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STABILITY OF A TIMOSHENKO SYSTEM WITH CONSTANT DELAY

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Abstract: The aim of this work is to develop a detail analysis of a Timoshenko type beam model taking into account a delay. We prove the well-posedness and regularity of solution, explained using the theory of the Faedo-Galerkin scheme. Namely, under a suitable choice of Lyapunov functional, exponential decay of the whole energy holds.

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Key Words: Timoshenko system; exponential stability; delay; Fadeo-Galerkin method

1. Introduction

Here we consider the following Shear beam model and new facts related to the classical Timoshenko system with internal delay

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - \kappa(\varphi_x(x,t)+\psi(x,t))_x + \mu_0\varphi_t(x,t) \\ \qquad +\mu_1\varphi_t(x,t-\tau) = 0 \quad \text{in } \ ]0,L[\times(0,+\infty), \\ -b\psi_{xx}(x,t) + \kappa(\varphi_x(x,t)+\psi(x,t)) = 0 \text{ in } \ ]0,L[\times(0,+\infty). \end{cases} \tag{1}$$

Additionally, we consider initial conditions given by

$$\varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \quad \psi(x,0)=\psi_0(x), \quad x\in]0,L[,$$

where $\varphi_0, \varphi_1, \psi_0$ are given functions, and the boundary conditions of Dirichlet

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are given by

$$\psi_x(0,t) = \psi_x(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \quad t > 0.$$

Note that the functions φ and ψ describe the transverse displacement of the beam and the rotation angle of a filament of the beam, respectively. $\rho_1, \kappa, \mu_0, \mu_1, b$ and τ are positive constants.

It is well known that if $\mu_1 = 0$, that is in absence of delay, the corresponding following system:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - \kappa(\varphi_x(x,t) + \psi(x,t))_x + \mu\varphi_t(x,t) = 0 \\ \qquad\qquad\qquad\qquad\qquad \text{in } \ ]0,L[\times(0,+\infty), \\ -b\psi_{xx}(x,t) + \kappa(\varphi_x(x,t) + \psi(x,t)) = 0 \text{ in } \ ]0,L[\times(0,+\infty), \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x) \quad \text{in }]0,L[, \\ \psi_x(0,t) = \psi_x(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \quad t > 0, \end{cases} \tag{2}$$

was analyzed in a recent paper by Almeida and al. [1], where it was shown that the energy of the system decays exponentially to zero.

In the case of the wave equations, Nicaise and al. [8] investigated exponential stability results with delay concentrated at τ for the system

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega \times (0,+\infty), \\ u(x,t) = 0 \text{ on } \Gamma_D \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0 \text{ on } \Gamma_N \times (0,+\infty), \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 \text{ in } \Omega, \\ u(x,t-\tau) = f_0(x,t-\tau) \text{ on } \Gamma_N \times (0,\tau), \end{cases} \tag{3}$$

under the condition $\mu_2 < \mu_1$, by combining inequalities due to Carleman estimates and compactness-uniqueness arguments.

Recently Bayili and al. [2] in the case of the wave equation with dynamical control, studied a wave equation set in a bounded domain with a dynamical control. For the strong stability result, they use the spectral decomposition theory of Sz-Nagy-Foias Parrott [9], Claude, Chen and al [5],[3]. Then they prove that if the delay term is small enough, then the system with delay

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega \times (0,+\infty), \\ u(x,t) = 0 \text{ on } \Gamma_D \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu}(x,t) + \eta(x,t) = 0 \text{ on } \Gamma_N \times (0,+\infty), \\ \eta_t(x,t) - u_t(x,t) + \beta_1 \eta(x,t) + \beta_2 \eta(x,t-\tau) = 0 \\ \qquad\qquad\qquad\qquad\qquad \text{on } \Gamma_N \times (0,+\infty), \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 \text{ in } \Omega, \\ \eta(x,0) = \eta_0 \text{ on } \Gamma_N, \\ \eta(x,t-\tau) = f_0(x,t-\tau) \text{ on } \Gamma_N \times (0,\tau), \end{cases} \tag{4}$$

has the same (polynomial) decay rate than the one without delay. In this paper, we assume that there exists a positive constant ζ verifying

$$\tau\mu_0 < \zeta < \tau(2\mu_0 - \mu_1). \tag{5}$$

The remaining of the paper is organized as follows. In Section 2 the well-posedness of the problem (1) is analyzed using the Faedo-Galerkin method. And finally in Section 3 we prove the exponential decay of the energy when time goes to infinity using a Lyapounov function.

2. Well-posedness of the problem

In this section we will give well-posedness results for problem (1) using Faedo-Galerkin method. To this aim, we introducing the following auxiliary change of variable

$$z(x,\rho,t) = \varphi_t(x,t-\tau\rho), \quad x \in]0,L[, \ \rho \in (0,1), \ t > 0. \tag{6}$$

The problem (1) is now equivalent to

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - \kappa(\varphi_x(x,t) + \psi(x,t))_x + \mu_0 \varphi_t(x,t) + \mu_1 z(x,1,t) = 0 \\ \qquad\qquad\qquad\qquad \text{in }]0,L[\times(0,+\infty), \\ -b\psi_{xx}(x,t) + \kappa(\varphi_x(x,t) + \psi(x,t)) = 0 \quad \text{in }]0,L[\times(0,+\infty), \\ \tau z_t(\rho,t) + z_\rho(\rho,t) = 0 \quad \text{in } (0,1)\times(0,+\infty); \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x) \quad \text{in }]0,L[, \\ z(x,\rho,0) = f_0(x,-\rho\tau) \ \ \forall \rho \in (0,1), \\ z(x,0,t) = \varphi_t(x,t) \ \ \forall t \in (0,+\infty) \\ \psi_x(0,t) = \psi_x(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \quad t > 0. \end{cases} \tag{7}$$

Let us consider the Hilbert spaces

$$\mathcal{H} = H^1(0,L) \times L^2(0,L) \times H^1_*(0,L) \times L^2\Big((0,L)\times(0,1)\Big),$$

and

$$\mathcal{H}_1 = (H^2(0,L)\cap H^1_0(0,L))^2 \times H^2_*(0,L) \times H^1\Big((0,L)\times(0,1)\Big),$$

where

$$L^2_*(0,L) = \left\{u \in L^2(0,L), \int_0^L u(x)dx = 0\right\},$$

$$H^1_*(0,L) = H^1(0,L)\cap L^2_*(0,L),$$

and

$$H^2_*(0,L) = H^2(0,L)\cap H^1_*(0,L).$$

We equipped $\mathcal{H}$ with the norm

$$\begin{aligned}\left\|(u,v,w,z)^{\mathsf{T}}\right\|^2_{\mathcal{H}} &= \frac{\rho_1}{2}\int_0^L |v|^2dx + \frac{b}{2}\int_0^L |w_x|^2dx \\ &+\frac{\kappa}{2}\int_0^L |u_x+w|^2dx + \frac{\zeta}{2}\int_0^L\int_0^1 |z|^2d\rho dx, .\end{aligned}$$

where ζ is a positive constant verifying

$$\tau\mu_0 < \zeta < \tau(2\mu_0-\mu_1). \tag{8}$$

Let (φ,ψ,z) be a solution of (7), the corresponding energy is given by

$$\begin{aligned}E(t) &= \frac{\rho_1}{2}\int_0^L |\varphi_t|^2dx + \frac{b}{2}\int_0^L |\psi_x|^2dx + \frac{\kappa}{2}\int_0^L |\varphi_x+\psi|^2dx \\ &+\frac{\zeta}{2}\int_0^L\int_0^1 |z|^2d\rho dx.\end{aligned}$$

Theorem 1. *Under the assumption* (5) *we have*

$$\frac{d}{dt}E(t) \le 0.$$

Proof. Multiplying $(7)_1$ by φ_t and integrating on $[0;L]$, we have

$$\rho_1\int_0^L \varphi_{tt}(x,t)\varphi_t(x,t)dx - \kappa\int_0^L \left[\varphi_x(x,t)+\psi(x,t)\right]_x \varphi_t(x,t)dx$$

$$+\mu_0 \int_0^L \varphi_t(x,t)\varphi_t(x,t)dx + \mu_1 \int_0^L z(x,1,t)\varphi_t(x,t)dx = 0.$$

