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EXISTENCE AND UNIQUENESS OF SOLUTION FOR SPECIAL NONLOCAL DIFFUSION MODEL WITH NONLINEAR FLUX

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Abstract: Using the Banach Fixed Point Theorem, we show the existence and uniqueness of solution for the following nonlocal diffusion problem with Neumann conditions

$$\begin{cases} u_t(x,t) = (J * u(x,t)) - u(x,t) + \displaystyle\int_{\partial\Omega} G(x-y)g(u(y,t))dS_y \\ (x,t) \in \overline{\Omega} \times (0,T), \\ \\ u(x,0) = u_0(x), \qquad x \in \overline{\Omega}, \end{cases}$$

in which, we consider for the boundary conditions the border of a special domain $\partial\Omega \subseteq \mathbb{R}^{\mathbb{N}}$, instead of the complement used previously by other authors. With this, we show the calculations are reduced in a satisfactory way and therefore, we have a new model which is a complement of a previous work where the authors use a linear flux. In this new model, the flux in the border is considered as no lineal. Finally, a comparison principle is considered.

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1. Introduction

The classical (local) diffusion model is represented by the semilineal equation

$$u_t = \triangle u, \tag{1}$$

where u is the density function. In [18], the author obtain an analogous nonlocal of (1) described as

$$u_t = J * u - u, \tag{2}$$

where $*$ is the convolution between J and u, and J is a symmetric ($J(x) = J(-x)$) and positive function which satisfies $\int J(x)dx = 1$. Thinking as in [18], if $u(x,t)$ is the density function at point x in the time t, and if $J(x-y)$ represent the probability distribution of jump from y to x, then $(J * u)(x,t)$ is the rate at which the individuals arrive to x from all other places y and $-u(x,t) = -\int_{\mathbb{R}^N} J(x-y)u(x,t)dy$ is the rate at which the individuals travel from x to all other places y. In absence of external forces, the density u satisfies Eq. (2), which is called as *nonlocal diffusion equation.*

Many diffusion processes have been modeled by equations of the form (2), and variants of these. Analytic studies for this class of models have been made recently, see for instance the references [6], [9], [17], [19], [1],[2], [3],[4],[5],[7], [8], [10],[11], [12], [13], [14],[15], [16], [23], [24].

The authors in [20] have studied the problem

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)(u(y,t)-u(x,t))dy + \int_{\partial\Omega} G(x-y)g(y,t)dS_y, \\ u(x,0) = u_0(x), \\ (x,t) \in \Omega \times (0,T), \end{cases} \tag{3}$$

considered as analogues of the local problem ([22])

$$\begin{cases} u_t(x,t) - \triangle u(x,t) = 0 & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial u(x,t)}{\partial \eta} = g(x,t) & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{4}$$

In (3), $\Omega \subseteq \mathbb{R}^N$ is a bounded, connected and smooth domain, $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative, bounded, smooth and symmetric function $J(z) = J(-z)$, supported in the unit ball such that

$$\int_{\mathbb{R}^N} J(z)dz = 1,$$

$G(x)$ with the same characteristics of J, the initial datum $u_0(x)$, non negative and $g(y,t)$, a regular function. In this model, the individuals only can jump in Ω (the individuals may not enter or leave Ω) this fact is determined in the firs integral. On the frontier of Ω the number of individuals that can enter (if g is positive), is determined by $G(x-y)g(x,y)$. This fact is called in the literature as homogeneous Neumann boundary conditions for nonlocal problems. The authors have showed existence and uniqueness of solution for the model, and they have studied a comparison principle as well as the asymptotic behavior of the solutions. In [21], the authors have studied the following family of nonlocal models

$$\begin{cases} u_t^\epsilon(x,t) = \dfrac{1}{\epsilon^2}\displaystyle\int_\Omega J_\epsilon(x-y)(u^\epsilon(y,t)-u^\epsilon(x,t))dy \\ +\dfrac{1}{\epsilon}\displaystyle\int_{\partial\Omega} G_\epsilon(x-y)g(y,t)dS_y, \\ u^\epsilon(x,0)=u_0(x), \end{cases} \tag{5}$$

where

$$J_\epsilon(z) = C_1\frac{1}{\epsilon^N}J(\frac{z}{\epsilon}), \qquad G_\epsilon(x) = C_1\frac{1}{\epsilon^N}G(\frac{x}{\epsilon}),$$

