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ON THE COMMUTATIVITY DEGREE OF CERTAIN FINITE GROUPS

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Abstract: The commutativity degree $P(G)$ of a finite group G, is the probability that two arbitrary elements in G commute. The commutativity degree of a group G can be used to measure how close a group is to be commutative. In this article, the commutativity degree of some finite groups is configured. Furthermore, upper and lower bounds of the commutativity degree of p-groups have been computed.

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Key Words: commutativity degree; conjugacy classes; p-group; 2-generators group; symmetric group

1. Introduction

The commutativity degree $P(G)$ of a finite group G, is the probability that two arbitrary elements in G commute. To determine this probability we need to count the number of choices for two elements being commute. This will be configured as the cardinality of A, where $A = \{(x, y) \in G \times G \mid xy = yx\}$. In [4], Erdős and Turán found the commutativity degree for symmetric groups. Next, Erfanian et al. in [5] generalized the notion of the commutativity degree of a group to the relative commutativity degree, which is defined as the probability of an element of a subgroup commutes with an element of the group. In [3], Castelaz found bounds on the commutativity degree of finite groups and found

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the exact value for some groups. Furthermore, she addressed the case of equality in the commutativity degree of a group and subgroup.

2. Preliminaries

Our notions are fairly standard, for a group G the identity element is e, the center $Z(G)$ of G is the set of all $x \in G$ such that $xy = yx$ for all $y \in G$. Two elements x and y in G are conjugate if there is an element $g \in G$ such that $x = gyg^{-1}$, the set of all such elements y is called the conjugacy class of x and denoted by x^G. The number of the conjugacy classes of a group G will be denoted by $k(G)$. Recall that, the lower central series of a group G is $G = \gamma_0(G) \geq \gamma_1(G) \geq \cdots \gamma_c(G) \geq \cdots$, where $\gamma_i(G) = [\gamma_{i-1}(G), G]$ (the commutator subgroup of $\gamma_{i-1}(G)$ and G), for $i = 1, 2, 3, \cdots$. A group G is called nilpotent if there exists c such that $\gamma_c(G) = [\gamma_{c-1}(G), G] = \{e\}$, and the smallest such c is the class of nilpotency.

The commutativity degree $P(G)$ of a group G is given by

$$\begin{aligned} P(G) &= \frac{|\{(x,y) \in G \times G \mid xy = yx\}|}{|G \times G|} \\ &= \frac{|\{(x,y) \in G \times G \mid xy = yx\}|}{|G|^2}. \end{aligned}$$

Clearly, the commutativity degree of any abelian group is 1. Therefore, $P(\mathbb{Z}_n) = 1$ for all $n \in \mathbb{N}$ and $P(G) < 1$ for any non-abelian group G. On the other hand, if $A = \{(x,y) \in G \times G \mid xy = yx\}$, then $(e, x) \in A$, $x \in G$. Thus $P(G) \geq \frac{|G|}{|G|^2} = \frac{1}{|G|}$, which is the ordinary lower bound of $P(G)$ for any finite group G. The next proposition gives a better lower bound of the commutativity degree for any finite group.

Proposition 1. *Let G be a finite group of order n and center $Z(G)$ with $|Z(G)| = k$. Then $P(G) \geq \frac{2nk - k^2 + n - k}{n^2}$.*

Proof. Let G be a finite group of order n and center $Z(G)$ with $|Z(G)| = k$. Let $B = \{(x,y) \mid x \in Z(G),\ y \in G\}$, $C = \{(y,x) \mid x \in Z(G),\ y \in G \setminus Z(G)\}$ and $D = \{(y,y) \mid y \in G \setminus Z(G)\}$. Then $A \cup B \cup C \subseteq \{(x,y) \in G \times G \mid xy = yx\}$.