Integrating by parts, we obtain

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^L |\varphi_t(x,t)|^2 dx - \kappa\left[(\varphi_x+\psi)\varphi_t(x,t)\right]_0^L$$

$$+\kappa \int_0^L \left[\varphi_x(x,t)+\psi(x,t)\right]\varphi_{tx}(x,t)dx + \mu_0 \int_0^L |\varphi_t(x,t)|^2 dx$$

$$+\mu_1 \int_0^L z(x,1,t)\varphi_t(x,t)dx = 0.$$

Since φ is zero at 0 and L, we have

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^L |\varphi_t(x,t)|^2 dx + \kappa \int_0^L \left[\varphi_x(x,t)+\psi(x,t)\right]\varphi_{tx}(x,t)dx$$
$$+\mu_0 \int_0^L |\varphi_t(x,t)|^2 dx + \mu_1 \int_0^L z(x,1,t)\varphi_t(x,t)dx = 0. \tag{9}$$

Multiply $(7)_2$ by ψ_t and integrate on $[0;L]$ we have

$$-b\int_0^L \psi_{xx}(x,t)\psi_t(x,t)dx + \kappa \int_0^L \left[\varphi_x(x,t)+\psi(x,t)\right]\psi_t(x,t)dx = 0.$$

Integrating by parts, we obtain

$$-b\left[\psi_x\psi_t(x,t)\right]_0^L + b\int_0^L \psi_x(x,t)\psi_{xt}(x,t)dx$$

$$+\kappa \int_0^L \left[\varphi_x(x,t)+\psi(x,t)\right]\psi_t(x,t)dx = 0.$$

Since ψ_x is zero at 0 and L, we have

$$b\int_0^L \psi_x(x,t)\psi_{xt}(x,t)dx + \kappa \int_0^L \left[\varphi_x(x,t)+\psi(x,t)\right]\psi_t(x,t)dx = 0,$$

which is written as

$$\frac{b}{2}\frac{d}{dt}\int_0^L |\psi_x(x,t)|^2 dx + \kappa \int_0^L \left[\varphi_x(x,t)+\psi(x,t)\right]\psi_t(x,t)dx = 0. \tag{10}$$

Multiply $(7)_3$ by $\dfrac{\zeta}{\tau}z$ and integrate on $[0;L]\times[0;1]$ we have

$$\zeta\int_0^L\int_0^1 z_t(x,\rho,t)z(x,\rho,t)d\rho dx+\frac{\zeta}{\tau}\int_0^L\int_0^1 z_\rho(x,\rho,t)z(x,\rho,t)dx=0,$$

written as

$$\frac{\zeta}{2}\frac{d}{dt}\int_0^L\int_0^1|z(x,\rho,t)|^2d\rho dx+\frac{\zeta}{2\tau}\int_0^L\left[|z(x,\rho,t)|^2\right]_0^1dx=0.$$

Finally,

$$\begin{aligned}&\frac{\zeta}{2}\frac{d}{dt}\int_0^L\int_0^1|z(x,\rho,t)|^2d\rho dx\\&+\frac{\zeta}{2\tau}\int_0^L\left[|z(x,1,t)|^2-|z^2(x,0,t)|^2\right]dx=0.\end{aligned}\tag{11}$$

By adding $(9)-(11)$ we obtain

$$\begin{aligned}&\frac{\rho_1}{2}\frac{d}{dt}\int_0^L|\varphi_t(x,t)|^2dx+\kappa\int_0^L\left[\varphi_x(x,t)+\psi(x,t)\right]\varphi_{tx}(x,t)dx\\&+\mu_0\int_0^L|\varphi_t(x,t)|^2dx+\mu_1\int_0^L z(x,1,t)\varphi_t(x,t)dx\\&+\frac{b}{2}\frac{d}{dt}\int_0^L|\psi_x(x,t)|^2dx+\kappa\int_0^L\left[\varphi_x(x,t)+\psi(x,t)\right]\psi_t(x,t)dx\\&+\frac{\zeta}{2}\frac{d}{dt}\int_0^L\int_0^1|z(x,\rho,t)|^2d\rho dx\\&+\frac{\zeta}{2\tau}\int_0^L\left[|z(x,1,t)|^2-|z(x,0,t)|^2\right]dx=0.\end{aligned}$$

This means that

$$\begin{aligned}&\frac{\rho_1}{2}\frac{d}{dt}\int_0^L|\varphi_t(x,t)|^2dx+\mu_0\int_0^L|\varphi_t(x,t)|^2dx\\&+\kappa\int_0^L\left[\varphi_x(x,t)+\psi(x,t)\right]\left[\varphi_x(x,t)+\psi(x,t)\right]_t dx\\&+\mu_1\int_0^L z(x,1,t)\varphi_t(x,t)dx+\frac{b}{2}\frac{d}{dt}\int_0^L|\psi_x(x,t)|^2dx\\&+\frac{\zeta}{2}\frac{d}{dt}\int_0^L\int_0^1|z(x,\rho,t)|^2d\rho dx\end{aligned}$$

$$+\frac{\zeta}{2\tau}\int_0^L\left[|z(x,1,t)|^2-|z(x,0,t)|^2\right]dx=0.$$

So,

$$\begin{aligned}\frac{d}{dt}\left\{\frac{\rho_1}{2}\int_0^L|\varphi_t(x,t)|^2dx+\frac{\kappa}{2}\int_0^L|\varphi_x(x,t)+\psi(x,t)|^2dx\right.\\ \left.+\frac{b}{2}\int_0^L|\psi_x(x,t)|^2dx+\frac{\zeta}{2}\int_0^L\int_0^1|z(x,\rho,t)|^2d\rho dx\right\}\\ +\mu_0\int_0^L|\varphi_t(x,t)|^2dx+\mu_1\int_0^L z(x,1,t)\varphi_t(x,t)dx\\ +\frac{\zeta}{2\tau}\int_0^L\left[|z(x,1,t)|^2-|z(x,0,t)|^2\right]dx=0.\end{aligned}$$

And this leads to

$$\begin{aligned}\frac{d}{dt}E(t) &= -\mu_0\int_0^L|\varphi_t(x,t)|^2dx-\mu_1\int_0^L z(x,1,t)\varphi_t(x,t)dx\\ &- \frac{\zeta}{2\tau}\int_0^L\left[|z(x,1,t)|^2-|z(x,0,t)|^2\right]dx.\end{aligned}$$

Since $\varphi_t(x,t)=z(x,0,t)$, we have

$$\begin{aligned}\frac{d}{dt}E(t) &= -\mu_0\int_0^L|z(x,0,t)|^2dx-\mu_1\int_0^L z(x,1,t)z(x,0,t)dx\\ &- \frac{\zeta}{2\tau}\int_0^L\left[|z(x,1,t)|^2-|z(x,0,t)|^2\right]dx. \qquad (12)\end{aligned}$$

We also know that

$$\begin{aligned}z(x,1,t)z(x,0,t) &= \frac{1}{2}|z(x,1,t)+z(x,0,t)|^2-\frac{1}{2}|z(x,0,t)|^2\\ &- \frac{1}{2}|z(x,1,t)|^2. \qquad (13)\end{aligned}$$

Using (13) in (12), we have

$$\begin{aligned}\frac{d}{dt}E(t) &= \left[\frac{\zeta-\tau(2\mu_0-\mu_1)}{2\tau}\right]\int_0^L|z(x,0,t)|^2dx\\ &- \frac{\mu_1}{2}\int_0^L|z(x,1,t)+z(x,0,t)|^2dx\end{aligned}$$

$$+ \quad \left[\frac{\tau\mu_1 - \zeta}{2\tau}\right] \int_0^L |z(x,1,t)| dx. \tag{14}$$

We have $-\dfrac{\mu_1}{2} < 0$ and by hypothesis (5), we get that

$$\frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} < 0 \text{ and } \frac{\tau\mu_1 - \zeta}{2\tau} < 0.$$

Therefore $\dfrac{d}{dt}E(t) \leq 0$, hence the system (7) is dissipative. □

3. Existence and uniqueness

In order to state our main result in this section, we start by defining that we mean by a weak solution of the problem (7) as follows:

Definition 2. Given an initial data $U_0 = (\varphi_0, \varphi_1, \psi_0, z_0) \in \mathcal{H}$, a function $U = (\varphi, \varphi_t, \psi, z) \in C([0,T]; \mathcal{H})$ is said to be weak solution of (7) if for almost everywhere $t \in [0,T]$,

$$\begin{cases} \rho_1 \dfrac{d}{dt}(\varphi_t, u) + \kappa(\varphi_x(x,t) + \psi(x,t), u_x) + \mu_0(\varphi_t, u) \\ \qquad\qquad +\mu_1(z(.,1,t), u) = 0, \\ b(\psi_x, v_x) + \kappa(\varphi_x + \psi, v) = 0, \\ \tau(z_t, w) + (z_\rho, w) = 0, \end{cases} \tag{15}$$

for all $u \in H_0^1(0,L), w \in H_*^2(0,L), w \in L^2((0,L) \times (0,1)$ and $(\varphi(0), \varphi_t(0), \psi(0), z(0)) = (\varphi_0, \varphi_1, \psi_0, z_0)$.