C_1 a normalization constant. Here, J, G are the kernels given in Eq. (3). For this problem, they have used the Banach Fixed Point Theorem to show that, for each $\epsilon > 0$, there exist an one unique solution. The author in [22], showed that the solutions u^ϵ of (5), converges to solution of Eq.(4), as the parameter ϵ go to zero, in the sense of the convergence weak*. Is a relevant fact that the solutions of local problems can be approximate by means of solutions of nonlocal problems.

The main objective of this work is the study of the nonlocal model with nonlinear flux

$$\begin{cases} u_t(x,t) = (J * u(x,t)) - u(x,t) \\ +\displaystyle\int_{\partial\Omega} G(x-y)g(u(y,t))dS_y, \quad (x,t)\in\overline{\Omega}\times(0,T), \\ u(x,0)=u_0(x), \qquad x\in\overline{\Omega}. \end{cases} \tag{6}$$

As was mentioned previously, Ω is a connected and bounded domain of $\mathbb{R}^N$ with smooth frontier. $J * u - u$ the nonlocal diffusion operator with $J * u(x,t)$ the classical convolution

$$(J * u(x,t)) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy.$$

Additionally, we consider the function g, to satisfy the following two conditions:

1. (C1): $g : \mathbb{R} \longrightarrow [0, \infty)$ a Lipchitz function with Lipchitz constant k_g, increasing and convex function.

2. (C2): $\displaystyle\int^{\infty} \frac{1}{g(s)} ds < \infty.$

The initial datum $u_0(x) \in C^1(\overline{\Omega})$ nonnegative,

$$J, G : \mathbb{R}^N \to \mathbb{R}^N,$$

are functions in $C^1(\Omega)$, supported in the unit ball $B_1(x)$ of $\mathbb{R}^N$, and positive in B_1, radially symmetric

$$J(-x) = J(x), \quad G(-x) = G(x),$$

$$\int_{\mathbb{R}^N} J(x)dx = 1, \ \int_{\mathbb{R}^N} G(x)dx = 1,$$

and strictly decreasing. The interpretation of the model (6), is similar to those given for (3). In this case, the individuals that enter from $\partial\Omega$ is determined by

$$\int_{\partial\Omega} G(x-y)g(u(y,t))dS_y, \qquad g(s) > 0.$$

As a solution of (6), we will consider a function $u \in C[\overline{\Omega} \times [0, t_0]]$ satisfying (6) c.t.p.

In this sense, first, we prove existence and uniqueness of solutions for (6), and after, we will give a comparison principle.

2. Existence and uniqueness of solution

We will use the Banach fixed point theorem to show the existence and uniqueness of solution for Eq.(6). With this in mind, we fixed $t_0 > 0$, and use the Banach space given by

$$X_{t_0} = C([0, t_0]; C(\overline{\Omega})) = C([0, t_0] \times \overline{\Omega}),$$

with the norm

$$|||w||| = \max_{0 \le t \le t_0} ||w(\cdot, t)||_{\infty} = \max_{0 \le t \le t_0} \max_{x \in \overline{\Omega}} |w(x,t)| = \max_{(x,t) \in \overline{\Omega} \times [0, t_0]} |w(x,t)|.$$

Using w instead of u, and s intead of t, in (6), we have

$$w_s(x,s) = \int_\Omega J(x-y)[w(y,s)-w(x,s)]dy + \int_{\partial\Omega} G(x-y)g(w(y,s))dS_y. \quad (7)$$

