International Journal of Applied Mathematics

Volume 36 No. 5 2023, 685-697

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

doi: http://dx.doi.org/10.12732/ijam.v36i5.8

NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS BASED ON QUASI-SUBORDINATION

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Abstract: In this paper, we introduce a new subclass $\mathcal{M}_{\Sigma,q}^{\lambda,\mu,\nu}(\gamma,\delta,\varphi)$ of analytic and bi-univalent functions involving a certain fractional integral operator which is defined based on quasi-subordination. For this class, we estimate the second and third coefficients of the Taylor-Maclaurin series expansions and upper bounds for Feketo-Szegö inequality. Furthermore, some relevant connections of certain special cases of the main results with those in several earlier works are also pointed out.

AMS Subject Classification: 30C45, 30C50

Key Words: analytic functions, bi-univalent functions, coefficient estimates, Feketo-Szegö inequality, subordination, fractional integral operator

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. We also

Received: May 26, 2023

denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in the unit disk $\mathbb{U}$. Let $h(z)$ be an analytic function in $\mathbb{U}$ and $|h(z)| \leq 1$, such that

$$h(z) = h_0 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots , \tag{2}$$

which all coefficients are real. Also let φ be an analytic and univalent function with positive real part in $\mathbb{U}$ with $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Taylor's series expansion of such function is of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots , \tag{3}$$

where all coefficients are real and $B_1 > 0$. Throughout this paper we assume that the function h and φ satisfy the above conditions one or otherwise stated.

For analytic functions f and g with $f(0) = g(0)$, f is said to be subordinate to g if there exists an analytic function ω on $\mathbb{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for $z \in \mathbb{U}$. The subordination will be denoted by

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \qquad \text{in } \mathbb{U}.$$

Note that $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ when g is univalent in $\mathbb{U}$.

For analytic functions f and g, the function f is quasi-subordinate to g in $\mathbb{U}$ if there exist analytic functions h and ω, with $|h(z)| \leq 1$, $\omega(0) = 0$, and $|\omega(z)| < 1$, such that $f(z)/h(z)$ is analytic in $\mathbb{U}$ and written as

$$\frac{f(z)}{h(z)} \prec g(z) \qquad (z \in \mathbb{U}). \tag{4}$$

We also denote the above expression by

$$f(z) \prec_q g(z) \qquad (z \in \mathbb{U}) \tag{5}$$

and this is equivalent to $f(z) = h(z)g(\omega(z))$ $(z \in \mathbb{U})$.

Observe that if $h(z) \equiv 1$, then $f(z) = g(\omega(z))$, so that $f(z) \prec g(z)$ in $\mathbb{U}$. Also notice that if $\omega(z) = z$, then $f(z) = h(z)g(z)$ and it is said that f is majorized by g and written by $f(z) \ll g(z)$ in $\mathbb{U}$. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization, see [20].

The well known Koebe one-quarter theorem [9] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius 1/4. Hence every function $f \in \mathcal{S}$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ $(z \in U)$ and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f); r_0(f) \geq 1/4),$$

where

$$\begin{aligned} g(w) &= f^{-1}(w) \\ &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \end{aligned} \tag{6}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-uninalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples of the class Σ, see [25].

In 1967, Lewin [13] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$. Subsequently, Netanyahu [18] showed that $\max_{f\in\Sigma} |a_2| = 4/3$ and Suffridge [27] has given an example of $f \in \Sigma$ for which $|a_2| = 4/3$. Later, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. A brief summery of functions in the family Σ can be found in the study of Srivastava et al. [25], which is a basic research on the bi-univalent function family Σ (also, see the references cited therein). In a number of sequels to [25], bounds for the first two coefficients $|a_2|$ and $|a_3|$ of different subclasses of bi-univalent functions were given, for example, see [1, 8, 16, 23, 27]. But the coefficient estimate problem for each of $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2\}; \mathbb{N} = \{1, 2, 3, \cdots\})$ is still an open problem. In recent years, Srivastava et al.'s pioneering research on the subject [25] has successfully revitalized the study of bi-univalent functions to have produced numerous bi-univalent function papers. There are also several papers dealing with bi-univalent functions defined by subordination, for example, see [2, 3, 17].