The main result of this section is the following:

Theorem 3.
Assume that (5) holds. Then for any data $U_0 = (\varphi_0, \varphi_1, \psi_0, f_0)$ belongs to $\mathcal{H}$, the problem (7) has one and only one weak solution $U = (\varphi, \varphi_t, \psi, z)$ verifying:

$$\begin{cases} \varphi \in L^\infty\left(0, T, H_0^1(0,L)\right), \\ \varphi_t \in L^\infty\left(0, T, L^2(0,L)\right), \\ \psi \in L^\infty\left(0, T, H_*^1(0,L)\right), \\ z \in L^\infty\left(0, T, L^2((0,L) \times (0,1))\right). \end{cases} \tag{16}$$

Moreover, if $U_0 = (\varphi_0, \varphi_1, \psi_0, f_0)$ *belongs to* $\mathcal{H}_1$*, then problem* (7) *has one and only one strong weak solution* $U = (\varphi, \varphi_t, \psi, z)$ *which satisfies*

$$\begin{cases} \varphi \in L^\infty\left(0, T, H^2(0,L) \cap H_0^1(0,L)\right), \\ \varphi_t \in L^\infty\left(0, T, H_0^1(0,L)\right), \\ \psi \in L^\infty\left(0, T, H_*^2(0,L)\right), \\ z \in L^\infty\left(0, T, H^1((0,L) \times (0,1))\right). \end{cases} \tag{17}$$

Proof. The Faedo-Galerkin method will be the key to prove the existence of a global solution.

Step 1. Let us consider initial data $(\varphi_0, \varphi_1, \psi_0, f_0) \in \mathcal{H}$.

Let $\{u^k\}, k \in \mathbb{N}^\star$ and $\{v^k\}, k \in \mathbb{N}^\star$ basics formed by eigenfunctions of $-\partial_{xx}$. This bases can be considered orthogonal in $H^2(\Omega) \cap H_0^1(\Omega)$ and $H_*^2(0,L)$ respectively, and both orthogonormal in $L^2(0;L)$.

As Yazid, Chen et al. in [10],[4], we also define the sequence $\{w^k\}, k \in \mathbb{N}^\star$ in the following way

$$w^k(x,0) = u^k(x) \text{ then we extend } w^k(x,0) \text{ by } w^k(x,\rho)$$

on $L^2\Big((0,L) \times (0,1)\Big)$.

Approximation spaces H_n, V_n and W_n of finite dimensions are given by

$$H_n = span\left\{u^1, u^2, ..., u^n\right\}, \quad V_n = span\left\{v^1, v^2, ..., v^n\right\}$$

and $W_n = span\left\{w^1, w^2, ..., w^n\right\}, n \in \mathbb{N}^\star$.

We will find an approximate solution of the form:

$$\varphi^n(t,x) = \sum_{j=1}^n a^{jn}(t) u^j(x),$$

$$\psi^n(t,x) = \sum_{j=1}^n b^{jn}(t) v^j(x),$$

$$z^n(x,t,\rho) = \sum_{j=1}^n c^{jn}(t) w^j(x,\rho),$$

to the following approximate problem

$$\begin{cases} \rho_1 \displaystyle\int_0^L \varphi_{tt}^n(x,t)udx - \kappa \int_0^L \left[\varphi_x^n(x,t) + \psi^n(x,t)\right]_x udx \\ \qquad +\mu_0 \displaystyle\int_0^L \varphi_t^n(x,t)udx + \mu_1 \int_0^L z^n(x,1,t)udx = 0, \\ -b \displaystyle\int_0^L \psi_{xx}^n(x,t)vdx + \kappa \int_0^L \left[\varphi_x^n(x,t) + \psi^n(x,t)\right] vdx = 0, \\ \zeta \displaystyle\int_0^L \int_0^1 z_t^n(x,\rho,t)wd\rho dx + \frac{\zeta}{\tau} \int_0^L \int_0^1 z_\rho^n(x,\rho,t)wd\rho dx = 0, \end{cases} \tag{18}$$

for all $u \in H_n, v \in V_n, w \in W_n$, with initial conditions such that

$$(\varphi^n(0), \varphi_t^n(0), \psi^n(0), z^n(0)) = (\varphi_0^n, \varphi_1^n, \psi_0^n, z_0^n) \to (\varphi_0, \varphi_1, \psi_0, f_0), \tag{19}$$

strongly in $\mathcal{H}$.

Note that a^{jn}, b^{jn} and $c^{jn}, 1 \leq j \leq n$ form the temporal weighting coefficients.

According to the standard theory of ordinary differential equations, the finite dimensional problem $(18) - (19)$ has a solution $(a^{jn}, b^{jn}, c^{jn}), 1 \leq j \leq n$ defined on $[0, t_n)$ for every $n \in \mathbb{N}^*$.

Then the a priori estimates that follow imply that in fact $t_n = T, \forall T > 0$.

Step 2. A Priori Estimate I

Replacing u by φ_t^n in $(18)_1$, v by ψ_t^n in $(18)_2$ and w by z^n in $(18)_3$, we obtain

$$\begin{cases} \dfrac{\rho_1}{2}\dfrac{d}{dt} \displaystyle\int_0^L |\varphi_t^n(x,t)|^2 dx + \kappa \int_0^L \left[\varphi_x^n(x,t) + \psi^n(x,t)\right] \varphi_{tx}^n(x,t)dx \\ \qquad +\mu_0 \displaystyle\int_0^L |\varphi_t^n(x,t)|^2 dx + \mu_1 \int_0^L z^n(x,1,t)\varphi_t^n(x,t)dx = 0, \\ \dfrac{b}{2}\dfrac{d}{dt} \displaystyle\int_0^L |\psi_x^n(x,t)|^2 dx + \kappa \int_0^L \left[\varphi_x^n(x,t) + \psi^n(x,t)\right] \psi_t^n(x,t)dx = 0, \\ \dfrac{\zeta}{2}\dfrac{d}{dt} \displaystyle\int_0^L \int_0^1 |z^n(x,\rho,t)|^2 d\rho dx \\ \qquad +\dfrac{\zeta}{2\tau} \displaystyle\int_0^L \left[|z^n(x,1,t)|^2 - |z^n(x,0,t)|^2\right] dx = 0. \end{cases} \tag{20}$$

By making the same transformations as in the session of the dissipative character, we obtain

$$\frac{d}{dt}E^n(t) \quad = \quad \left[\frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau}\right] \int_0^L |z^n(x,0,t)|^2 dx$$

$$
\begin{aligned}
&- \frac{\mu_1}{2}\int_0^L |z^n(x,1,t)+z^n(x,0,t)|^2 dx \\
&+ \left[\frac{\tau\mu_1-\zeta}{2\tau}\right]\int_0^L |z^n(x,1,t)|^2 dx,
\end{aligned} \tag{21}
$$

where

$$
\begin{aligned}
E^n(t) &= \frac{\rho_1}{2}\int_0^L |\varphi_t^n|^2 dx + \frac{b}{2}\int_0^L |\psi_x^n|^2 dx + \frac{\kappa}{2}\int_0^L |\varphi_x^n+\psi^n|^2 dx \\
&+ \frac{\zeta}{2}\int_0^L\int_0^1 |z^n|^2 d\rho dx.
\end{aligned}
$$