After integration of (7), from 0 to t, we obtained

$$\begin{aligned} w(x,t) = \quad & w_0(x) + \int_0^t \int_\Omega J(x-y)[w(y,s)-w(x,s)]\,dy\,ds \\ & + \int_0^t \int_{\partial\Omega} G(x-y)g(w(y,s))\,dS_y\,ds. \end{aligned} \quad (8)$$

We define $D : X_{t_0} \longrightarrow X_{t_0}$ as

$$\begin{aligned} D_{w_0,g}(w(x,t)) = \quad & w_0(x) + \int_0^t \int_\Omega J(x-y)[w(y,s)-w(x,s)]\,dy\,ds \\ & + \int_0^t \int_{\partial\Omega} G(x-y)g(w(y,s))\,dS_y\,ds. \end{aligned} \quad (9)$$

With respect to D, we can enunciate the following lemma:

Lemma 1. *The operator $D_{w_0,g}$ in (9) is well defined.*

Proof. Let $0 < t_1 < t_2 \leq t_0$, $w \in X_{t_0}$, $||J||_\infty = K_1$, and $||G||_\infty = K_2$. Then we have:

$$
\begin{aligned}
&\Big| D_{w_0,g}[w(x,t_1)] - D_{w_0,g}[w(x,t_2)] \Big| \\
&= \Big| \int_{t_1}^{t_2} \int_{\Omega} J(x-y)[w(y,s)-w(x,s)]dyds \\
&\qquad + \int_{t_1}^{t_2} \int_{\partial\Omega} G(x-y)g(w(y,s))dS_y ds \Big| \\
&\leq \int_{t_1}^{t_2} \int_{\Omega} |J(x-y)||w(y,s)-w(x,s)|dyds \\
&\qquad + \int_{t_1}^{t_2} \int_{\partial\Omega} |G(x-y)||g(w(y,s))|dyds \\
&\leq (t_2-t_1)K_1|\Omega| \, |||w(y,s)-w(x,s)||| \\
&\qquad +(t_2-t_1)K_2 \\
&\leq \int_{\partial\Omega} |g(w(y,s))|dS_y \\
&\leq (t_2-t_1)\max\{1, K_1|\Omega|, K_2|\partial\Omega|\}\Big\{ 2|||w||| + ||g||_{\infty} \Big\}.
\end{aligned}
$$

With the previous estimates, $D_{w_0,g}$ is continuous in $t \in (0, t_0]$. The continuity for the case $t = 0$, can be obtained from the following calculations:

$$
\begin{aligned}
&\Big| D_{w_0,g}[w(x,t)] - w_0(x) \Big| \\
&= \Big| \int_{0}^{t} \int_{\Omega} J(x-y)[w(y,s)-w(x,s)]dyds \\
&\qquad + \int_{0}^{t} \int_{\partial\Omega} G(x-y)g(w(y,s))dS_y ds \Big| \\
&\leq (t)\max\{1, K_1|\Omega|, K_2|\partial\Omega|\}\Big\{ 2|||w||| + ||g||_{\infty} \Big\}.
\end{aligned}
$$

Therefore, we have continuity in $t \in [0, t_0]$. Clearly, as function of variable x, the operator is continuous, due to J is uniformly continuous ([20]). $\square$

Theorem 2. *For all $u_0 \in C(\overline{\Omega})$, there exists a one unique solution $u \in$*

$C[[0,\infty);C(\overline{\Omega})]$ *of the problem (6). Moreover, the* total mass *in* Ω *satisfies*

$$\int_\Omega u(y,t)dy = \int_\Omega u_0(y)dy + \int_0^t \int_\Omega \int_{\partial\Omega} G(x-y)g(u(y,s))dS_y\,dx\,ds. \tag{10}$$