Let a, b and c be complex numbers with $c \neq 0, -1, -2, \cdots$. Then the Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{7}$$

where $(\eta)_k$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\eta)_k = \frac{\Gamma(\eta + k)}{\Gamma(\eta)} = \begin{cases} 1 & (k = 0) \\ \eta(\eta + 1) \cdots (\eta + k - 1) & (k \in \mathbb{N}). \end{cases}$$

The hypergeometric function ${}_2F_1(a, b; c; z)$ is analytic in $\mathbb{U}$ and if a or b is a negative integer, then it reduces to a polynomial.

Various definitions of operators of fractional calculus are available in the literature (cf., e.g. [10, 22, 24]). Let us mention the Saigo hypergeometric operators. Below is their definition due to Saigo [21] (see also [5, 11, 19]).

Definition 1. For $\lambda > 0$, μ, $\nu \in \mathbb{R}$, the fractional integral operator $\mathcal{I}_{0,z}^{\lambda,\mu,\nu}$ is defined by

$$\begin{aligned}&\mathcal{I}_{0,z}^{\lambda,\mu,\nu} f(z)\\ &\qquad = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-\zeta)^{\lambda-1} {}_2F_1\left(\lambda+\mu, -\nu; \lambda; 1-\frac{\zeta}{z}\right) f(\zeta)\, d\zeta, \qquad (8)\end{aligned}$$

where ${}_2F_1$ is the Gaussian hypergeometric function defined by (1.3) and $f(z)$ is taken to be an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = \mathcal{O}(|z|^\epsilon) \quad (z \to 0)$$

for $\epsilon > \max\{0, \mu-\nu\} - 1$, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring that $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

The definition (8) is an interesting extension of both Riemann-Liouville and Erdélyi-Kober fractional operators including the Gauss hypergeometric function in the kernel.

Based on this definition, Owa et al. [19] (see also in [11], [12]) defined a modification (normalized version) of the fractional integral operator $\mathcal{I}$ as follows:

$$\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(2-\mu+\nu)} z^\mu\, \mathcal{I}_{0,z}^{\lambda,\mu,\nu} f(z)$$

for $f(z) \in \mathcal{A}$ and $\mu - \nu < 2$. Then in the above mentioned works it is observed that $\mathcal{J}_{0,z}^{\lambda,\mu,\nu}$ maps $\mathcal{A}$ into itself, and the image of a power series (1) has the form:

$$\mathcal{J}_{0,z}^{\lambda,\mu,\nu} f(z) = z + \sum_{n=2}^{\infty} \phi_n a_n z^n,$$

where

$$\phi_n = \frac{(2-\mu+\nu)_{n-1}(2)_{n-1}}{(2-\mu)_{n-1}(\lambda+\nu+2)_{n-1}} \quad (f \in \mathcal{A}; \lambda > 0; \mu-\nu < 2). \qquad (9)$$

We note that

$$\begin{aligned}&\mathcal{J}_{0,z}^{0,0,\nu} f(z) = f(z)\\ &\mathcal{J}_{0,z}^{\gamma-\delta+1,0,\delta-2} f(z) = \mathcal{I}_{\gamma,\delta} f(z) \quad (f \in \mathcal{A}; \gamma+1 > \delta > 0)\\ &\mathcal{J}_{0,z}^{\gamma,0,\delta-1} f(z) = \mathcal{Q}_\delta^\gamma(f)(z) \quad (f \in \mathcal{A}; \gamma \geq 0; \delta > -1),\end{aligned}$$

where $\mathcal{I}_{\gamma,\delta}$ and $\mathcal{Q}_{\delta}^{\gamma}(f)$ are the integral operators introduced by Choi et al. [6] and Liu [14]. Also, Kiryakova [11] considered the properties of this modification in the class of analytic functions (see also [12]).

Now we define the following subclass of function class Σ.

