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AN EFFICIENT WAVELET ALGORITHM FOR THE FRACTIONAL VIEW ANALYSIS OF BAGLEY-TORVIK EQUATION

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Abstract

The Hermite wavelet collocation approach with fractional functional matrices of derivatives is availed to solve the fractional nonlinear Bagley-Torvik issue in this paper. These Hermite wavelets functional matrices are exploited to turn the fractional differential equations towards algebraic equation systems. Using numerical examples, the Hermite wavelet solutions are contrasted with the accurate and numerical solutions. They are discovered to be in accord. The results demonstrate the efficacy and value wavelet approach.

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Key Words and Phrases: fractional order Bagley-Torvik equation, Hermite wavelets, functional matrices of fractional derivatives, modified Riemann-Liouville derivative

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1. **Introduction**

The movement of rigid plate immersed in Newtonian fluid expressed by fractional order Bagley-Torwik equation is represented by

$$\frac{d^2y}{dt^2}+A\frac{d^{\frac{3}{2}}y}{dt^{\frac{3}{2}}}+By(t)=g(t),\quad 0\leq t\leq T, \tag{1}$$

the boundary conditions are

$$y=a \text{ at } t=0 \text{ and } y=b \text{ at } t=T. \tag{2}$$

In this study, a brand-new technique is presented to improve the Bagley-Torvik analysis with fractional derivatives. Due to its use in the domains of engineering and science, fractional calculus and its applications have grown in significance [2, 3, 4, 9, 12].

From the mother wavelet, we can construct a family of wavelet functions by changing the dilation parameter $a\in R$ and translation parameter $b\in R$. The continuous mother wavelet function is given by

$$\Psi_{a,b}(t)=\frac{\Psi\left(\frac{t-b}{a}\right)}{a^{\frac{1}{2}}}. \tag{3}$$

Here $a^{-\frac{1}{2}}$ is a constant of normalization. The discrete wavelet functions are represented through

$$\Psi_{m,n}=|a_0|^{\frac{m}{2}}\Psi(a_0^m t-nb_0),\quad m,n\in Z. \tag{4}$$

The effectiveness of wavelet approaches in solving boundary value problems (BVPs) has been well demonstrated [1, 5, 6, 7, 8, 10, 11]. This study suggests using the Hermite wavelet approach to solve the fractional Bagley-Torvik equation.

2. **Analysis of wavelets**

2.1. **Hermite Wavelets.** The Hermite wavelets defined in $[0,1)$ are

$$\Psi_{m,n}(t)=\begin{cases}\frac{2^{\frac{p+1}{2}}}{\sqrt{\pi}}H_m(2^pt-2n+1), & t\in\left[\frac{n-1}{2^{p-1}},\frac{n}{2^{p-1}}\right],\\ 0 & \text{otherwise,}\end{cases} \tag{5}$$

where $0\leq m\leq M$ and $0\leq n\leq 2^{p-1}$. Here $H_m(t)$ denotes m^{th} order Hermite polynomials. The weight function of Hermite polynomials is given by

$$w(x)=\sqrt{1-t^2}. \tag{6}$$

2.2. **Hermite Wavelet Operational Matrices of Derivatives.** The functional matrices of derivatives D and D^2 for $M = 2$ and $k = 1$ are

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{pmatrix} \quad \text{and} \quad D^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 32 & 0 & 0 \end{pmatrix}. \tag{7}$$

Then the Hermite wavelet matrix is

$$\psi(x) = \frac{2}{\sqrt{\pi}} \begin{pmatrix} 1 \\ 4t - 2 \\ 16t^2 - 16t + 2 \end{pmatrix}. \tag{8}$$

The modified derivative of Riemann-Liouville is defined by

$$D_x^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int\limits_0^x (t-\xi)^{-\alpha-1}[f(\xi) - f(0)]d\xi, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int\limits_0^x (t-\xi)^{-\alpha}[f(\xi) - f(0)]d\xi, & 0 < \alpha < 1, \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x (t-\xi)^{n-\alpha-1}[f(\xi) - f(0)]d\xi, & n \le \alpha < n+1, n \ge 1, \end{cases} \tag{9}$$

with the properties

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \gamma > 0. \tag{10}$$

$$D_t^\alpha c = \frac{ct^{-\alpha}}{\Gamma(1-\alpha)}, \tag{11}$$

where c is a constant.