From the definition of $P(G)$, it follows that

$$\begin{aligned} P(G) &= \frac{|\{(x,y) \in G \times G \mid xy = yx\}|}{|G|^2} \\ &\geq \frac{|B \cup C \cup D|}{|G|^2} \\ &= \frac{kn + k(n-k) + n - k}{n^2} \\ &= \frac{kn + kn - k^2 + n - k}{n^2} \\ &= \frac{2nk - k^2 + n - k}{n^2}. \end{aligned}$$

Hence the claim follows. It is obvious that, the equality holds when G is abelian group. □

The number of the conjugacy classes of a finite group G is a significant quantity, which is denoted by $k(G)$. It is used to measure the probability of two elements to commute. See the next theorem.

Theorem 2. *[3] Let G be a finite group. Then the commutativity degree of G is*

$$P(G) = \frac{k(G)}{|G|}.$$

3. Main results

In this section the commutativity degree of certain finite groups will be computed.

3.1. Commutativity degree of non-abelian 2-generators 2-groups

For every natural number $n \geq 8$ there are exactly 4 non-abelian groups (up to isomorphism) of order $2n = 2(2^i)$, $i \geq 3$, with a cyclic subgroup of index 2, and they are:

- The dihedral group D_{2n}, which is a non-abelian 2-generator group of order $2n$, has a subgroup $\mathbb{Z}_n$ of index 2.

- The quasidihedral group QD_{2n}, which is a non-abelian 2-generator group of order $2n$, has a subgroup $\mathbb{Z}_n$ of index 2.

- The generalized quaternion group GQ_{2n}, which is a non-abelian 2-generator group of order $2n$, has a subgroup $\mathbb{Z}_n$ of index 2.

- A 2-generator group G described as:

$$\left\langle r, s \mid r^n = s^2 = e, sr = r^{n/2+1}s \right\rangle,$$

is isomorphic to the semidirect product of $\mathbb{Z}_n \rtimes \mathbb{Z}_2$, it has $\mathbb{Z}_n$ as a normal subgroup of index 2. We call this group by a Nondihedral group and denoted by ND_{2n}.

For more information about these groups one may refer to [2].

Next, we will find the commutativity degree of this family of groups.

Recall that the dihedral group D_{2n} of order $2n$ is the group of symmetries of a regular n-polygon, which includes rotations and reflections. Dihedral groups are among the simplest examples of finite groups. The presentation of the dihedral group in terms of rules and generators can be given by:

$$D_{2n} = \left\langle r, s \mid r^n = s^2 = e = \left(r^k s\right)^2 , \text{ for } k = 1, 2, \ldots, n \right\rangle.$$

Remark 3. Let $G = D_{2n}$ be the dihedral group of order $2n$. Then the commutativity degree of G is

$$P(G) = \begin{cases} \dfrac{n+3}{4n} & , \ n \text{ is odd} \\ \\ \dfrac{n+6}{4n} & , \ n \text{ is even} \end{cases} .$$

Proof. Let $G = D_{2n}$ be the dihedral group of order $2n$. The commutativity degree of G is

$$P(G) = \frac{|A|}{|G|^2} = \frac{|A|}{(2n)^2} \text{ where } A = \{(x, y) \mid x, y \in G \text{ and } xy = yx\}.$$

We need to find $|A|$, which can be easily done when A is written as a union of disjoint subsets. To do this, we have two cases:

Case 1. If n is odd, then e commutes with all $x \in G$, the element r^k commutes with r^m for any $k, m = 1, 2, 3, \ldots, n$ and any reflection $r^k s$ commutes only with itself. So

$$\begin{aligned} A &= \{(e,x) \mid x \in G\} \\ &\cup \{(x,e) \mid x \in G \setminus \{e\}\} \\ &\cup \{(x,y) \mid x = r^k\ ,\ y = r^m\ ,\ k,m = 1,2,\ldots,n-1\} \\ &\cup \{(x,x) \mid x = r^k s\ ,\ k = 1,2,\ldots,n\}. \end{aligned}$$

Then, it follows that

$$\begin{aligned} |A| &= 2n + (2n-1) + (n-1)(n-1) + n \\ &= n^2 + 3n. \end{aligned}$$

Hence, $P(G) = \dfrac{n^2+3n}{4n^2} = \dfrac{n+3}{4n}$ for n is odd.