Thus integrating (21) from 0 to $t < t_n$, we obtain from our choice of initial data that for all $t \in [0; t_n]$ and for every $n \in \mathbb{N}^*$,

$$
\begin{aligned}
E^n(t) - E^n(0) &= \left[\frac{\zeta-\tau(2\mu_0-\mu_1)}{2\tau}\right]\int_0^t\int_0^L |z^n(x,0,s)|^2 dxds \\
&+ \left[\frac{\tau\mu_1-\zeta}{2\tau}\right]\int_0^t\int_0^L |z^n(x,1,s)|^2 dxds \\
&- \frac{\mu_1}{2}\int_0^t\int_0^L |z^n(x,1,t)+z^n(x,0,s)|^2 dxds
\end{aligned}
$$

which means

$$
\begin{aligned}
E^n(t) &- \left[\frac{\zeta-\tau(2\mu_0-\mu_1)}{2\tau}\right]\int_0^t\int_0^L |z^n(x,0,s)|^2 dxds \\
&+ \frac{\mu_1}{2}\int_0^t\int_0^L |z^n(x,1,s)+z^n(x,0,s)|^2 dxds \\
&- \left[\frac{\tau\mu_1-\zeta}{2\tau}\right]\int_0^t\int_0^L |z^n(x,1,s)|^2 dxds = E^n(0),
\end{aligned}
$$

where

$$
\begin{aligned}
E^n(0) &= \frac{\rho_1}{2}\int_0^L |\varphi_t^n(x,0)|^2 dx + \frac{b}{2}\int_0^L |\psi_x^n(x,0)|^2 dx \\
&+ \frac{\kappa}{2}\int_0^L |\varphi_x^n(x,0)+\psi^n(x,0)|^2 dx \\
&+ \frac{\zeta}{2}\int_0^L\int_0^1 |z^n(x,\rho,0)|^2 d\rho dx.
\end{aligned}
$$

As $(\varphi^n(0), \varphi_t^n(0), \psi^n(0), z^n(0)) = (\varphi_0^n, \varphi_1^n, \psi_0^n, z_0^n) \to (\varphi_0, \varphi_1, \psi_0, f_0)$ strongly in $\mathcal{H}$, then there exists a positive constant C_1 such that $E^n(0) \leq C_1$. Hence

$$\begin{aligned} E^n(t) &- \left[\frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau}\right]\int_0^t\int_0^L |z^n(x,0,s)|^2 dxds \\ &+ \frac{\mu_1}{2}\int_0^t\int_0^L |z^n(x,1,s) + z^n(x,0,s)|^2 dxds \\ &- \left[\frac{\tau\mu_1 - \zeta}{2\tau}\right]\int_0^t\int_0^L |z^n(x,1,s)|^2 dxds \leq C_1, \end{aligned}$$

which means that

$$\begin{aligned} &\frac{\rho_1}{2}\int_0^L |\varphi_t^n|^2 dx + \frac{b}{2}\int_0^L |\psi_x^n|^2 dx + \frac{\kappa}{2}\int_0^L |\varphi_x^n + \psi^n|^2 dx \\ &+\frac{\zeta}{2}\int_0^L\int_0^1 |z^n|^2 d\rho dx \\ &-\left[\frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau}\right]\int_0^t\int_0^L |z^n(x,0,s)|^2 dxds \\ &+\frac{\mu_1}{2}\int_0^t\int_0^L |z^n(x,1,s) + z^n(x,0,s)|^2 dxds \\ &-\left[\frac{\tau\mu_1 - \zeta}{2\tau}\right]\int_0^t\int_0^L |z^n(x,1,s)|^2 dxds \leq C_1. \end{aligned} \tag{22}$$

As the constant C_1 does not depend on n, we can therefore take $t_n = T$, for all $T > 0$.

Step 3. A Priori Estimate II

Let us derive the equation $(18)_1$ with respect to t and then replacing u by φ_{tt}^n. We obtain

$$\begin{aligned} \rho_1 \int_0^L \varphi_{ttt}^n(x,t)\varphi_{tt}^n(x,t)dx - \kappa\int_0^L \left[\varphi_{xt}^n(x,t) + \psi_t^n(x,t)\right]_x \varphi_{tt}^n(x,t)dx \\ +\mu_0\int_0^L \varphi_{tt}^n(x,t)\varphi_{tt}^n(x,t)dx + \mu_1\int_0^L z_t^n(x,1,t)\varphi_{tt}^n(x,t)dx = 0. \end{aligned}$$

By integrating by parts, we obtain

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^L |\varphi_{tt}^n(x,t)|^2 dx - \kappa\left[(\varphi_{xt}^n + \psi_t^n)\varphi_{tt}^n(x,t)\right]_0^L$$

$$+\,\kappa\int_0^L\left[\varphi_{xt}^n(x,t)+\psi_t^n(x,t)\right]\varphi_{xtt}^n(x,t)dx$$
$$+\,\mu_0\int_0^L|\varphi_{tt}^n(x,t)|^2dx+\mu_1\int_0^L z_t^n(x,1,t)\varphi_{tt}^n(x,t)dx=0.$$

As φ is null at 0 and at L, we have

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^L|\varphi_{tt}^n(x,t)|^2dx+\kappa\int_0^L\left[\varphi_{xt}^n(x,t)+\psi_t^n(x,t)\right]\varphi_{xtt}^n(x,t)dx$$
$$+\mu_0\int_0^L|\varphi_{tt}^n(x,t)|^2dx+\mu_1\int_0^L z_t^n(x,1,t)\varphi_{tt}^n(x,t)dx=0. \tag{23}$$

Let us derive the equation $(18)_2$ with respect to t, and then replacing v by ψ_{tt}^n, we obtain

$$-b\int_0^L\psi_{xxt}^n(x,t)\psi_{tt}^n(x,t)dx$$
$$+\kappa\int_0^L\left[\varphi_{xt}^n(x,t)+\psi_t^n(x,t)\right]\psi_{tt}^n(x,t)dx=0.$$

Integrating by parts, we obtain

$$-b\left[\psi_{xt}^n\psi_{tt}^n(x,t)\right]_0^L+b\int_0^L\psi_{xt}^n(x,t)\psi_{xtt}^n(x,t)dx$$
$$+\kappa\int_0^L\left[\varphi_{xt}^n(x,t)+\psi_t^n(x,t)\right]\psi_{tt}^n(x,t)dx=0.$$

Since ψ_x is zero at the edge we have

$$b\int_0^L\psi_{xt}^n(x,t)\psi_{xtt}^n(x,t)dx$$
$$+\kappa\int_0^L\left[\varphi_{xt}^n(x,t)+\psi_t^n(x,t)\right]\psi_{tt}^n(x,t)dx=0.$$

This is written as

$$\frac{b}{2}\frac{d}{dt}\int_0^L|\psi_{xt}^n(x,t)|^2dx$$
$$+\kappa\int_0^L\left[\varphi_{xt}^n(x,t)+\psi_t^n(x,t)\right]\psi_{tt}^n(x,t)dx=0. \tag{24}$$

Summing (23) and (24), we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{\rho_1\int_0^L|\varphi^n_{tt}(x,t)|^2dx+k\int_0^L|\varphi^n_{xt}(x,t)+\psi^n_t(x,t)|^2dx\right.$$
$$\left.+b\int_0^L|\psi^n_{xt}(x,t)|^2dx\right\}+\mu_0\int_0^L|\varphi^n_{tt}(x,t)|^2dx$$
$$+\mu_1\int_0^L z^n_t(x,1,t)\varphi^n_{tt}(x,t)dx=0,$$

which means that

$$\frac{d}{dt}G^n(t)+\mu_1\int_0^L z^n_t(x,1,t)\varphi^n_{tt}(x,t)dx$$
$$+\mu_0\int_0^L|\varphi^n_{tt}(x,t)|^2dx=0, \tag{25}$$

where

$$\begin{aligned} G^n(t) &= \frac{\rho_1}{2}\int_0^L|\varphi^n_{tt}(x,t)|^2dx+\frac{k}{2}\int_0^L|\varphi^n_{xt}(x,t)+\psi^n_t(x,t)|^2dx \\ &+ \frac{b}{2}\int_0^L|\psi^n_{xt}(x,t)|^2dx. \end{aligned}$$