Definition 2. Let $0 \leq \delta \leq 1$, $\lambda > 0$, $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu - \nu < 2$. A function $f \in \Sigma$ is said to be in the subclass $\mathcal{M}_{\Sigma,q}^{\lambda,\mu,\nu}(\gamma,\delta,\varphi)$, if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z(\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z))'}{(1-\delta)z+\delta\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}-1\right) \prec_q \varphi(z)-1 \quad (z\in\mathbb{U})$$

and

$$\frac{1}{\gamma}\left(\frac{w(\mathcal{J}_{0,w}^{\lambda,\mu,\nu}g(w))'}{(1-\delta)w+\delta\mathcal{J}_{0,w}^{\lambda,\mu,\nu}g(w)}-1\right) \prec_q \varphi(w)-1 \quad (w\in\mathbb{U}),$$

where $g = f^{-1}$ is given by (6).

Remark 1. Taking $\lambda = \mu = 0$ and $\delta = 1$ in $\mathcal{M}_{\Sigma,q}^{\lambda,\mu,\nu}(\gamma,\delta,\varphi)$, we have

$$\mathcal{M}_{\Sigma,q}^{0,0,\nu}(\gamma,1,\varphi) = \mathcal{S}_{\Sigma,q}^{*}(\gamma,\varphi)$$

which was introduced and studied by Magesh *et al.* [15]. Also, we note that for $h(z) \equiv 1$ the class $\mathcal{S}_{\Sigma,q}^{*}(\gamma,\varphi) = \mathcal{S}_{\Sigma}^{*}(\gamma,\varphi)$ was introduced and studied by Deniz [7].

The object of the present paper is to investigate the coefficient estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to the subclass $\mathcal{M}_{\Sigma,q}^{\lambda,\mu,\nu}(\gamma,\delta,\varphi)$. Also, we determine the Feketo-Szegö inequality for the class $\mathcal{M}_{\Sigma,q}^{\lambda,\mu,\nu}(\gamma,\delta,\varphi)$.

2. Main results

In order to establish our results, we need the following lemma.

Lemma 1 ([9]). *Let $\mathcal{P}$ be the class of all functions h analytic in $\mathbb{U}$ of the form*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

which satisfy $\operatorname{Re}(h(z)) > 0$ *for all $z \in \mathbb{U}$. Then if $h \in \mathcal{P}$, then* $|c_n| \leq 2$ $(n \in \mathbb{N})$.

We begin by proving the following result.

Theorem 1. *Let* $0 \leq \delta \leq 1$, $\lambda > 0$, $\gamma \in \mathbb{C}\backslash\{0\}$ *and* $\max\{\mu, \mu-\nu, -\lambda-\nu\} < 2$. *If the function* $f(z)$ *given by* (1) *belongs to* $\mathcal{M}_{\Sigma,q}^{\lambda,\mu,\nu}(\gamma,\delta,\varphi)$, *then*

$$|a_2| \leq \frac{|\gamma||h_0|B_1\sqrt{B_1}}{\sqrt{|\gamma[(3-\delta)\phi_3+(\delta^2-2\delta)\phi_2^2]h_0B_1^2-(2-\delta)^2\phi_2^2(B_2-B_1)|}} \tag{10}$$

and

$$\begin{aligned}|a_3| \leq \frac{|\gamma||h_1|B_1}{(3-\delta)\phi_3} &+ \frac{|\gamma||h_0||B_2-B_1|}{|(3-\delta)\phi_3+(\delta^2-2\delta)\phi_2^2|} \\ &+\frac{|\gamma||h_0|B_1[(2\delta-\delta^2)\phi_2^2+|2(3-\delta)\phi_3+(\delta^2-2\delta)\phi_2^2|]}{2(3-\delta)\phi_3|(3-\delta)\phi_3+(\delta^2-2\delta)\phi_2^2|},\end{aligned} \tag{11}$$

where ϕ_2 *and* ϕ_3 *are given by* (9).