Case 2. If n is even, then there is a rotation $a = r^{n/2} \in G$ for which $a^2 = e$, so $ar^k s = r^k sa$ for all $k = 1, 2, \ldots, n$. Therefore, $(a, r^k s), (r^k s, a) \in A$ for all $k = 1, 2, \ldots, n$, the number of such choices is $n + n = 2n$. Also, $r^k s r^{k+\frac{n}{2}} s = r^{k+\frac{n}{2}} s r^k = r^{\frac{n}{2}}$, then $(r^k s, r^{k+\frac{n}{2}} s) \in A\ ,\ k = 1, 2, \ldots, n$ and the number of these choices is n. In addition to the same number of choices in the previous case. So,

$$\begin{aligned} A &= \{(e,x) \mid x \in G\} \\ &\cup \{(x,e) \mid x \in G \setminus \{e\}\} \\ &\cup \{(x,a) \mid x \in G \setminus \{e\}\} \qquad a = r^{\frac{n}{2}} \\ &\cup \{(a,x) \mid x \in G \setminus \{e,a\}\} \qquad a = r^{\frac{n}{2}} \\ &\cup \{(x,y) \mid x = r^k\ ,\ y = r^m\ ,\ k,m = 1,2,\ldots,n-1 \\ &\qquad\qquad \text{and} \quad k,m \neq \tfrac{n}{2}\} \\ &\cup \{(x,x) \mid x = r^k s\ ,\ k = 1,2,\ldots,n\} \\ &\cup \{(r^k s, r^{k+n/2} s) \mid k = 1,2,\ldots,n\}. \end{aligned}$$

Then

$$\begin{aligned} |A| &= (2n) + (2n-1) + (2n-1) + (2n-2) \\ &\qquad + (n-2)^2 + (n) + (n) \\ &= n^2 + 6n. \end{aligned}$$

Therefore, $P(G) = \dfrac{n^2+6n}{4n^2} = \dfrac{n+6}{4n}$ for n is even.

$\square$

Recall that, the quasidihedral groups QD_n, are also called semi-dihedral groups, are hence non-abelian groups of order $n = 2(2^i) = 2m$ for a positive integer i greater than or equal 3. The group presentation in terms of rules and generators can be given by:

$$QD_n = \left\langle r, s \mid r^m = s^2 = e \ , \ rsr = r^{m/2}s \right\rangle .$$

Using the previous description, the next remark gives the commutativity degree of the quasidihedral group.

Remark 4. Let $G = QD_n$ be the quasidihedral group of order $n = 2(2^i) = 2m$, $i \geq 3$. Then the commutativity degree of G is

$$P(G) = \frac{m+6}{4m}.$$

Proof. Let $G = QD_n$ be the quasidihedral group of order $n = 2(2^i) = 2m$, $i \geq 3$. The commutativity degree of G is

$$P(G) = \frac{|A|}{|G|^2} = \frac{|A|}{(2m)^2} \text{ where } A = \{(x, y) \mid x, y \in G \text{ and } xy = yx\}.$$

We need to find $|A|$, which can be easily done when A is written as a union of disjoint subsets. Set $B = \{(e, x) \mid x \in G\} \cup \{(x, e) \mid x \in G \setminus \{e\}\}$. Then $B \subset A$. Also, $r^i r^j = r^j r^i$ for all $i, j = 1, 2, \ldots, m$, so $C = \{(r^i, r^j) \mid i, j = 1, 2, \ldots, m - 1\} \subset A$. In addition, $r^{m/2}x = xr^{m/2}$ for all $x = r^k s$, $k = 1, 2, \ldots, m$. So, let $D = \{(x, r^{m/2}) \mid x = r^k s \ , \ k = 1, 2, \ldots, m\} \cup \{(r^{m/2}, x) \mid x = r^k s \ , \ k = 1, 2, \ldots, m\}$, then $D \subset A$. Similarly, $E = \{(r^k s, r^k s) \mid k = 1, 2, \ldots, m\} \subset A$. The last subset F can be constructed from all pairs $(r^k s, r^{k+m/2}s)$, $k = 1, 2, \ldots, m$. This implies that $A = B \cup C \cup D \cup E \cup F$, and since these subsets are disjoint, then

$$\begin{array}{rcccccccccc} |A| & = & |B| & + & |C| & + & |D| & + & |E| & + & |F| \\ & = & 2m + 2m - 1 & + & (m-1)^2 & + & m + m & + & m & + & m \\ & = & m^2 + 6m. & & & & & & & & \end{array}$$