By integrating (25) from 0 to t we have

$$G^n(t)+\mu_1\int_0^t\int_0^L z^n_s(x,1,s)\varphi^n_{ss}(x,s)dxds$$
$$+\mu_0\int_0^t\int_0^L|\varphi^n_{ss}(x,s)|^2dxds=G^n(0),$$

where

$$\begin{aligned} G^n(0) &= \frac{\rho_1}{2}\int_0^L|\varphi^n_{tt}(x,0)|^2dx+\frac{k}{2}\int_0^L|\varphi^n_{xt}(x,0)+\psi^n_t(x,0)|^2dx \\ &+ \frac{b}{2}\int_0^L|\psi^n_{xt}(x,0)|^2dx. \end{aligned}$$

As $(\varphi^n(0),\varphi^n_t(0),\psi^n(0),z^n(0))=(\varphi^n_0,\varphi^n_1,\psi^n_0,z^n_0)\to(\varphi_0,\varphi_1,\psi_0,f_0)$ strongly in $\mathcal{H}$, then there exists a positive constant C_2 such that $G^n(0)\le C_2$. Hence

$$G^n(t)+\mu_1\int_0^t\int_0^L z^n_s(x,1,s)\varphi^n_{ss}(x,s)dxds$$
$$+\mu_0\int_0^t\int_0^L|\varphi^n_{ss}(x,s)|^2dxds\le C_2. \tag{26}$$

Step 4. Passage to Limit

From (22) and (26) we have that:
$\{\varphi^n\}$ is bounded in $L^\infty\left(0,T,H_0^1(0,L)\right)$,
$\{\varphi_t^n\}$ is bounded in $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{\varphi_{tt}^n\}$ is bounded in $L^\infty\left(0,T,L^2(0,L)\right)$,
$\left(\psi^n\right)$ is bounded in $L^\infty\left(0,T,H_*^1(0,L)\right)$,
$\left(\psi_t^n\right)$ is bounded in $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{z^n\}$ is bounded in $L^\infty\left(0,T,L^2((0,L)\times(0,1))\right)$.

So we can extract subsequences $\{\varphi^n\}$,$\{\psi^n\}$ and $\{z^n\}$ such as
$\{\varphi^n\}\to\varphi$ weakly star in $L^\infty\left(0,T,H_0^1(0,L)\right)$,
$\{\varphi_t^n\}\to\varphi_t$ weakly star in $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{\varphi_{tt}^n\}\to\varphi_{tt}$ weakly star in $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{\psi^n\}\to\psi$ weakly star in $L^\infty\left(0,T,H_*^1(0,L)\right)$,
$\{\psi_t^n\}\to\psi_t$ weakly star in $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{z^n\}\to z$ weakly star in $L^\infty\left(0,T,L^2((0,L)\times(0,1))\right)$.
Moreover, from (22) we have
$\{\varphi^n\}$ is bounded in $L^2\left(0,T,H_0^1(0,L)\right)$,
$\{\varphi_t^n\}$ is bounded in $L^2\left(0,T,L^2(0,L)\right)$.

And since $H_0^1(0,L)$ is compactly injected into $L^2(0,L)$, see [6], we have by the Aubin-Lions theorem [7] that
$\{\varphi^n\}\to\varphi$ strongly in $L^\infty\left(0,T,L^2(0,L)\right)$.

We also show that
$\{\varphi_t^n\}\to\varphi_t$ strongly in $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{\psi^n\}\to\psi$ strongly in $L^\infty\left(0,T,H_0^1(0,L)\right)$.

Then we can pass to limit the approximate problem (18) – (19) in order to get a weak solution of problem (7).

And we use density arguments to get problems (7) that admit a global weak solution satisfying

$$\begin{cases} \varphi\in L^\infty\left(0,T,H_0^1(0,L)\right), \\ \varphi_t\in L^\infty\left(0,T,L^2(0,L)\right), \\ \psi\in L^\infty\left(0,T,H_\star^1(0,L)\right), \\ z\in L^\infty\left(0,T,L^2((0,L)\times(0,1))\right). \end{cases} \tag{27}$$

Step 5. A Priori Estimate III

Suppose that the initiale data in the approximate problem (18) satisfies $(\varphi_0, \varphi_1, \psi_0, f_0) \in \mathcal{H}_1$ and

$$(\varphi_0^n, \varphi_1^n, \psi_0^n, z_0^n) \to (\varphi_0, \varphi_1, \psi_0, f_0) \tag{28}$$

strongly in $\mathcal{H}_1$.

Replacing u by $-\varphi_{xxt}^n$ in $(18)_1$, v by $-\psi_{xxt}^n$ in $(18)_2$ and w by z_{xx}^n in $(18)_3$ we arrive at

$$\begin{aligned} \frac{d}{dt}F^n(t) &= \left[\frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau}\right] \int_0^L |z_x^n(x,0,t)|^2 dx \\ &- \frac{\mu_1}{2}\int_0^L |z_x^n(x,1,t) + z_x^n(x,0,t)|^2 dx \\ &+ \left[\frac{\tau\mu_1 - \zeta}{2\tau}\right] \int_0^L |z_x^n(x,1,t)|^2 dx, \end{aligned} \tag{29}$$

where

$$\begin{aligned} F^n(t) &= \frac{\rho_1}{2}\int_0^L |\varphi_{tx}^n|^2 dx + \frac{b}{2}\int_0^L |\psi_{xx}^n|^2 dx + \frac{\kappa}{2}\int_0^L |\varphi_{xx}^n + \psi_x^n|^2 dx \\ &+ \frac{\zeta}{2}\int_0^L \int_0^1 |z_x^n|^2 d\rho dx. \end{aligned}$$

Thus integrating (29) from 0 to t, we obtain from our choice of initial data that for all $t \in [0;T]$ and for every $n \in \mathbb{N}^\star$,

$$\begin{aligned} F^n(t) - F^n(0) &= \left[\frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau}\right] \int_0^t \int_0^L |z_x^n(x,0,s)|^2 dx ds \\ &+ \left[\frac{\tau\mu_1 - \zeta}{2\tau}\right] \int_0^t \int_0^L |z_x^n(x,1,s)|^2 dx ds \\ &- \frac{\mu_1}{2}\int_0^t \int_0^L |z_x^n(x,1,s) + z_x^n(x,0,s)|^2 dx ds. \end{aligned}$$

This means

$$\begin{aligned} F^n(t) &- \left[\frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau}\right] \int_0^t \int_0^L |z_x^n(x,0,s)|^2 dx ds \\ &+ \frac{\mu_1}{2}\int_0^t \int_0^L |z_x^n(x,1,s) + z_x^n(x,0,s)|^2 dx ds \end{aligned}$$

$$- \left[\frac{\tau\mu_1 - \zeta}{2\tau}\right] \int_0^t \int_0^L |z_x^n(x,1,s)|^2 dxds = F^n(0),$$

where

$$\begin{aligned} F^n(0) &= \frac{\rho_1}{2}\int_0^L |\varphi_{tx}^n(x,0)|^2 dx + \frac{b}{2}\int_0^L |\psi_{xx}^n(x,0)|^2 dx \\ &+ \frac{\kappa}{2}\int_0^L |\varphi_{xx}^n(x,0) + \psi_x^n(x,0)|^2 dx \\ &+ \frac{\zeta}{2}\int_0^L \int_0^1 |z_x^n(x,\rho,0)|^2 d\rho dx. \end{aligned}$$

As $(\varphi^n(0), \varphi_t^n(0), \psi^n(0), z^n(0)) = (\varphi_0^n, \varphi_1^n, \psi_0^n, z_0^n) \to (\varphi_0, \varphi_1, \psi_0, f_0)$ strongly in $\mathcal{H}_1$ we can deduce that each of the sequences $\{\varphi^n(0)\}$,$\{\varphi_t^n(0)\}$, $\{\psi^n(0)\}$ and $\{z^n(0)\}$ is bounded.