Proof. Let

$$\begin{aligned} D_{w_0,g}(w(x,t)) = w_0(x) &+ \int_0^t \int_\Omega J(x-y)[w(y,s)-w(x,s)]\,dy\,ds \\ &+ \int_0^t \int_{\partial\Omega} G(x-y)g(w(y,s))\,dS_y\,ds, \end{aligned}$$

and

$$\begin{aligned} D_{w_0,g}(z(x,t)) = w_0(x) &+ \int_0^t \int_\Omega J(x-y)[z(y,s)-z(x,s)]\,dy\,ds \\ &+ \int_0^t \int_{\partial\Omega} G(x-y)g(z(y,s))\,dS_y\,ds, \end{aligned}$$

then

$$\begin{aligned} &\Big|D_{w_0,g}[w(x,t)] - D_{w_0,g}[z(x,t)]\Big| \\ &\le \Big|\int_0^t \int_\Omega J(x-y)[w(y,s)-z(y,s)]\,dy\,ds\Big| \\ &\qquad + \Big|\int_0^t \int_\Omega J(x-y)[w(x,s)-z(x,s)]\,dy\,ds\Big| \\ &\qquad + \Big|\int_0^t \int_{\partial\Omega} G(x-y)[g(w(y,s))-g(z(y,s))]\,dS_y\,ds\Big| \\ &\le t_0C_1||w-z||_\infty + t_0C_2||w-z||_\infty + t_0C_3||w-z||_\infty \\ &\le Ct_0||w-z||_\infty, \end{aligned} \tag{11}$$

where $C_1 = C_2 = K_1|\Omega|$, $C_3 = K_2|\partial\Omega|k_g$ y $C = \max\{C_1, C_3\}$. We have

$$\Big|\Big|\Big|D_{w_0,g}[w(x,t)] - D_{w_0,g}[z(x,t)]\Big|\Big|\Big| \le Ct_0|||w-z|||.$$

If we take $t_0C \le \dfrac{1}{2}$, by the Banach Fixed Point Theorem, we conclude the result.

Finally, if u is a solution of (6), using w instead of u in (8), we have

$$\begin{aligned} u(x,t) - u_0(x) = \int_0^t \int_\Omega J(x-y)[u(y,s) - u(x,s)]dy\, ds \\ + \int_0^t \int_{\partial\Omega} G(x-y)g(u(y,s))dS_y\, ds. \end{aligned} \tag{12}$$

After integration of (12) respect to x, we obtain

$$\begin{aligned} \textstyle\int_\Omega u(x,t)dx - \int_\Omega u_0(x)dx = \int_\Omega \int_0^t \int_\Omega J(x-y)u(y,s)dy\, ds\, dx \\ - \int_\Omega \int_0^t \int_\Omega J(x-y)u(x,s)dy\, ds\, dx \\ + \int_0^t \int_\Omega \int_{\partial\Omega} G(x-y)g(u(y,s))dS_y\, ds\, dx, \end{aligned}$$

from which

$$\int_\Omega u(x,t)dx - \int_\Omega u_0(x)dx = \int_0^t \int_\Omega \int_{\partial\Omega} G(x-y)g(u(y,s))dS_y\, dx\, ds, \tag{13}$$

obtaining in this way (10). □

We have the following corollary.

Corollary 3. *Let u, w be solutions of (6), with initial data $u_0 \in C(\overline{\Omega})$, $w_0 \in C(\overline{\Omega}$ and with boundary condition g. Then, for all $t_0 > 0$, such that $Ct_0 \leq \frac{1}{2}$, being C as in the previous theorem, there exists $\tilde{C}$ (depending on t_0) such that*

$$\max_{0\leq t\leq t_0} ||u(\cdot,t) - w(\cdot,t)||_\infty \quad \leq \tilde{C}||u(\cdot,0) - w(\cdot,0)||_\infty.$$

Proof. We have

$$\begin{aligned} u(x,t) = u_0(x) + \int_0^t \int_\Omega J(x-y)[u(y,s) - u(x,s)]dy\, ds \\ + \int_0^t \int_{\partial\Omega} G(x-y)g(u(y,s))dS_y\, ds, \end{aligned}$$

and

$$\begin{aligned} w(x,t) = w_0(x) + \int_0^t \int_\Omega J(x-y)[w(y,s) - w(x,s)]dy\, ds \\ + \int_0^t \int_{\partial\Omega} G(x-y)g(w(y,s))dS_y\, ds. \end{aligned}$$