Now we have the following fractional Hermite wavelet operational matrices

$$D^{\frac{1}{2}}\psi(t) = \begin{pmatrix} 1.13t^{-\frac{1}{2}} \\ 9.04t^{\frac{1}{2}} - 2.26t^{-\frac{1}{2}} \\ 48.15t^{\frac{3}{2}} - 36.16t^{\frac{1}{2}} + 2.26t^{-\frac{1}{2}} \end{pmatrix}, \tag{12}$$

$$D^{\frac{3}{2}}\psi(x) = \begin{pmatrix} -0.56t^{-\frac{3}{2}} \\ 4.51t^{-\frac{1}{2}} + 1.13t^{-\frac{3}{2}} \\ 72.22t^{\frac{1}{2}} - 18.05t^{-\frac{1}{2}} - 1.13t^{-\frac{3}{2}} \end{pmatrix}. \tag{13}$$

3. **Numerical Illustrations**

PROBLEM 3.1. *Bagley-Torvik fractional differential equation is*

$$\frac{d^2y}{dt^2} + \frac{d^{\frac{3}{2}}y}{dt^{\frac{3}{2}}} + y(t) = 1 + t, \quad 0 \le t \le 1, \tag{14}$$

with initial conditions

$$y = 1 \text{ and } \frac{dy}{dt} = 1 \text{ at } t = 0. \tag{15}$$

The explicit solution is

$$y = 1 + t. \tag{16}$$

The Hermite wavelet algorithm is applied to solve above equation which corresponds to $k = 1$ and $M = 2$.

If we set the connection coefficients

$$C = \frac{\sqrt{\pi}}{2}(C_0, C_1, C_2), \tag{17}$$

the scheduled wavelet scheme is

$$\frac{d^2}{dt^2}(C^T\psi(t)) + \frac{d^{\frac{3}{2}}}{dt^{\frac{3}{2}}}(C^T\psi(t)) + C^T(\psi(t)) - g(t) = 0. \tag{18}$$

Now we choose a point $x = 0.5$ and then we gain

$$82.34c_2 + 9.58c_1 + 0.42c_0 = 1.5. \tag{19}$$

Using the initial conditions, we have

$$2c_0 - 4c_1 + 4c_2 = 1, \tag{20}$$

$$8c_1 - 32c_2 = 1. \tag{21}$$

The above algebraic equations are solved to get

$$C_0 = 0.7494,\ C_1 = 0.1246, |C_2 = 0. \tag{22}$$

Then,

$$y(t) = 1.0004 + 0.9968t. \tag{23}$$

Problem 3.2. *The other fractional Bagley-Torvik equation is*

$$\frac{d^2y}{dt^2} + \frac{1}{2}\frac{d^{\frac{3}{2}}y}{dt^{\frac{3}{2}}} + \frac{1}{2}y(t) = 8, \quad 0 \le t \le 1, \tag{24}$$

with initial conditions

$$y(0) = 0,\ y'(0) = 0. \tag{25}$$

Applying the Hermite wavelet scheme, we get the following algebraic equations

$$146.34c_2 + 9.58c_1 + 0.42c_0 = 16, \tag{26}$$

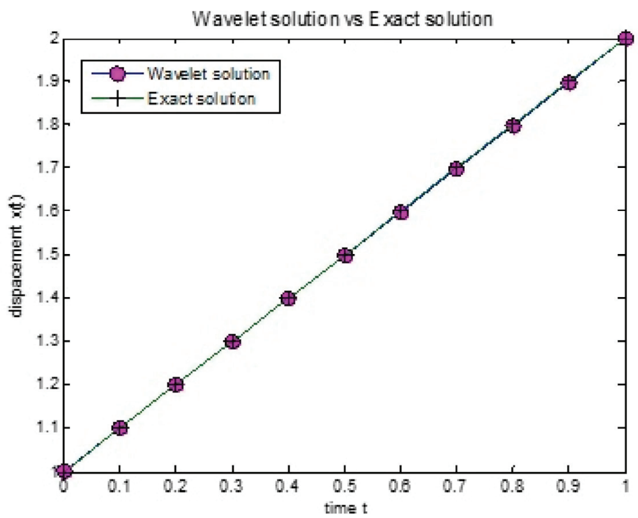

FIGURE 1. Contrast between wavelet solution and exact solution for Eq. (14) and Eq. (15)