Thus there exists a positive constant C_3 such that $F^n(0) \leq C_3$. Hence

$$\begin{aligned} F^n(t) &- \left[\frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau}\right] \int_0^t \int_0^L |z_x^n(x,0,s)|^2 dxds \\ &+ \frac{\mu_1}{2}\int_0^t \int_0^L |z_x^n(x,1,s) + z_x^n(x,0,s)|^2 dxds \\ &- \left[\frac{\tau\mu_1 - \zeta}{2\tau}\right] \int_0^t \int_0^L |z_x^n(x,1,s)|^2 dxds \leq C_3. \end{aligned}$$

Which means that

$$\begin{aligned} &\frac{\rho_1}{2}\int_0^L |\varphi_{tx}^n|^2 dx + \frac{b}{2}\int_0^L |\psi_{xx}^n|^2 dx + \frac{\kappa}{2}\int_0^L |\varphi_{xx}^n + \psi_x^n|^2 dx \\ + &\frac{\zeta}{2}\int_0^L \int_0^1 |z_x^n|^2 d\rho dx \\ &- \left[\frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau}\right] \int_0^t \int_0^L |z_x^n(x,0,s)|^2 dxds \\ + &\frac{\mu_1}{2}\int_0^t \int_0^L |z_x^n(x,1,s) + z_x^n(x,0,s)|^2 dxds \\ &- \left[\frac{\tau\mu_1 - \zeta}{2\tau}\right] \int_0^t \int_0^L |z_x^n(x,1,s)|^2 dxds \leq C_3, \end{aligned} \tag{30}$$

where C_3 is a positive constant independent of t and n but depending on initial data. Then we can conclude that

$\{\varphi^n\}$ is bounded in $L^\infty\left(0, T, H^2(0,L) \cap H_0^1(0,L)\right)$,
$\{\varphi_t^n\}$ is bounded in $L^\infty\left(0, T, H_0^1(0,L)\right)$,
$\{\psi^n\}$ is bounded in $L^\infty\left(0, T, H_\star^2(0,L)\right)$,
$\{z^n\}$ is bounded in $L^\infty(0, T, H^1((0,L) \times (0,1)))$.

This implies that
$\{\varphi^n\} \to \varphi$ weakly star in $L^\infty\left(0, T, H^2(0,L) \cap H_0^1(0,L)\right)$,
$\{\varphi_t^n\} \to \varphi_t$ weakly star in $L^\infty\left(0, T, H_0^1(0,L)\right)$,
$\{\psi^n\} \to \psi$ weakly star in $L^\infty\left(0, T, H_\star^2(0,L)\right)$,
$\{z^n\} \to z$ weakly star in $L^\infty\left(0, T, H^1((0,L) \times (0,1))\right)$.

From the above limits, we conclude that $(\varphi, \varphi_t, \psi, z)$ is a strong weak solution satisfying

$$\begin{cases} \varphi \in L^\infty\left(0, T, H^2(0,L) \cap H_0^1(0,L)\right), \\ \varphi_t \in L^\infty\left(0, T, H_0^1(0,L)\right), \\ \psi \in L^\infty\left(0, T, H_\star^2(0,L)\right), \\ z \in L^\infty\left(0, T, H^1((0,L) \times (0,1))\right). \end{cases} \tag{31}$$

Step 6. Continuous dependence

Let $U(t) = (\varphi, \varphi_t, \psi, z)$ and $V(t) = (\varphi', \varphi'_t, \psi', z')$ be the stronger weak solutions of the problem (7) corresponding to initial data

$$U(0) = (\varphi_0, \varphi_1, \psi_0, z_0), \quad V(0) = \left(\varphi'_0, \varphi'_1, \psi'_0, z'_0\right) \in \mathcal{H}_1.$$

Then $(\Phi, \Phi_t, \Psi, Z) = U(t) - V(t)$ is solution of the system

$$\begin{cases} \rho_1 \Phi_{tt}(x,t) - \kappa(\Phi_x(x,t) + \Psi(x,t))_x + \mu_0 \Phi_t(x,t) + \mu_1 Z(x,1,t) = 0 \\ \qquad\qquad \text{in }]0, L[\times(0,+\infty), \\ -b\Phi_{xx}(x,t) + \kappa(\Phi_x(x,t) + \Psi(x,t)) = 0 \text{ in }]0, L[\times(0,+\infty), \\ \tau Z_t(x,\rho,t) + Z_\rho(x,\rho,t) = 0 \ (x,\rho,t) \in (0,L) \times (0,1) \times (0,+\infty), \end{cases} \tag{32}$$

with initial data $\Big(\Phi(0), \Phi_t(0), \Psi(0), Z(0)\Big) = U(0) - V(0)$.

Multiplying $(32)_1$ by Φ_t and $(32)_2$ by Ψ_t and integrating, we obtain

$$\frac{d}{dt}\mathcal{G}(t) + \mu_1 \int_0^L \Phi_t(x,t) Z(x,1,t)\,dx + \mu_0 \int_0^L |\Phi_t(x,t)|^2 dx = 0$$

with

$$\begin{aligned}\mathcal{G}(t) &= \frac{\rho_1}{2}\int_0^L |\Phi_t(x,t)|^2dx + \frac{k}{2}\int_0^L |\Phi_x(x,t)+\Psi(x,t)|^2dx\\ &+ \frac{b}{2}\int_0^L |\Psi_x(x,t)|^2dx.\end{aligned}$$

Applying Young's inequality, we obtain the existence of a constant M_1 such that

$$\begin{aligned}\frac{d}{dt}\mathcal{G}(t) &\leq M_1\int_0^L |\Phi_t(x,t)|^2dx\\ &\leq M_1\Big[\int_0^L |\Phi_t(x,t)|^2dx + \int_0^L |\Phi_x(x,t)+\Psi(x,t)|^2dx\\ &+ \int_0^L |\Psi_x(x,t)|^2dx\Big]. \qquad (33)\end{aligned}$$

And by integrating (33) from 0 to t we get

$$\begin{aligned}\mathcal{G}(t) &\leq \mathcal{G}(0) + M_1\int_0^t\Big[\int_0^L |\Phi_\lambda(x,\lambda)|^2dx\\ &+ \int_0^L |\Phi_x(x,\lambda)+\Psi(x,\lambda)|^2dx + \int_0^L |\Psi_x(x,\lambda)|^2dx\Big]d\lambda. \qquad (34)\end{aligned}$$

On the other hand, we know that for $M_2 = \min\left\{\frac{\rho_1}{2},\frac{k}{2},\frac{b}{2}\right\}$, we have

$$\begin{aligned}\mathcal{G}(\lambda) &\geq M_2\Big[\int_0^L |\Phi_\lambda(x,\lambda)|^2dx + \int_0^L |\Phi_x(x,\lambda)+\Psi(x,\lambda)|^2dx\\ &+ \int_0^L |\Psi_x(x,\lambda)|^2dx\Big]. \qquad (35)\end{aligned}$$

From (34) and (35) we have

$$\mathcal{G}(t) \leq \mathcal{G}(0) + \frac{M_1}{M_2}\int_0^t \mathcal{G}(\lambda)d\lambda. \qquad (36)$$

Applying Gronwall's inequality we have

$$\mathcal{G}(t) \leq \mathcal{G}(0)e^{\frac{M_1.t}{M_2}}. \qquad (37)$$

So we obtain the continuous dependence of solution on the initial data. In particular, the solution is unique.

The proof of theorem is thus completed. $\square$

4. Exponential stability

Theorem 4. *Let the assumption* (5) *be satisfied. Then, there exist positive constants M and K such that, for any solution of* (7)

$$E(t) \leq ME(0)e^{-Kt}, \quad \forall t \geq 0.$$

Proof. Let $\mathcal{F}(t) = 2c\rho_1 \int_0^L \varphi_t \varphi dx + \mu_0 c \int_0^L |\varphi|^2 dx$,
where c is a constant whose conditions we will specify later.