Therefore, $P(G) = \dfrac{m^2 + 6m}{(2m)^2} = \dfrac{m+6}{4m}.$ $\square$

Recall that, the generalized quaternion group GQ_n is a non-abelian 2-generator group of order $n = 2(2^i) = 2m$ for a positive integer i , $i \geq 3$. The group presentation in terms of rules and generators can be given by:

$$GQ_n = \left\langle r, s \mid r^m = s^4 = e, r^{m/2} = s^2, rs = sr \right\rangle .$$

Using the previous description, the next remark gives the commutativity degree of the generalized quaternion group.

Remark 5. Let $G = GQ_n$ be the generalized quaternion group of order $n = 2(2^i) = 2m$, $i \geq 3$. Then the commutativity degree of G is

$$P(G) = \frac{m+6}{4m}.$$

Proof. Let $G = GQ_n$ be the generalized quaternion group of order $n = 2(2^i) = 2m$, $i \geq 3$. Let $A = \{(x,y) \in G \times G \mid xy = yx\}$, to consider $P(G)$ we first list the elements structure of A:

- The identity element e commutes with every element in G. So, $(e,x),(x,e) \in A$ for all $x \in G$. There are $2m$ and $2m-1$ elements of such pairs.
- Since $Z(G) = \{e, r^{m/2}\}$, then $r^{m/2}$ commutes with every element in G. Thus, $(r^{m/2}, x) \in A$ for $x \in G \setminus \{e\}$. There are $2m-1$ elements of such pairs, these pairs are listed below.
- r^k , $\frac{m}{2} \neq k = 1,2,\ldots,m-1$ commutes with $x = r^i$, $i = 1,2,\ldots,m-1$. So, $(r^k, r^i) \in A$, $\frac{m}{2} \neq k = 1,2,\ldots,m-1$, $i = 1,2,\ldots,m-1$. There are $(m-2)(m-1) = m^2 - 3m + 2$ elements of such pairs.
- $r^k s$ for $k = 1,2,\ldots,m$ commutes with $r^{m/2}$, $r^k s$ and $\left(r^k s\right)^{-1}$. There are $3m$ elements of such pairs in A.

Using the previous list, one obtains:

$$\begin{aligned} |A| &= (2m + 2m - 1) + (2m-1) + (m^2 - 3m + 2) + (3m) \\ &= m^2 + 6m. \end{aligned}$$

Therefore, $P(G) = \dfrac{|A|}{|G|^2} = \dfrac{m^2 + 6m}{4m^2} = \dfrac{m+6}{4m}.$ □

Recall that, the Nondihedral group ND_{2n} is a non-abelian 2-generator group of order $n = 2(2^i) = 2m$ for a positive integer i , $i \geq 3$. The group presentation in terms of rules and generators can be given by:

$$ND_n = \left\langle r, s \mid r^m = s^2 = e, sr = r^{m+1}s \right\rangle.$$

Using the previous description, the next remark gives the commutativity degree of the Nondihedral group.

Remark 6. Let $G = ND_n$ be the Nondihedral group of order $n = 2(2^i) = 2m$, $i \geq 3$. Then the commutativity degree of G is
$$P(G) = \frac{5}{8}.$$

Proof. Let $G = ND_n$ be the Nondihedral group of order $n = 2(2^i) = 2m$, $i \geq 3$. Let $A = \{(x, y) \in G \times G \mid xy = yx\}$, to consider $P(G)$ we first list the elements structure of A:

