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ANALYSIS OF INTERFACIAL FRACTAL CONTACTS IN 3-DIMENSIONAL PACKING

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Abstract: Asymptotic analysis of problems in domains with fractal interfaces or fractal limits is important for the study of phenomena related to heterogeneous materials and in particular to know the behavior of the faults in the geological zones.

We consider a dense network of elastic materials modelled by a dense network of elastic balls in the unit ball $B(0;1)$ of $\mathbb{R}^n$; $n = 3$, obtained from an 3-dimensional packing of elastic spherical balls. The purpose is to use Γ-convergence methods in order to study the asymptotic behaviour of the structure and deriving contact laws on the residual fractal interface. The problem considered here has other implications, such as the modeling of the behavior of composite materials or the study of displacement of a geological fault causing a generation of a new fractures.

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1. Introduction

The wall rocks in most geological fault zones contain a layer cataclastic rock

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called gouge commonly attributed to wear and attrition of the sliding walls, or to implosive loading during earthquakes (see for example [23] and [24]). The material around these complexities is subjected, during the continuous deformations of the medium, to a concentration of great stresses, which leads to a generation of new fractures with a change in the elasticity, the permeability and the geometry of the medium [28]. The asymptotic analysis of mathematical problems in domains with fractal interfaces or fractal edges is relatively new and recent. It has been considered in the study of certain physical phenomena in disordered media (see for example [30] and [32]).

For the main work, we consider here a packing of the unit ball $\Omega = B(0,1)$. The principle of construction of Apollonian spheres packing of the unit ball is very similar to the construction of Apollonian circles packing of the unit disk. This construction consists in considering three pairwise externally tangent balls B_1, B_2, B_3 of respective radius $\rho_1, \rho_2; \rho_3 < 1$, which are internally tangent to Ω, and then repeat the process by adding more balls B_i, of radii $\rho_i; i > 3$, in B_0 such that $B_i \cap B_j = \emptyset; \quad j = 1, ..., i-1$ (see for instance [8]). We denote $\{B_k\}_{k=1}^{\infty}$ the network of isotropic and homogeneous linear elastic open balls that are removed from Ω to obtain the residual set Λ defined in [1], which is a fractal whose fractal dimension is $d \sim 2.47$ (see for instance [5], [6] and [8]).

We consider the same problem as in the 2-dimensional case [1] replacing discs by balls and lines by discs. For every $h \in \mathbb{N}^*$, we define the space

$$\mathbf{V}_{2,h}(\Omega_h) = \left\{ v \in \mathbf{H}^1\left(\Omega_h, \mathbb{R}^3\right); [v]_{T_h} = 0 \text{ and } v = 0 \text{ on } \Gamma_0 \cap \bar{\Omega}_h \right\} \tag{1}$$

where, using the same notation as in the above section,

$$\Omega_h = \bigcup_{k=1}^{N(h)} B^*_{k,h},$$
$$T_h = \bigcup_{k=1}^{N(h)} \bigcup_{j=1}^{n_k(h)} B\left(x^*_{j,k}, r_h\right) \cap Z_{j,k}(h),$$

where $B\left(x^*_{j,k}, r_h\right)$ is a ball centred at the contact point $x_{j,k}$, of radius r_h with $\lim_{h\to\infty} \frac{r_h}{\rho_h^{\epsilon d}} = 0; \frac{d+1}{2d} < \epsilon < 1$, and $Z_{j,k}(h); j \in \{1, \ldots, n_k(h)\}$ is the thin contact disk of radius $\rho_h^{\epsilon d}$ centred at the contact point $x^*_{j,k}$.

We define the energy functional $F_{2,h}$ on $\mathbf{L}^2\left(\Omega, \mathbb{R}^3\right)$ through

$$F_{2,h}(u) = \begin{cases} \int_{\Omega_h} \sigma_{ij}(u) e_{ij}(u) dx, & \text{if } u \in \mathbf{V}_{2,h}(\Omega_h), \\ +\infty, & \text{otherwise.} \end{cases} \tag{2}$$

The equilibrium of the material is described through the minimization problem

$$\min_{u\in\mathbf{L}^2(\Omega,\mathbb{R}^3)}\left\{F_{2,h}(u)-2\int_{\Omega_h}f.udx\right\}. \tag{3}$$

Performing some estimates as in the above tree-dimensional case, one can prove that problem (3) has a unique solution $u^h\in\mathbf{V}_{2,h}\left(\Omega_h\right)$ such that

$$\sup_{h\in\mathbb{N}^*}\left\|u^h\right\|_{\mathbf{H}^1(\Omega_h)}<+\infty \tag{4}$$

We use spherical coordinates (r,θ,ϕ) to define the map Υ_2^h from $\bigcup_{k=1}^{N(h)}B_k$ to the set Ω_h by

$$\Upsilon_2^h(r,\theta,\phi)=\begin{cases}\left(\left(R_k-d_k(h)\right)\frac{r}{R_k},\theta,\phi\right) & \text{in the cone of apex the point,}\\ (r,\theta,\phi) & c_k\text{and of base the disk } Z_{j,k}(h),\\ \text{elsewhere,} & \end{cases} \tag{5}$$

for every $k\in\{1,2,\dots,N(h)\}$ and $j\in\{1,\dots,n_k(h)\}$.