Multiply $(7)_1$ by $2c\varphi$ and integrate from 0 to L. We obtain

$$2c\rho_1 \int_0^L \varphi_{tt}(x,t)\varphi(x,t)dx - 2c\kappa \int_0^L \left[\varphi_x(x,t) + \psi(x,t)\right]_x \varphi(x,t)dx$$
$$+2c\mu_0 \int_0^L \varphi_t(x,t)\varphi(x,t)dx + 2c\mu_1 \int_0^L z(x,1,t)\varphi(x,t)dx = 0.$$

As $\varphi_{tt}\varphi = \dfrac{\partial}{\partial t}(\varphi_t\varphi) - |\varphi_t|^2$, we have

$$2c\rho_1 \int_0^L \frac{\partial}{\partial t}[\varphi_t(x,t)\varphi(x,t)]dx - 2c\rho_1 \int_0^L |\varphi_t|^2 dx$$
$$+2c\kappa \int_0^L \left[\varphi_x(x,t) + \psi(x,t)\right] \varphi_x(x,t)dx + c\mu_0 \int_0^L \frac{\partial}{\partial t}|\varphi(x,t)|^2 dx$$
$$+2c\mu_1 \int_0^L z(x,1,t)\varphi(x,t)dx = 0,$$

which means that

$$\frac{\partial}{\partial t}\int_0^L \left[2c\rho_1\varphi_t(x,t)\varphi(x,t) + c\mu_0|\varphi(x,t)|^2\right]dx$$
$$-2c\rho_1 \int_0^L |\varphi_t|^2 dx + 2c\kappa \int_0^L \left[\varphi_x(x,t) + \psi(x,t)\right] \varphi_x(x,t)dx$$
$$+2c\mu_1 \int_0^L z(x,1,t)\varphi(x,t)dx = 0. \tag{38}$$

Multiply $(7)_2$ by $2c\psi$ and integrate over 0 to L. We obtain

$$-2cb\int_0^L \psi_{xx}(x,t)\psi dx + 2c\kappa\int_0^L \left[\varphi_x(x,t)+\psi(x,t)\right]\psi dx = 0.$$

Integrating by parts, we have

$$-2cb\left[\psi_x\psi\right]_0^L + 2cb\int_0^L |\psi_x|^2 dx + 2c\kappa\int_0^L \left[\varphi_x(x,t)+\psi(x,t)\right]\psi dx = 0.$$

Since ψ_x is zero at 0 and L, we have

$$2cb\int_0^L |\psi_x|^2 dx + 2c\kappa\int_0^L \left[\varphi_x(x,t)+\psi(x,t)\right]\psi dx = 0. \tag{39}$$

Now summing (38) and (39), we obtain that

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t) &= 2c\rho_1\int_0^L |\varphi_t|^2 dx - 2c\kappa\int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 dx\\ &- 2c\mu_1\int_0^L z(x,1,t)\varphi(x,t)dx - 2cb\int_0^L |\psi_x|^2 dx.\end{aligned}$$

Applying Young's inequality to $-2c\mu_1\int_0^L z(x,1,t)\varphi(x,t)dx$, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t) &\le 2c\rho_1\int_0^L |\varphi_t|^2 dx - 2c\kappa\int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 dx\\ &+ \frac{(2c\mu_1)^2}{4\varepsilon}\int_0^L |z(x,1,t)|^2 dx + \varepsilon\int_0^L |\varphi(x,t)|^2 dx\\ &- 2cb\int_0^L |\psi_x|^2 dx, \forall\varepsilon>0\\ &\le 2c\rho_1\int_0^L |\varphi_t|^2 dx - 2c\kappa\int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 dx\\ &+ \varepsilon\int_0^L |\varphi(x,t)|^2 dx + \frac{(c\mu_1)^2}{\varepsilon}\int_0^L |z(x,1,t)|^2 dx\\ &- 2cb\int_0^L |\psi_x|^2 dx, \quad \forall\varepsilon>0.\end{aligned}$$

Applying Poincare's inequality at $\varepsilon\int_0^L |\varphi(x,t)|^2 dx$, we obtain

$$\frac{d}{dt}\mathcal{F}(t) \le 2c\rho_1\int_0^L |\varphi_t|^2 dx - 2c\kappa\int_0^L |\varphi_x(x,t)+\psi(x,t)|^2 dx$$

$$\begin{aligned} &+ \varepsilon L^2 \int_0^L |\varphi_x(x,t)|^2 dx + \frac{(c\mu_1)^2}{\varepsilon} \int_0^L |z(x,1,t)|^2 dx \\ &- 2cb \int_0^L |\psi_x|^2 dx, \quad \forall \varepsilon > 0. \end{aligned} \tag{40}$$

We also know that

$$\int_0^L |\varphi_x(x,t)|^2 dx \leq 2 \int_0^L |\varphi_x(x,t) + \psi(x,t)|^2 dx + 2 \int_0^L |\psi(x,t)|^2 dx.$$

By Poincare's inequality we have

$$\begin{aligned} \int_0^L |\varphi_x(x,t)|^2 dx &\leq 2 \int_0^L |\varphi_x(x,t) + \psi(x,t)|^2 dx \\ &+ 2L^2 \int_0^L |\psi_x(x,t)|^2 dx. \end{aligned} \tag{41}$$

Multiplying (41) by εL^2 we have

$$\begin{aligned} \varepsilon L^2 \int_0^L |\varphi_x(x,t)|^2 dx &\leq 2\varepsilon L^2 \int_0^L |\varphi_x(x,t) + \psi(x,t)|^2 dx \\ &+ 2\varepsilon L^4 \int_0^L |\psi_x(x,t)|^2 dx \\ &\leq \frac{2\varepsilon L^2}{k}.k \int_0^L |\varphi_x(x,t) + \psi(x,t)|^2 dx \\ &+ \frac{2\varepsilon L^4}{b}.b \int_0^L |\psi_x(x,t)|^2 dx. \end{aligned}$$

If we put $c = \max\left\{ \frac{2\varepsilon L^2}{k}, \frac{2\varepsilon L^4}{b} \right\}$ we obtain

$$\begin{aligned} \varepsilon L^2 \int_0^L |\varphi_x(x,t)|^2 dx &\leq ck \int_0^L |\varphi_x(x,t) + \psi(x,t)|^2 dx \\ &+ cb \int_0^L |\psi_x(x,t)|^2 dx. \end{aligned} \tag{42}$$

From (40) and (42) we can write

$$\frac{d}{dt}\mathcal{F}(t) \leq 2c\rho_1 \int_0^L |\varphi_t|^2 dx - 2c\kappa \int_0^L |\varphi_x(x,t) + \psi(x,t)|^2 dx$$

$$\begin{aligned}
&+ \quad ck \int_0^L |\varphi_x(x,t) + \psi(x,t)|^2 dx \\
&+ \quad cb \int_0^L |\psi_x(x,t)|^2 dx + \frac{(c\mu_1)^2}{\varepsilon} \int_0^L |z(x,1,t)|^2 dx \\
&- \quad 2cb \int_0^L |\psi_x|^2 dx, \quad \forall \varepsilon > 0,
\end{aligned}$$

which means that

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}(t) \quad &\leq \quad 2c\rho_1 \int_0^L |\varphi_t|^2 dx - c\kappa \int_0^L |\varphi_x(x,t) + \psi(x,t)|^2 dx \\
&+ \quad \frac{(c\mu_1)^2}{\varepsilon} \int_0^L |z(x,1,t)|^2 dx \\
&- \quad cb \int_0^L |\psi_x|^2 dx, \quad \forall \varepsilon > 0. \qquad (43)
\end{aligned}$$

Let us put $I(t) = c\zeta e^{2\tau} \int_0^L \int_0^1 e^{-2\tau\rho} |z|^2 d\rho dx$, we have

$$\frac{d}{dt} I(t) = 2c\zeta e^{2\tau} \int_0^L \int_0^1 e^{-2\tau\rho} z z_t d\rho dx.$$