Proof. Let $f \in \mathcal{M}_{\Sigma,q}^{\lambda,\mu,\nu}(\gamma,\delta,\varphi)$ and g be the analytic function of f^{-1} to $\mathbb{U}$. Then there exist two functions r and s, analytic in $\mathbb{U}$ with $r(0) = s(0) = 0$, $|r(z)| < 1$ and $|s(w)| < 1$ $(z, w \in \mathbb{U})$ such that

$$\frac{1}{\gamma}\left(\frac{z(\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z))'}{(1-\delta)z+\delta\mathcal{J}_{0,z}^{\lambda,\mu,\nu}f(z)}-1\right) = h(z)\left(\varphi(r(z))-1\right) \tag{12}$$

and

$$\frac{1}{\gamma}\left(\frac{w(\mathcal{J}_{0,w}^{\lambda,\mu,\nu}g(w))'}{(1-\delta)w+\delta\mathcal{J}_{0,w}^{\lambda,\mu,\nu}g(w)}-1\right) = h(w)\left(\varphi(s(w))-1\right). \tag{13}$$

Next, we define the function $p, q \in \mathcal{P}$ by

$$p(z) = \frac{1+r(z)}{1-r(z)} = 1+p_1z+p_2z^2+\cdots$$

and

$$q(w) = \frac{1+s(w)}{1-s(w)} = 1+q_1w+q_2w^2+\cdots$$

or equivalently,

$$r(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}p_1z+\frac{1}{2}\left(p_2-\frac{1}{2}p_1^2\right)z^2+\cdots \tag{14}$$

and

$$s(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}q_1w+\frac{1}{2}\left(q_2-\frac{1}{2}q_1^2\right)w^2+\cdots. \tag{15}$$

Using (14) and (15) along with (3), it follows that

$$h(z)\left[\varphi\left(\frac{p(z)-1}{p(z)+1}\right)-1\right]=\frac{1}{2}h_0B_1p_1z$$
$$+\left(\frac{1}{2}h_1B_1p_1+\frac{1}{2}h_0B_1\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}h_0B_2p_1^2\right)z^2+\cdots \tag{16}$$

and

$$h(w)\left[\varphi\left(\frac{q(w)-1}{q(w)+1}\right)-1\right]=\frac{1}{2}h_0B_1q_1w$$
$$+\left(\frac{1}{2}h_1B_1q_1+\frac{1}{2}h_0B_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}h_0B_2q_1^2\right)w^2+\cdots . \tag{17}$$

By equating the coefficients from (12), (13), (16) and (17), we have

$$\frac{(2-\delta)\phi_2}{\gamma}a_2=\frac{1}{2}h_0B_1p_1, \tag{18}$$

$$\frac{(\delta^2-2\delta)\phi_2^2}{\gamma}a_2^2+\frac{(3-\delta)\phi_3}{\gamma}a_3$$
$$=\frac{1}{2}h_1B_1p_1+\frac{1}{2}h_0B_1\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}h_0B_2p_1^2, \tag{19}$$

$$-\frac{(2-\delta)\phi_2}{\gamma}a_2=\frac{1}{2}h_0B_1q_1, \tag{20}$$

and

$$\frac{(\delta^2-2\delta)\phi_2^2}{\gamma}a_2^2+\frac{(3-\delta)\phi_3}{\gamma}(2a_2^2-a_3)$$
$$=\frac{1}{2}h_1B_1q_1+\frac{1}{2}h_0B_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}h_0B_2q_1^2. \tag{21}$$

From (18) and (20), we find that

$$p_1=-q_1 \tag{22}$$

$$8(2-\delta)^2\phi_2^2a_2^2=\gamma^2h_0^2B_1^2(p_1^2+q_1^2). \tag{23}$$

If we add (19) to (21) and substitute (22), we obtain

$$\frac{2}{\gamma}\left[(3-\delta)\phi_3+(\delta^2-2\delta)\phi_2^2\right]a_2^2$$

$$= \frac{1}{2}h_0B_1(p_2+q_2) + \frac{1}{4}h_0(B_2-B_1)(p_1^2+q_1^2). \tag{24}$$

Substituting (23) into (24), we observe that

$$a_2^2 = \frac{\gamma^2 h_0^2 B_1^3 (p_2+q_2)}{4\gamma[(3-\delta)\phi_3 + (\delta^2-2\delta)\phi_2^2]h_0B_1^2 - 4(2-\delta)^2\phi_2^2(B_2-B_1)}. \tag{25}$$

By applying Lemma 1 in (25), we get the desired estimate of $|a_2|$ as asserted in (10).

