{*}{-} N$. Therefore, for any $c \in N$, there is an element $a \in M$ such that the equality $z + c = a$ holds. From this we get the expression

$$c = a - z \in N. \tag{22}$$

By condition, since $N \subset M_\alpha$, relation $M_\alpha \overset{*}{-} N \neq \emptyset$ is valid. Let $z_2 \in M_\alpha \overset{*}{-} N$. It follows that $z_2 + N \subset M_\alpha$. This relation holds for all elements of the set N. Hence, according to (22), we can write the relation

$$z_2 + a - z \in M_\alpha. \tag{23}$$

According to the condition, $M \overset{*}{-} M_\alpha \neq \emptyset$. Let $z_1 \in M \overset{*}{-} M_\alpha$. Then, $z_1 + M_\alpha \subset M$ is appropriate. Since this relation holds for all elements of the set M_α, it also holds for the element $z_2 + a - z$ in expression (23)

$$z_2 + z_1 + a - z \in M.$$

Since $a \in M,\ z_1 + z_2 - z = 0$ and hence, the equality $z_1 + z_2 = z$ holds true.

Now, let $z \in \big(M \overset{*}{-} M_\alpha\big) + \big(M_\alpha \overset{*}{-} N\big)$, then there are elements $z_1 \in M \overset{*}{-} M_\alpha$ and $z_2 \in M_\alpha \overset{*}{-} N$ such that $z_1 + z_2 = z$. According to the definition of Minkowski difference from relation $z_1 \in M \overset{*}{-} M_\alpha$, we can write relation $z_1 + M_\alpha \subset M$, similarly, we get the expression $z_2 + N \subset M_\alpha$ from the relation $z_2 \in M_\alpha \overset{*}{-} N$. From these two expressions we get $z_1 + z_2 + N \subset M$, which leads to $z_1 + z_2 \subset M \overset{*}{-} N$. The theorem is proved. □

Definition 6.1. A compact set N is said to be embedded in a foliation F if such a leaf $L_\alpha = \partial M_\alpha,\ \alpha \in A$ and an element $z \in \mathbb{R}^n$ are found for which the relation $z + N \subset M_\alpha$ holds.

Definition 6.2. A compact set N is said to be densely embedded in a foliation F if $z + N \subset M_{\alpha_0}$ and the index α_0 is the smallest among the numbers $\alpha \in A$ for which the relation $z + N \subset M_\alpha$ holds.

It is easy to understand from this definition that if the compact set N is densely embedded in foliation F, the dimension of the geometric difference $M_\alpha \overset{*}{-} N$ is smaller than the dimension of the space $\mathbb{R}^n$.

Definition 6.3. A compact set N is said to be completely densely embedded in a foliation F if Minkowski difference $M_\alpha \overset{*}{-} N = \{a\}$ consists of a single point.

Theorem 6.2. *If compact set N completely densely embedded in a foliation F, then the equality*

$$M \overset{*}{-} N = \big(M \overset{*}{-} M_\alpha\big) + a \tag{24}$$

holds.

Using the concept of "complete dense embedding", we can write the following results for cases where the "subtrahend" set in the theorems 3.1, 4.1, 5.1 above is an arbitrary compact set N.

THEOREM 6.3. *For polygons P^A and P^B in the Euclidean plane $\mathbb{R}^2$, condition (6) holds. If compact set N is completely dense embedded in set P^B, then the equality $P^A \overset{*}{-} N = P^A \overset{*}{-} P^B$ holds.*

THEOREM 6.4. *For tetrahedrons T^A and T^B in the Euclidean space $\mathbb{R}^3$, condition (15) holds. If compact set N is completely dense embedded in set T^B, then the equality $T^A \overset{*}{-} N = T^A \overset{*}{-} T^B$ holds.*

THEOREM 6.5. *For the given cubes C^A and C^B in the Euclidean space $\mathbb{R}^n$, let the lengths of the orthogonal projections of the vectors $\vec{d_i}, i = \overline{1, 2^{n-1}}$ corresponding to all the diagonals of the cube C^B onto the vectors $\vec{a}_1$,$\vec{a}_2$,...,$\vec{a}_n$ not be greater than the length of the vector $\vec{a}_1$. If a compact set N is completely dense embedded in set C^B, then the equality $C^A \overset{*}{-} N = C^A \overset{*}{-} C^B$ holds.*

7. Conclusion

The Minkowski difference is actually useful as a research and conceptual tool. But, unfortunately, it is well known that there are serious difficulties in finding the Minkowski difference for given arbitrary forms of sets. This is the main obstacle for using the Minkowski difference in various practical applications. The results of the analysis of the work done by experts so far on finding the Minkowski difference and sum have shown that the Minkowski sum of sets is sufficiently studied, but there is a lack of data and literature on the Minkowski difference and its calculation.

Above, we introduced new methods for finding Minkowski differences of regular polygons given by vertices in the plane $\mathbb{R}^2$, regular tetrahedron given by vertices in space $\mathbb{R}^3$. Taking these results, we came to the conclusion that the form of the Minkowski difference of these sets will be similar to the "minuend" set.

But we cannot state this conclusion for the Minkowski difference of n-dimensional cubes in $\mathbb{R}^n$. Because the Minkowski difference of two cubes can also be a rectangular parallelepiped whose edges are parallel to the edges of the "minuend" cube. At the end of the article, we stated a theorem that helps to calculate the Minkowski difference of arbitrary convex compact sets in $\mathbb{R}^n$ using the elements of the theory of foliation.

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