From $(7)_3$ we have $z_t = -\frac{1}{\tau} z_\rho$, so

$$\begin{aligned}
\frac{d}{dt} I(t) \quad &= \quad \frac{-2c\zeta e^{2\tau}}{\tau} \int_0^L \int_0^1 e^{-2\tau\rho} z z_\rho d\rho dx \\
&= \quad - \quad 2c\zeta e^{2\tau} \int_0^L \int_0^1 e^{-2\tau\rho} z^2 d\rho dx \\
&\quad - \quad \frac{c\zeta e^{2\tau}}{\tau} \int_0^L \int_0^1 \frac{d}{d\rho}\Big(e^{-2\tau\rho} z^2\Big) dx d\rho \\
&= \quad - \quad 2c\zeta e^{2\tau} \int_0^L \int_0^1 e^{-2\tau\rho} z^2 d\rho dx - \frac{c\zeta e^{2\tau}}{\tau} \int_0^L e^{-2\tau} z^2(x,1,t) dx \\
&\quad + \quad \frac{c\zeta e^{2\tau}}{\tau} \int_0^L e^0 z^2(x,0,t) dx \\
&= \quad - \quad 2c\zeta e^{2\tau} \int_0^L \int_0^1 e^{-2\tau\rho} z^2 d\rho dx - \frac{c\zeta}{\tau} \int_0^L z^2(x,1,t) dx \\
&\quad + \quad \frac{c\zeta e^{2\tau}}{\tau} \int_0^L z^2(x,0,t) dx
\end{aligned}$$

$$\begin{aligned}
\leq\ & - \ c\zeta e^{2\tau}\int_0^L\int_0^1 e^{-2\tau\rho}z^2 d\rho dx - \frac{c\zeta}{\tau}\int_0^L z^2(x,1,t)dx \\
& + \ \frac{c\zeta e^{2\tau}}{\tau}\int_0^L z^2(x,0,t)dx \\
& - \ c\zeta e^{2\tau}\int_0^L\int_0^1 e^{-2\tau}z^2 d\rho dx - \frac{c\zeta}{\tau}\int_0^L z^2(x,1,t)dx \\
& + \ \frac{c\zeta e^{2\tau}}{\tau}\int_0^L z^2(x,0,t)dx,
\end{aligned}$$

however

$$\begin{aligned}
\frac{d}{dt}I(t) \ \leq\ & -c\zeta\int_0^L\int_0^1 z^2 d\rho dx - \frac{c\zeta}{\tau}\int_0^L z^2(x,1,t)dx \\
& + \ \frac{c\zeta e^{2\tau}}{\tau}\int_0^L z^2(x,0,t)dx. \qquad (44)
\end{aligned}$$

Summing (43) and (44) we have

$$\begin{aligned}
\frac{d}{dt}\Big(I(t)+\mathcal{F}(t)\Big) \ \leq\ & 2c\rho_1\int_0^L|\varphi_t|^2dx - c\kappa\int_0^L|\varphi_x(x,t)+\psi(x,t)|^2dx \\
& + \ \frac{(c\mu_1)^2}{\varepsilon}\int_0^L|z(x,1,t)|^2dx - cb\int_0^L|\psi_x|^2dx \\
& - \ c\zeta\int_0^L\int_0^1|z|^2d\rho dx - \frac{c\zeta}{\tau}\int_0^L|z(x,1,t)|^2dx \\
& + \ \frac{c\zeta e^{2\tau}}{\tau}\int_0^L|z(x,0,t)|dx \\
\leq\ & - \ c\Big[\rho_1\int_0^L|\varphi_t|^2dx + b\int_0^L|\psi_x|^2dx + \zeta\int_0^L\int_0^1|z|^2d\rho dx \\
& + \ \kappa\int_0^L|\varphi_x(x,t)+\psi(x,t)|^2dx\Big] \\
& + \ 3c\rho_1\int_0^L|\varphi_t|^2dx + \Big[\frac{(c\mu_1)^2}{\varepsilon} - \frac{c\zeta}{\tau}\Big]\int_0^L|z(x,1,t)|^2dx \\
& + \ \frac{c\zeta e^{2\tau}}{\tau}\int_0^L|z(x,0,t)|^2dx \\
\leq\ & + \ 3c\rho_1\int_0^L|\varphi_t|^2dx + \Big[\frac{(c\mu_1)^2}{\varepsilon} - \frac{c\zeta}{\tau}\Big]\int_0^L|z(x,1,t)|^2dx \\
& + \ \frac{c\zeta e^{2\tau}}{\tau}\int_0^L|z(x,0,t)|^2dx - 2cE(t)
\end{aligned}$$

$$
\begin{aligned}
\leq & - 2cE(t) + 3c\rho_1 \int_0^L |z(x,0,t)|^2 dx \\
& + \Big[\frac{(c\mu_1)^2}{\varepsilon} - \frac{c\zeta}{\tau}\Big] \int_0^L |z(x,1,t)|^2 dx \\
& + \frac{c\zeta e^{2\tau}}{\tau} \int_0^L |z(x,0,t)|^2 dx.
\end{aligned}
$$

Thereby,

$$
\begin{aligned}
\frac{d}{dt}\Big(I(t)+\mathcal{F}(t)\Big) \leq & -2cE(t) + \Big(\frac{c\zeta e^{2\tau}}{\tau} + 3c\rho_1\Big) \int_0^L |z(x,0,t)|^2 dx \\
& + \Big[\frac{(c\mu_1)^2}{\varepsilon} - \frac{c\zeta}{\tau}\Big] \int_0^L |z(x,1,t)|^2 dx. \qquad (45)
\end{aligned}
$$

Setting

$$\mathcal{L}(t) = NE(t) + I(t) + \mathcal{F}(t), \quad \forall t > 0,$$

where N is a constant whose conditions we will specify later. We obtain from (14) and (45)

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq & \Big[N\frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} + \frac{c\zeta e^{2\tau}}{\tau} + 3c\rho_1\Big] \int_0^L |z(x,0,t)|^2 dx \\
& - \frac{N\mu_1}{2} \int_0^L |z(x,1,t) - z(x,0,t)|^2 dx - 2cE(t) \\
& + \Big[N\frac{\tau\mu_1 - \zeta}{2\tau} + \frac{(c\mu_1)^2}{\varepsilon} - \frac{c\zeta}{\tau}\Big] \int_0^L |z(x,1,t)|^2 dx.
\end{aligned}
$$

As $\dfrac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} \leq 0; -\dfrac{\mu_1}{2} \leq 0$ and $\dfrac{\tau\mu_1 - \zeta}{2\tau} \leq 0$, just take N very large enough such that

$$N\frac{\zeta - \tau(2\mu_0 - \mu_1)}{2\tau} + \frac{c\zeta e^{2\tau}}{\tau} + 3c\rho_1 < 0,$$

and

$$N\frac{\tau\mu_1 - \zeta}{2\tau} + \frac{(c\mu_1)^2}{\varepsilon} - \frac{c\zeta}{\tau} < 0.$$

Now we can take

$$N > \max\left\{\frac{6c\tau\rho_1 + 2c\zeta e^{2\tau}}{\tau(2\mu_0 - \mu_1) - \zeta}, \frac{2\tau c^2\mu_1^2 - 2c\zeta\varepsilon}{(\zeta - \tau\mu_1)\varepsilon}\right\}.$$

For this value of N we obtain that

$$\frac{d}{dt}\mathcal{L}(t) \leq -2cE(t). \tag{46}$$

Moreover, it is easy to see the existence of two constants α and β such that

$$\alpha E(t) \leq \mathcal{L}(t) \leq \beta E(t), \forall t \geq 0. \tag{47}$$

By (46) and (47) we obtain

$$\frac{\frac{d}{dt}\mathcal{L}(t)}{\mathcal{L}(t)} \leq \frac{-2cE(t)}{\mathcal{L}(t)} \leq \frac{-2cE(t)}{\beta E(t)},$$

which means that

$$\frac{\frac{d}{dt}\mathcal{L}(t)}{\mathcal{L}(t)} \leq \frac{-2c}{\beta}. \tag{48}$$

Integrating (48) from 0 to t we have

$$ln\mathcal{L}(t) - ln\mathcal{L}(0) \leq \frac{-2c}{\beta}t.$$

This means that

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{\frac{-2c}{\beta}t}. \tag{49}$$

By (47) we obtain

$$\alpha E(t) \leq \mathcal{L}(t),$$

and

$$\mathcal{L}(0) \leq \beta E(0).$$

However (49) can be written as

$$E(t) \leq \frac{\beta E(0)}{\alpha}e^{\frac{-2ct}{\beta}}.$$

We obtain, by taking $M = \dfrac{\beta}{\alpha}$ and $K = \frac{2c}{\beta}$, that

$$E(t) \leq ME(0)e^{-Kt}, \forall t \geq 0.$$

The proof of theorem is thus completed. □

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