

WEAKLY-H CONTRACTION FIXED POINT THEOREM IN b -Menger SPACES

Abderrahim Mbarki^{1 §}, Jamal Hlal²

¹ANO Laboratory

National School of Applied Sciences P.O. Box 669
Mohammed First University Oujda, MOROCCO

²ANO Laboratory, Faculty of Sciences

Mohammed First University
60000 Oujda, MOROCCO

Abstract: In this paper we study the problem of existence of fixed points for weakly Hick contraction in b -Menger spaces. Some examples are furnished to demonstrate the validity of the obtained results.

AMS Subject Classification: 54E70, 54H25, 47S50, 34B15

Key Words: b -Menger space; weakly Hick contraction; fixed point

1. Introduction and preliminaries

In 2017, Mbarki et al. [5] introduced a new notion called probabilistic b -metric by generalizing the (probabilistic) triangle inequality axiom in the definition of standard probabilistic metric. Then they obtained some important fixed point results in a series of works (see [6], [7], [8], [9], [10]).

Received: May 7, 2021

© 2022 Academic Publications

[§]Correspondence author

Our aim is to continue the research on the fixed point in b-Menger spaces. In this work, we study the problem of the existence of a fixed point for weak-Hicks contraction maps. Some non-trivial examples have been presented to illustrate the results.

Here, we state some basic definitions and expressions, necessary to use in the main part of the paper.

Definition 1. A distance distribution function (briefly, a d.d.f.) is a nondecreasing function F defined on $R^+ \cup \{\infty\}$ that satisfies $f(0) = 0$ and $f(\infty) = 1$, and is left continuous on $(0, \infty)$. The set of all d.d.f.'s will be noted by Δ^+ ; and the set of all F in Δ^+ for which $\lim_{t \rightarrow \infty} f(t) = 1$ by D^+ .

A simple example of distribution function is the Heaviside function in D^+ ,

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

A commutative, associative and nondecreasing mapping $T : [0, 1] \rightarrow [0, 1]$ is called a t -norm, if and only if

1. $T(a, 1) = a$, for all $a \in [0, 1]$,
2. $T(0, 0) = 0$.

As examples we mention the three typical examples of continuous t -norms as follows:

$$T_p(a, b) = ab, T_M(a, b) = \min(a, b) \text{ and } T_L(a, b) = \max\{a + b - 1, 0\}.$$

Let T be a t -norm for the sake of brevity,

$$T^1(x, y) = T(x, y), \quad T^2(x, y, z) = T(x, T^1(y, z))$$

and for every $n \geq 3$,

$$T^n(x_1, x_2, \dots, x_{n+1}) = T(x_1, T(x_2, x_3, \dots, x_{n+1})).$$

A t -norm T can be extended to a countable infinitely operation taking for any $\{x_n\}$ in $[0, 1]$ the values

$$T_{i=1}^{\infty} x_i = \lim_{n \rightarrow \infty} T^n(x_1, \dots, x_{n+1}),$$

(this limit always exists since the sequence $\{T^n(x_1, \dots, x_{n+1})\}$ is nondecreasing and bounded from below) and $T_{i=n}^{\infty} x_i = T_{i=1}^{\infty} x_{n+i}$.

We shall make use of the following definitions.

Definition 2. ([1]) We say that a t -norm T is of H-type, if the family $\{T^n(t)\}$ is equicontinuous at $t = 1$, that is,

$$\forall \epsilon \in (0, 1) \quad \exists \lambda \in (0, 1) : t > 1 - \lambda \Rightarrow T^n(t) > 1 - \epsilon \quad \text{for all } n \geq 1.$$

Definition 3. ([2]) A t -norm T is geometrically convergent (g -convergent) if, for all $q \in (0, 1)$,

$$\lim_{n \rightarrow \infty} T_{i=n}^{\infty}(1 - q^i) = 1.$$

It should be noticed (see [2]), that if $\lim_{n \rightarrow \infty} T_{i=n}^{\infty}(1 - q_0^i) = 1$ for some $q_0 \in (0, 1)$, then T is g -convergent t -norm. Also, note that if T is g -convergent, then $\sup_{a < 1} T(a, a) = 1$. Some sufficient condition for T is given by Hadzic et al. [2] to ensure that T is a g -convergent t -norm. Also, note that T_L and the t -norms of H -type are g -convergent t -norms.