and using the above initial conditions, we have

$$2c_0 - 4c_1 + 4c_2 = 0, \tag{27}$$

$$8c_1 - 32c_2 = 0. \tag{28}$$

The above algebraic equations are solved to obtain

$$C_0 = 0.5129, \ C_1 = 0.3419, \ C_2 = 0.0855. \tag{29}$$

Then

$$y(t) = 2.736t^2. \tag{30}$$

PROBLEM 3.3. *Yet another fractional Bagley-Torvik equation is the following*

$$\frac{d^2y}{dt^2} + \frac{d^{\frac{3}{2}}y}{dt^{\frac{3}{2}}} + y(t) = t^3 + 5\pi + \frac{8t^{\frac{3}{2}}}{\sqrt{\pi}}, \quad 0 \leq t \leq 1, \tag{31}$$

with boundary conditions

$$y = 0 \ at \ t = 0 \ \ and \ \ y = 0 \ at \ t = 1. \tag{32}$$

Applying the Hermite wavelet scheme we get the following algebraic equations

$$82.34c_2 + 9.58c_1 + 0.42c_0 = 4.23. \tag{33}$$

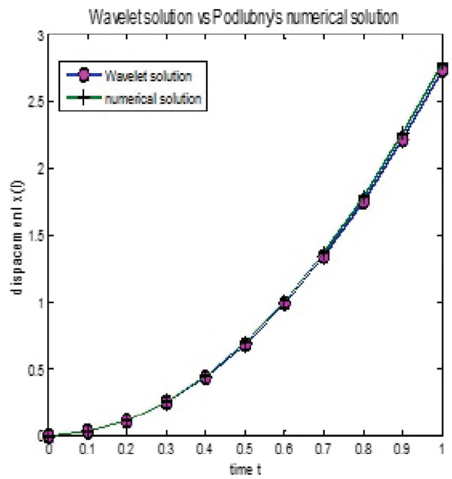

FIGURE 2. Contrast between wavelet and Podlubny's numerical solutions [4] for Eq. (24) and Eq. (25)

Using the boundary conditions, we have

$$2c_0 - 4c_1 + 4c_2 = 0, \tag{34}$$

$$2c_0 + 4c_1 + 4c_2 = 0. \tag{35}$$

The above algebraic equations are solved to obtain

$$c_0 = -0.1038, \;\; c_1 = 0, \;\; c_2 = 0.0519. \tag{36}$$

Then

$$y(t) = 1.66t^2 - 1.66t. \tag{37}$$

4. **Conclusion**

In this study, Hermite wavelet functional operational matrices of derivatives are exploited to find the solutions of fractional Bagley-Torvik problem. The suggested wavelet approach is very straightforward and simple to use. The available precise answers and alternative numerical solutions are compared to the wavelet solutions. Good agreement has been demonstrated with minimal computing effort. It should be mentioned that depending on the collocation points, this approach converts fractional differential equations towards a set of linear algebraic equations. As a result, it may offer certain benefits while creating computer codes for the required system.

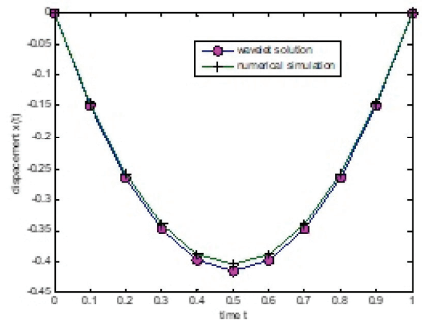

Figure 3. Contrast between wavelet solution and numerical simulation for Eq. (31) and Eq. (32)

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