Next, if we subtract (21) from (19) and a computation using (22) finally lead to

$$a_3 = a_2^2 + \frac{\gamma h_1 B_1 p_1}{2(3-\delta)\phi_3} + \frac{\gamma h_0 B_1}{4(3-\delta)\phi_3}(p_2-q_2). \tag{26}$$

Hence, from (24) and Lemma 1, we obtain the desired estimate of $|a_3|$ as asserted in (11). This completes the proof of Theorem 1.

Theorem 2. *Let* $0 \le \delta \le 1$, $\lambda > 0$, $\gamma \in \mathbb{C}\backslash\{0\}$ *and* $\max\{\mu, \mu-\nu, -\lambda-\nu\} < 2$, *and let* $\eta \in \mathbb{R}$. *If the function* $f(z)$ *given by* (1) *belongs to* $\mathcal{M}_{\Sigma,q}^{\lambda,\mu,\nu}(\gamma,\delta,\varphi)$, *then*

$$|a_3 - \eta a_2^2| \le \begin{cases} \dfrac{|\gamma|(|h_0|+|h_1|)B_1}{(3-\delta)\phi_3}, \\ \qquad\qquad |\eta - 1| \le \dfrac{|\psi(\gamma,\delta)|}{(3-\delta)|\gamma||h_0|\phi_3 B_1^2}\ , \\ \dfrac{|\gamma||h_1|B_1}{(3-\delta)\phi_3} + \dfrac{|\gamma|^2|h_0|^2|1-\eta|B_1^3}{|\psi(\gamma,\delta)|}, \\ \qquad\qquad |\eta - 1| \ge \dfrac{|\psi(\gamma,\delta)|}{(3-\delta)|\gamma||h_0|\phi_3 B_1^2}\ , \end{cases} \tag{27}$$

where ϕ_2 *and* ϕ_3 *are given by* (9), *and*

$$\psi(\gamma,\delta) = \gamma h_0 B_1^2[(3-\delta)\phi_3 + (\delta^2-2\delta)\phi_2^2] - (2-\delta)^2\phi_2^2(B_2-B_1).$$

Proof. From (25) and (26), it follows that

$$a_3 - \eta a_2^2 = (1-\eta)a_2^2 + \frac{\gamma h_1 B_1 p_1}{2(3-\delta)\phi_3} + \frac{\gamma h_0 B_1(p_2-q_2)}{4(3-\delta)\phi_3}$$

$$= \frac{\gamma h_1 B_1 p_1}{2(3-\delta)\phi_3} + \gamma h_0 B_1 \left[\left(u(\eta) + \frac{1}{4(3-\delta)\phi_3} \right) p_2 \right.$$
$$\left. + \left(u(\eta) - \frac{1}{4(3-\delta)\phi_3} \right) q_2 \right],$$

where

$$u(\eta) = \frac{(1-\eta)\gamma h_0 B_1^2}{4[\gamma h_0 B_1^2\{(3-\delta)\phi_3 + (\delta^2 - 2\delta)\phi_2^2\} - (2-\delta)^2\phi_2^2(B_2 - B_1)]}.$$

Then, by using Lemma 1, we conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \dfrac{|\gamma| B_1(|h_0| + |h_1|)}{(3-\delta)\phi_3}, \\ \qquad\qquad 0 \leq |u(\eta)| \leq \dfrac{1}{4(3-\delta)\phi_3}, \\ \dfrac{|\gamma||h_1|B_1}{(3-\delta)\phi_3} + 4|\gamma||h_0|B_1|u(\eta)|, \\ \qquad\qquad |u(\eta)| \geq \dfrac{1}{4(3-\delta)\phi_3}. \end{cases} \tag{28}$$

So (27) can be easily obtained from (28). This evidently completes the proof of Theorem 2.

By taking $\delta = 0$ in Theorem 1 and 2, we have the following corollary.