Definition 4. ([5]) A b -Menger space is a quadruple (M, F, T, s) where M is a nonempty set, F is a function from $M \times M$ into Δ^+ , T is a t -norm, $s \geq 1$ is a real number, and the following conditions are satisfied:

For all $p, r, q \in M$ and $x, y > 0$,

1. $F_{pp} = H$,
2. $F_{pr} = H \Rightarrow p = r$,
3. $F_{pr} = F_{rp}$,
4. $F_{pr}(s(x + y)) \geq T(F_{pq}(x), F_{qr}(y))$.

It should be noted that a Menger space is a b -Menger space with $s = 1$.

Definition 5. Let $\{x_n\}$ be a sequence in a probabilistic semimetric space (M, F) (i.e., (1), (2) and (3) of Definition 4).

1. A sequence $\{x_n\}$ in M is said to be convergent to x in M if, for every $\epsilon > 0$ and $\delta \in (0, 1)$, there exists a positive integer $N(\epsilon, \delta)$ such that $F_{x_n x}(\epsilon) > 1 - \delta$, whenever $n \geq N(\epsilon, \delta)$.
2. A sequence $\{x_n\}$ in M is called Cauchy sequence if, for every $\epsilon > 0$ and $\delta \in (0, 1)$, there exists a positive integer $N(\epsilon, \delta)$ such that $F_{x_n x_m}(\epsilon) > 1 - \delta$, whenever $n, m \geq N(\epsilon, \delta)$.

3. (M, F) is said to be complete if every Cauchy sequence has a limit.

2. Main result

The following type probabilistic contractions, intensively studied in the fixed point theory in PM space (see e.g. [3], [11]), have been introduced by Hicks [4]:

Definition 6. A mapping f of a probabilistic semimetric space (M, F) into itself is Hicks contraction (C -contraction for short) if there is a k in $(0, 1)$ such that the following implication holds for every $p, q \in M$ and all $t > 0$:

$$(H) : F_{pq}(t) > 1 - t \implies F_{fpfq}(kt) > 1 - kt.$$

Mihet ([11]) considered a generalization of the C -contraction, known as weak-Hicks contraction:

Definition 7. A mapping f of a probabilistic semimetric space (M, F) into itself is said to be a weak-Hicks contraction ($w-H$ contraction for short) if there exists $k \in (0, 1)$ such that, for all $p, q \in S$,

$$(w-H) : t \in (0, 1), F_{pq}(t) > 1 - t \implies F_{f(p)f(q)}(kt) > 1 - kt.$$

Now, we can state and prove the main fixed point theorem of this paper.

Theorem 8. Let (M, F, T, s) be a complete b -Menger space such that T is a g -convergent t -norm and f be a $w-H$ contraction with $k \in (0, \frac{1}{s})$. Then f has a fixed point iff there exists $x \in M$ such that $F_{xfx}(1) > 0$.

Proof. Suppose that f has a fixed point x , then $F_{xfx}(1) = F_{xx}(1) = 1 > 0$.

For the converse implication: First, we show that there exists $\delta \in (0, 1)$ such that $F_{xfx}(\delta) > 1 - \delta$. Indeed, if we have $F_{xfx}(\delta) \leq 1 - \delta$ for all $\delta \in (0, 1)$, letting $\delta \rightarrow 1$ and using the fact that F_{xfx} is left continuous, we obtain $F_{xfx}(1) \leq 0$, which is a contradiction. Hence there exists $\delta \in (0, 1)$ such that $F_{xfx}(\delta) > 1 - \delta$. Now, using (w-H), we can prove by induction that

$$F_{f^n x f^{n+1} x}(k^n \delta) > 1 - k^n \delta$$

for every $n \in N$. We shall prove that the sequence $\{x_n\}$ is a Cauchy sequence, i.e. that for every $\epsilon, \gamma > 0$ there exists $n''_0 \in N$ such that

$$F_{x_n x_{n+m}}(\epsilon) > 1 - \gamma$$

for every $n \geq n''_0$ and every $m \in N$. Since T is a g -convergent t -norm, then exists $n_1 \in N$ such that