- The identity element e commutes with every element in G. So, $(e, x), (x, e) \in A$ for all $x \in G$. There are $2m$ and $2m - 1$ of elements such pairs .
- Since $r^{m/2} \in Z(G)$, then r^k for $k = 2, 4, \ldots, m - 2$ commutes with every element in G. Thus, $(r^k, x), (x, r^k) \in A$ for $k = 2, 4, \ldots, m - 2$. There are $(\frac{m}{2} - 1)(2m - 1)$ and $(\frac{m}{2} - 1)(\frac{3}{2}m)$ elements of such pairs (note that $e = r^m$ has been already used in the previous item).
- r^k for odd k commutes with $x = r^i$, $i = 1, 2, \ldots, m$. So, $(r^k, r^i) \in A$, $k = 1, 3, \ldots, m - 1$, $i = 1, 2, \ldots, m$. There are $\frac{1}{4}m^2$ elements of such pairs (note that (r^k, r^i) , $k = 1, 3, \ldots, m-1$, $i = 2, 4, \ldots, m$ have already been used above).
- $r^k s$ for even k commutes with $x = r^i s$, $i = 2, 4, \ldots, m$. So, $(r^k s, r^i s) \in A$, $k = 2, 4, \ldots, m$, $i = 2, 4, \ldots, m$. There are $\frac{1}{4}m^2$ elements of such pairs.
- $r^k s$ for odd k commutes with $x = r^i s$, $i = 1, 3, \ldots, m-1$. So, $(r^k s, r^i s) \in A$, $k = 1, 3, \ldots, m - 1$, $i = 1, 3, \ldots, m - 1$. There are $\frac{1}{4}m^2$ elements of such pairs.

Using the previous list, we has:
$$\begin{aligned} |A| &= 2m + 2m - 1 + (\tfrac{m}{2} - 1)(2m - 1) + (\tfrac{m}{2} - 1)(\tfrac{3}{2}m) + \tfrac{1}{4}m^2 \\ &\quad + \tfrac{1}{4}m^2 + \tfrac{1}{4}m^2 \\ &= 4m - 1 + m^2 - \tfrac{5}{2}m + 1 + \tfrac{3}{4}m^2 - \tfrac{3}{2}m + \tfrac{3}{4}m^2 \\ &= \tfrac{5}{2}m^2. \end{aligned}$$

Therefore, $P(G) = \dfrac{|A|}{|G|^2} = \dfrac{\frac{5}{2}m^2}{4m^2} = \dfrac{5}{8}$. □

The number of the conjugacy classes of $G = ND_n$ can be easily considered from the previous result and Theorem 2. That is:
$$k(G) = P(G) \cdot |G| = \frac{5}{8}n.$$

3.2. Commutativity degree of the symmetric group S_n

In this part we will use the number of conjugacy classes of the symmetric group $G = S_n$ to find $P(G)$. So we need the following theorem.

Theorem 7. *[6] For the symmetric group S_n of cycle lengths $L = \{0, 2, 3, \cdots, n\}$. The number of conjugacy classes uniquely determined by a cycle of length $k \in L$ is M_k, where M_k is the number of solutions of $k_i \in L$ for the inequality $k + \sum_{k \geq k_i \in L} k_i \leq n$. Therefore, $k(G) = \sum_{k \in L} M_k$.*

Now, we find $P(S_n)$ using the following remark.

Remark 8. Let $G = S_n$ be the symmetric group of cycle lengths $L = \{0, 2, 3, \cdots, n\}$, and let M_k be the number of solutions of $k_i \in L$ for the inequality:

$$k + \sum_{k \geq k_i \in L} k_i \leq n. \tag{1}$$

Then

$$P(G) = \frac{\sum_{k \in L} M_k}{|G|}.$$

Proof. The proof is straight forward using Theorem 2 and Theorem 7. $\square$

Example 9. Let $G = S_{12}$. The next list shows the number of solutions for Inequality (1).

- For $k = 0$: The cases of $k_i \in L$; $k_i \leq 0$ is only 0. Thus the choices of the sum ≤ 12 is $0 + 0$, implies that $M_0 = 1$.
- For $k = 2$: The cases of $k_i \in L$; $k_i \leq 2$ are $0, 2$. Thus the choices of the sum ≤ 12 are $2+0, 2+2, 2+2+2, 2+2+2+2, 2+2+2+2+2, 2+2+2+2+2+2$, implies that $M_2 = 6$.
- For $k = 3$: The cases of $k_i \in L$; $k_i \leq 3$ are $0, 2, 3$. Thus the choices of the sum ≤ 12 are $3+0,\ 3+2,\ 3+2+2,\ 3+2+2+2,\ 3+2+2+2+2,\ 3+3,\ 3+3+2,\ 3+3+2+2,\ 3+3+2+2+2, 3+3+3,\ 3+3+3+2,\ 3+3+3+3$, implies that $M_3 = 12$.