Proceeding as in Theorem 2, we obtain

$$|u(x,t) - w(x,t)| \leq ||u_0(x,0) - w_0(x,0)||_\infty + Ct_0|u - w|.$$

We can conclude

$$(1 - Ct_0)|u(\cdot,t) - w(\cdot,t)| \leq ||u(\cdot,0) - w(\cdot,0)||_\infty,$$

and taking $\tilde{C} = \frac{1}{1-Ct_0}$, the result is obtained. □

Corollary 4. *Let $u \in X_{t_0}$. Then, u is solution of (6), if and only if*

$$\begin{aligned} u(x,t) = e^{-A(x)t}u_0(x) + \int_0^t \int_\Omega e^{-A(x)(t-s)} J(x-y)u(y,s)dy\,ds \\ + \int_0^t \int_{\partial\Omega} e^{-A(x)(t-s)} G(x-y)g(u(y,s))dS_y\,ds, \end{aligned} \tag{14}$$

with

$$A(x) = \int_\Omega J(x-y)dy.$$

Proof. Using u instead of w in (7), we have

$$u_s(x,s) = |1 \int_\Omega J(x-y)[u(y,s) - u(x,s)]dy + \int_{\partial\Omega} G(x-y)g(u(y,s))dS_y.$$

Multiplying both sides of previous equation by $e^{A(x)s}$, we have

$$\begin{aligned} &u_s(x,s)e^{A(x)s} + \int_\Omega e^{A(x)s} J(x-y)u(x,s)dy \\ &= \int_\Omega e^{A(x)s} J(x-y)u(y,s)dy + \int_{\partial\Omega} e^{A(x)s} G(x-y)g(u(y,s))dS_y. \end{aligned}$$

In a similar way,

$$\begin{aligned} \tfrac{d}{ds}(u(x,s)e^{A(x)s}) &= \int_\Omega e^{A(x)s} J(x-y)[u(y,s)dy \\ &+ \int_{\partial\Omega} e^{A(x)s} G(x-y)g(u(y,s))dS_y. \end{aligned}$$

Integrating this last equation from 0 to t, we have

$$\begin{aligned} u(x,t)e^{A(x)t} - u_0(x) &= \int_0^t \int_\Omega e^{A(x)s} J(x-y)[u(y,s)dy\,ds \\ &+ \int_0^t \int_{\partial\Omega} e^{A(x)s} G(x-y)g(u(y,s))dS_y\,ds, \end{aligned}$$

from which we obtain (14). □

Remark 5. Note that from conditions on J, there exists a constant α such that $A(x) \geq \alpha > 0$, for each $x \in \overline{\Omega}$.

Remark 6. (With respect to local existence and uniqueness). We consider (6), and g, Lipchitz locally and $u_0 \in C(\overline{\Omega})$. Then, for any $t_1 > 0$, there exists one unique solution $u(x,t) \in C(\overline{\Omega} \times [0,t_1])$.

In fact, consider

$$M = ||u_0||_{L^\infty} + r,$$

$r > 0$, and g Lipchitz on $[-M, M]$. Let $\tilde{g}$ the extension of g to all $\mathbb{R}$, in a such way that $\tilde{g}$, continue being Lipchitz on $\mathbb{R}$. We consider the problem

$$\begin{cases} u_t(x,t) = (J(x) * u(x,t)) - u(x,t) + \displaystyle\int_{\partial\Omega} G(x-y)\tilde{g}(u(x,t))dS_y, \\ (x,t) \in \overline{\Omega} \times (0,T), \\ u(x,0) = u_0(x), \qquad x \in \overline{\Omega}. \end{cases}$$

As in Theorem 2, there exists one unique solution $\tilde{u}(x,t) \in C(\overline{\Omega} \times [0,t_0])$ of that problem ($\tilde{u}(x,0) = u_0(x)$). Additionally

$$-M \leq |||\tilde{u}(x,t)||| \leq M,$$

for $t \in [0,t_1]$, any $t_1 < t_0$. Furthermore,

$$\tilde{g}(\tilde{u}) = g(\tilde{u})$$

for $(x,t) \in \overline{\Omega} \times [0,t_1]$. Therefore, taking $u = \tilde{u}$, we have the solution of the initial problem. The uniqueness is obtained from the respective uniqueness of $\tilde{u}$.