$$T_{i=n}^\infty(1 - k^i) > 1 - \gamma$$

for all $n \geq n_1$. Let $n_0 \in N$ be such that

$$\sum_{i \geq n_0} (sk)^i \delta < \epsilon.$$

Then for $n \geq n''_0 = \max\{n_0, n_1\}$, we have

$$\begin{aligned} F_{x_n x_{n+m}}(\epsilon) &\geq F_{x_n x_{n+m}}\left(\sum_{i=n}^{n+m-1} (sk)^i \delta\right) \\ &\geq T(F_{x_n x_{n+1}}(s^{n-1} k^n \delta), F_{x_{n+1} x_{n+m}}(\sum_{i=n+1}^{n+m-1} (s^{i-1} k)^i \delta)) \\ &\cdot \\ &\cdot \\ &\cdot \\ &\geq T^{m-1}(F_{x_n x_{n+1}}(s^{n-1} k^n \delta), F_{x_{n+1} x_{n+2}}(s^{n-1} k^{n+1} \delta), \dots, \\ &F_{x_{n+m-1} x_{n+m}}(s^{n-1} k^{n+m-1} \delta)) \\ &\geq T^{m-1}(F_{x_n x_{n+1}}(k^n \delta), F_{x_{n+1} x_{n+2}}(k^{n+1} \delta), \dots, F_{x_{n+m-1} x_{n+m}}(k^{n+m-1} \delta)) \\ &\geq T^{m-1}(1 - k^n \delta, \dots, 1 - k^{n+m-1} \delta) \\ &\geq T^{m-1}(1 - k^n, \dots, 1 - k^{n+m-1}) \\ &\geq T_{i=n}^\infty(1 - k^i) \\ &> 1 - \gamma. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in (M, F, T) . By completeness, $x_n \rightarrow z$ for some z in M . Now we show that $x_n \rightarrow fz$. Indeed, if $\epsilon > 0$ and $\lambda \in (0, 1)$ are given, we choose $\sigma \in (0, 1)$ such that $0 < k\sigma < \min\{\epsilon, \lambda\}$. Then there exists N for all $n \geq N$ that we have $F_{x_n z}(\sigma) > 1 - \sigma \implies F_{f x_n f z}(k\sigma) > 1 - k\sigma \implies F_{f x_n f z}(k\sigma) > 1 - \lambda \implies F_{f x_n f z}(\epsilon) > 1 - \lambda$. Therefore, $\lim_{n \rightarrow \infty} x_{n+1} = fz$.

By inequality (4) in Definition 4, we have

$$F_{f z z}(t) \geq T(F_{f z x_{n+1}}(\frac{t}{2s}), F_{x_{n+1} z}(\frac{t}{2s}))$$

for all $t > 0$, $n \geq 1$.

Since $\sup_{a < 1} T(a, a) = 1$, let $\sigma \in (0, 1)$, there exists $\rho \in (0, 1)$ such that

$$T(1 - \rho, 1 - \rho) > 1 - \sigma$$

and using $F_{f z x_{n+1}}(\frac{t}{2s}) \rightarrow 1$ and $F_{x_{n+1} z}(\frac{t}{2s}) \rightarrow 1$, there exists $N \in N$ such that

$$F_{fzx_{n+1}}(\frac{t}{2s}) > 1 - \rho, F_{x_{n+1}z}(\frac{t}{2s}) > 1 - \rho$$

for all $n \geq N$, which implies that

$$\begin{aligned} F_{fzz}(t) &\geq T(F_{fzx_{n+1}}(\frac{t}{2s}), F_{x_{n+1}z}(\frac{t}{2s})) \\ &\geq T(1 - \rho, 1 - \rho) \\ &\geq 1 - \sigma. \end{aligned}$$

Then

$$F_{fzz}(t) \geq 1 - \sigma, \forall \sigma \in (0, 1).$$

Letting $\sigma \rightarrow 0$ we get $F_{fzz}(t) = 1$, since this is for arbitrary $t > 0$, then $fz = z$. $\square$

It should be noted unlike other previous works as [5]-[10], that in the proof of Theorem 8 the conditions $Range(F) \subset D^+$, T is of H -type or continuous t -norm on the b -Menger metric are not necessary.

Example 9. Here we illustrate our results from Theorem 8 by the following example: let $M = \{a, b, c, d\}$ and define $F_{pq}(t) = 0$ for all $p, q \in M$, $p \neq q$ and $t > 0$. It is easy to see that $(M, F, T_L, 1)$ is a complete b -Menger space. Consider the mapping $f : M \rightarrow M$, defined by $f(a) = a$, $f(b) = b$, $f(c) = d$, $f(d) = c$. Since $t \in (0, 1)$, $F_{pq}(t) > 1 - t \implies p = q \implies F_{f(p)f(q)}(kt) = 1$, f is a weak Hicks contraction. Note that this mapping has two fixed points : a and b . But $F_{ab} \notin D^+$ and T_L is not a t -norme of type H .