Corollary 1. *Let* $\lambda > 0$, $\gamma \in \mathbb{C} \setminus \{0\}$ *and* $\max\{\mu, \mu - \nu, -\lambda - \nu\} < 2$, *and let* $\eta \in \mathbb{R}$. *If the function* $f(z)$ *given by* (1) *belongs to* $\mathcal{M}_{\Sigma,q}^{\lambda,\mu,\nu}(\gamma, 0, \varphi)$, *then*

$$|a_2| \leq \frac{|\gamma||h_0|B_1\sqrt{B_1}}{\sqrt{|3\gamma h_0\phi_3 B_1^2 - 4\phi_2^2(B_2 - B_1)|}},$$

$$|a_3| \leq \frac{|\gamma|}{3\phi_3}\left[|h_0|(B_1 + |B_2 - B_1|) + |h_1|B_1\right],$$

and

$$|a_3-\eta a_2^2| \leq \begin{cases} \dfrac{|\gamma|(|h_0|+|h_1|)B_1}{3\phi_3}, \\ \qquad |\eta-1| \leq \dfrac{|3\gamma h_0\phi_3 B_1^2-4\phi_2^2(B_2-B_1)|}{3|\gamma||h_0|\phi_3 B_1^2}, \\ \dfrac{|\gamma||h_1|B_1}{3\phi_3}+\dfrac{|\gamma|^2|h_0|^2|1-\eta|B_1^3}{|3\gamma h_0\phi_3 B_1^2-4\phi_2^2(B_2-B_1)|}, \\ \qquad |\eta-1| \geq \dfrac{|3\gamma h_0\phi_3 B_1^2-4\phi_2^2(B_2-B_1)|}{3|\gamma||h_0|\phi_3 B_1^2}, \end{cases}$$

where ϕ_2 and ϕ_3 are given by (9).

By putting $\delta=1$ in Theorem 1 and 2, we get the following result.

Corollary 2. *Let $\lambda>0$, $\gamma\in\mathbb{C}\setminus\{0\}$ and $\max\{\mu,\mu-\nu,-\lambda-\nu\}<2$, and let $\eta\in\mathbb{R}$. If the function $f(z)$ given by* (1) *belongs to $\mathcal{M}_{\Sigma,q}^{\lambda,\mu,\nu}(\gamma,1,\varphi)$, then*

$$|a_2| \leq \frac{|\gamma||h_0|B_1\sqrt{B_1}}{\sqrt{|\gamma(2\phi_3-\phi_2^2)h_0B_1^2-\phi_2^2(B_2-B_1)|}},$$

$$|a_3| \leq \frac{|\gamma||h_1|B_1}{2\phi_3}+\frac{|\gamma||h_0|}{|2\phi_3-\phi_2^2|}\left[|B_2-B_1|+\frac{(\phi_2^2+|4\phi_3-\phi_2^2|)B_1}{4\phi_3}\right],$$

and

$$|a_3-\eta a_2^2| \leq \begin{cases} \dfrac{|\gamma|(|h_0|+|h_1|)B_1}{2\phi_3}, \\ \qquad |\eta-1| \leq \dfrac{|\gamma h_0(2\phi_3-\phi_2^2)B_1^2-\phi_2^2(B_2-B_1)|}{2|\gamma||h_0|\phi_3 B_1}, \\ \dfrac{|\gamma||h_1|B_1}{2\phi_3}+\dfrac{|\gamma|^2|h_0|^2|1-\eta|B_1^3}{|\gamma h_0(2\phi_3-\phi_2^2)B_1^2-\phi_2^2(B_2-B_1)|}, \\ \qquad |\eta-1| \geq \dfrac{|\gamma h_0(2\phi_3-\phi_2^2)B_1^2-\phi_2^2(B_2-B_1)|}{2|\gamma||h_0|\phi_3 B_1}, \end{cases}$$

where ϕ_2 and ϕ_3 are given by (9).

Remark 2. Taking $\lambda = \mu = 0$ in Corollary 2, we obtain a recent result due to Magesh *et al.* [15, Corollary 9]. Also, putting $\lambda = \mu = 0$ and $h(z) \equiv 1$ in Corollary 2, we get the result of Deniz [7, Corollary 2.3].

Acknowledgements

This work was supported by Daegu National University of Education Research grant in 2022.

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