- Similarly, for $k = 4, 5, 6, 7, 8, 9, 10, 11$ and $k = 12$, one has $M_4 = 15, M_5 = 13, M_6 = 11, M_7 = 7, M_8 = 5, M_9 = 3, M_{10} = 2, M_{11} = 1$ and $M_{12} = 1$.

This implies that

$$\begin{aligned} k(G) &= \sum_{k\in L} M_k \\ &= M_0 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7 + M_8 + M_9 \\ &\quad + M_{10} + M_{11} + M_{12} \\ &= 77. \end{aligned}$$

Now, using Remark 8, one has:

$$P(G) = \frac{k(G)}{|G|} = \frac{77}{12!} = \frac{1}{6220800}.$$

The previous results can be used to find the number of order pairs $(x, y) \in S_{12} \times S_{12}$ such that $xy = yx$. That can be done using $P(G) = \frac{|A|}{|G|^2}$, where $A = \{(x, y) \mid xy = yx\ ,\ x, y \in S_{12}\}$. So, $|A| = P(G) \cdot |G|^2 = 36883123200$. This large number can not be estimated using any of the usual counting principles.

3.3. Commutativity degree of p-groups

As, any p-group is nilpotent [7]. This allow us to use the next theorem to find a bound for the commutativity degree of these groups.

Theorem 10. *[1] Let G be a finite group of nilpotency class c, and $Z(G)$ be the center of G. Let $u = |G|-r$, where $r = \left\lfloor \frac{|G| - |Z(G)|}{c} \right\rfloor$. Then $k(G) \leq u$.*

Now, we will use Theorem 2 and Theorem 10 in the following remark.

Remark 11. Let G be a finite p-group of nilpotency class c and center $Z(G)$, let $u = |G| - r$, where $r = \left\lfloor \frac{|G| - |Z(G)|}{c} \right\rfloor$. Then

$$P(G) \leq \frac{u}{|G|}.$$

Proof. The proof is an immediate consequence of Theorem 2 and Theorem 10. □

Example 12. The group

$$\begin{aligned} G &= \left\langle a,\ b \mid a^{5^2} = b^5 = [a,[b,a]] = [b,[b,a]] = [a,b]^5 = e \right\rangle \\ &\cong (\mathbb{Z}_{5^2} \times \mathbb{Z}_5) \rtimes \mathbb{Z}_5, \end{aligned}$$

is a none abelian p-group ($p = 5$) of nilpotency class 2 and $|Z(G)| = 5^2 = 25$. Using Remark 11, it follows that:

$$r = \left\lfloor \frac{|G| - |Z(G)|}{c} \right\rfloor = \left\lfloor \frac{625 - 25}{2} \right\rfloor = 300.$$

Therefore,

$$P(G) \leq \frac{u}{|G|} = \frac{625 - 300}{625} = \frac{325}{625} = \frac{13}{25}.$$

Using Proposition 1, one obtains:

$$\frac{1249}{15625} \leq P(G) \leq \frac{13}{25}$$

4. Conclusion

In this paper, the commutativity degree of non-abelian 2-generators groups has been computed. The investigated results can be used to find the commutativity degree of such groups using explicit and elementary rules. Also, the commutativity degree of symmetric group has been obtained. The results were obtained by the number of conjugacy classes of the symmetric group. The final results show the significance of the conjugacy classes on the group structure. Moreover, the number of the conjugacy classes has been also used to obtain an upper bound to the commutativity degree of p-groups.

Clearly, the considered groups in this paper are of the most important ones to compute their $P(G)$. Indeed, further considerations are needed to evaluate the commutativity degree for other groups.

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