By taking $s = 1$ in Theorem 8, we obtain the following result.

Corollary 10. ([11]) *Let (M, F, T) be a complete Menger space such that T is a g -convergent t -norm and f be a w - H contraction. Then f has a fixed point iff there exists $x \in M$ such that $F_{xfx}(1) > 0$.*

Example 11. Let $M = [0, \infty)$. Define $F : M \times M \rightarrow \Delta^+$ by

$$F_{pq}(t) = H(t - |p - q|^2).$$

We claim that $(M, F, T_L, 2)$ is a b -Menger space. Indeed, if

$$T_L(F_{pr}(t), F_{rq}(s)) = \max\{F_{pr}(t) + F_{rq}(s) - 1, 0\} > 0,$$

this implies that $H(t - |p - r|^2) = H(s - |r - q|^2) = 1$, then $t > |p - r|^2$ and $s > |r - q|^2$. So,

$$\begin{aligned} 2(s + t) &> 2(|p - r|^2 + |r - q|^2) \\ &\geq |p - q|^2. \end{aligned}$$

Hence

$$\begin{aligned} F_{pq}(2(s + t)) &\geq H(2(s + t) - |p - q|^2) \\ &\geq 1 \\ &\geq T_L(F_{pr}(t), F_{rq}(s)), \end{aligned}$$

which proves our claim. It is obvious that $(M, F, T_L, 2)$ is complete. Now consider the function $f : M \rightarrow M$ given by $f(x) = \frac{\sqrt{2}}{\sqrt{5}}x$. Clearly, all the conditions of Theorem 8 are satisfied and 0 is a fixed point of f . But we cannot apply the results of [11] here, since (M, F, T_L) is not a standard probabilistic metric space because it lacks the triangle inequality:

$$F_{13}(3) = 0 < T_L(F_{12}(\frac{3}{2}), F_{23}(\frac{3}{2})) = 1.$$

In the same way we cannot apply the results of [5]-[10] since T_L is not a t -norm of type H .

References

- [1] O. Hadžić, A fixed point theorem in Menger spaces, *Publ. Inst. Math. (Beograd)*, **20** (1979), 107–112.
- [2] O. Hadžić, E. Pap, M. Budincevic, Countable extension of triangular norms and their applications to fixed point theory in probabilistic metric spaces, *Kibernetika*, **38**, No 3 (2002), 363–381.
- [3] O. Hadžić, E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic Publishers, Dordrecht (2001).
- [4] T.L. Hicks, Fixed point theory in PM spaces, *Rev. Res. Novi Sad*, **13** (1983), 63–72.
- [5] A. Mbarki, R. Oubrahim, Probabilistic b -metric spaces and nonlinear contractions, *Fixed Point Theory Application*, **2017** (2017), 15–29.

- [6] A. Mbarki, R. Oubrahim, Fixed point theorems with cyclical contractive conditions in b -Menger spaces, *Results in Fixed Point Theory and Applications*, **2018** (2018); doi: 10.30697/rfpta-2018-005.
- [7] A. Mbarki, R. Oubrahim, Common fixed point theorems in b -Menger spaces, In: *Recent Advances in Intuitionistic Fuzzy Logic Systems, Studies in Fuzziness and Soft Computing*, Springer (2019), 283–289.
- [8] A. Mbarki, R. Oubrahim, Common fixed point theorem in b -Menger spaces with a fully convex structure, *International Journal of Applied Mathematics*, **32**, No 2 (2019), 219–238; doi: 10.12732/ijam.v32i5.1.
- [9] A. Mbarki, R. Oubrahim, Fixed point theorem satisfying cyclical conditions in b -Menger spaces, *Moroccan J. Pure. Appl. Anal.*, **5**, No 1 (2019), 31–36.
- [10] A. Mbarki, R. Oubrahim, Probabilistic φ -contraction in b -menger spaces with fully convex structure, *International Journal of Applied Mathematics*, **33**, No 4 (2020), 621–633; doi: 10.12732/ijam.v33i4.7.
- [11] D. Mihet, Weak-Hicks contractions, *Fixed Point Theory*, **6**, No 1 (2005), 71–78.
- [12] B. Schweizer, H. Sherwood and R.M. Tradif, Contractions on probabilistic metric spaces: examples and counterexamples, *Stochastica*, **XII**, No 1 (1988), 5–17.