Under regularity of the kernels and the initial datum, we can obtain solutions with more regularity. We can enunciate this fact in the following corollary which proof depends on Corollary 4, and equation (14), and we omit here:

Corollary 7. *1. If $u \in C[\Omega \times (0,T)]$, $u_0 \in C^k(\overline{\Omega})$, with $0 \leq k \leq \infty$, $g \in L^\infty(\mathbb{R})$, y $J, G \in W^{k,1}(\mathbb{R}^N)$, then $u(\cdot,t) \in C^k(\Omega \times [0,T])$.*

2. If $J, G \in L^\infty(\mathbb{R}^N)$, $u_0 \in C^\infty(\Omega)$ y $g \in L^\infty(\mathbb{R})$, then $u \in L^\infty[\Omega \times (0,T)]$.

We finish this work with following comparison principle:

Corollary 8. *Let u, v be solutions of (6), with initial datums $u_0, v_0 \in C^1(\Omega)$, and boundary condition g (increasing) respectively. Suppose that $u_0 > v_0$. Then $u > v$* c.t.p.

Proof. By Corollary 4, we have

$$u(x,t) = e^{-A(x)t}u_0(x) + \int_0^t \int_\Omega e^{-A(x)(t-s)} J(x-y)u(y,s)dy\, ds + \int_0^t \int_{\partial\Omega} e^{-A(x)(t-s)} G(x-y)g(u(y,s))dS_y\, ds,$$

and

$$v(x,t) = e^{-A(x)t}v_0(x) + \int_0^t \int_\Omega e^{-A(x)(t-s)} J(x-y)v(y,s)dy\, ds + \int_0^t \int_{\partial\Omega} e^{-A(x)(t-s)} G(x-y)g(v(y,s))dS_y\, ds.$$

Let $w(x,t) = u(x,t) - v(x,t)$. Therefore $w(x,0) = u(x,0) - v(x,0) = u_0 - v_0 = w_0 > 0$. In the case that the enunciate was false, there exists $\tilde{x}$ and $\tilde{t}$, such that $w(\tilde{x},\tilde{t}) = 0$ and $w(y,s) > 0$ for $(y,s) \in \overline{\Omega} \times [0,\tilde{t})$. Then,

$$\begin{aligned} 0 = w(\tilde{x},\tilde{t}) = e^{-A(x)t}w_0(x) &+ \int_0^{\tilde{t}} \int_\Omega e^{-A(\tilde{x})(\tilde{t}-s)} J(\tilde{x}-y)w(y,s)dy\, ds \\ &+ \int_0^{\tilde{t}} \int_{\partial\Omega} e^{-A(\tilde{x})(\tilde{t}-s)} G(\tilde{x}-y)g'(\xi)(w(y,s))dS_y\, ds \\ &> 0, \end{aligned}$$

which is a contradiction. So that, must be $w > 0$, for all $(x,t) \in \overline{\Omega} \times [0, t_0)$. $\square$

3. Conclusions

We have used the Banach Fixed Point Theorem for proved existence and uniqueness of solutions for a nonlocal problem with nonlinear flux. The work, is complementary to results presented in the reference [20], where we have used a linear flux. Comparing our results with those obtained by other authors [14], clearly we have obtained a reduction in all calculations, using $\partial\Omega$, instead of $\mathbb{R}^{\mathbb{N}} - \Omega$. The importance of the obtained results here, consist on the fact that several types of model as we have mentioned here, have very similarity with respective local problems, and as we have showed in [22], solutions of this last models, can be approximate using solutions of adequate nonlocal problems. In the references, we can found a variety of problems, which can be improved using the frontier used in our work.

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