Let u^h is the solution of problem (3). As in the 2-dimensional case [1], we can prove that there exists a subsequence of the sequence $\left(u^h\right)_h$, still denoted in the same way, and $u\in\mathbf{H}^1_{\Gamma_0}\left(\Omega\backslash\Sigma,\mathbb{R}^3\right)$, such that

$$\begin{array}{cccl}\nabla\left(u^h\circ\Upsilon_2^h\right)\mu_h & \underset{h\to\infty}{\to} & \nabla u\mathbf{1}_{\Omega\backslash\Sigma}dx & \text{in } \mathcal{M}_b\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)\text{-weak }*,\\ u^h & \underset{h\to\infty}{\to} & u & \text{in } \mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)\text{-weak.}\end{array} \tag{6}$$

Accordingly, we consider the following topology τ_2.

Definition 1. We say that a sequence $\left(u^h\right)_h$; $u^h\in\mathbf{V}_{2,h}\left(\Omega_h\right)$, τ_2-converges to u if $\sup_{h\in\mathbb{N}^*}\left\|u^h\right\|_{\mathbf{H}^1(\Omega_h)}<+\infty$, and

$$\nabla\left(u^h\circ\Upsilon_2^h\right)\mu_h\underset{h\to\infty}{\to}\nabla u\mathbf{1}_{\Omega\backslash\Sigma}dx\quad\text{in}\quad\mathcal{M}_b\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)\text{-weak}*,$$

$$u^h\underset{h\to\infty}{\to}u\quad\text{in}\quad\mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)-\text{ weak.}$$

2. The main result

Our main result in this section reads as follows.

Theorem 2. *Let $r_h = c\rho_h^d$, c being a positive constant. Then:*

1. (lim sup ineguality) For every $u \in \mathbf{H}^1_{\Gamma_0}\left(\Omega\backslash\Sigma, \mathbb{R}^3\right)$, there exists a sequence $\left(u^h\right)_h; u^h \in \mathbf{V}_{1,h}\left(\Omega_h\right)$, such that $\left(u^h\right)^0_h$ τ_1-converges to u and

$$\limsup_{h\to\infty} F_{2,h}\left(u^h\right) \leq F_{2,c}(u),$$

2. (lim inf ineguality) For every sequence $\left(u^h\right)^0_h$; $u^h \in \mathbf{V}_{1,h}\left(\Omega_h\right)$, such that $\left(u^h\right)^0_h$ τ_1-converges to u, we have $u \in \mathbf{H}^1_{\Gamma_0}\left(\Omega\backslash\Sigma, \mathbb{R}^3\right)$ and

$$\liminf_{h\to\infty} F_{2,h}\left(u^h\right) \geq F_{2,c}(u)$$

the sequence $\left(F_{2,h}\right)_h$, defined in (1), Γ-converges, with respect o the topology τ_2 defined in Definition 1, to the functional $F_{2,c}$ defined in (2).

The boundary value problem associated to the limit functional (2) is given in the following

Corollary 3. *Under the hypothesis of Theorem 2, the sequence $\left(u^h\right)_h$, with u^h the solution of problem (3), τ_1-converges to the solution u of the problem*

$$\begin{cases} -\operatorname{div}\sigma(u) = f & \text{in } \Omega\backslash\Sigma, \\ u = 0 & \text{on } \Gamma_0, \\ \sigma_{ij}(u)\nu_j = -\dfrac{c}{\mathcal{H}^d(\Lambda)} A_{ij}\left([u]_\Sigma\right)_j \mathcal{H}^d & \text{on } \Sigma. \end{cases} \tag{7}$$

Moreover, the convergence of the energy functionals $\lim_{h\to\infty} F_{2,h}\left(u^h\right) = F_{2,c}(u)$ holds true.

2.1. Proof of Theorem 2

2.1.1. First step: Boundary layer problems

We consider the following auxiliary linear elasticity problems posed in the half space $y_3 > 0$ wich is denoted here $\mathbb{R}^{3+}$:

$$\begin{cases} \operatorname{div} \sigma\left(w^m\right) = 0 \text{ in } \mathbb{R}^{3+} \backslash \bar{D}(0,1);\ m = 1,2,3, \\ w^m = e_m \quad \text{on} \quad D(0,1);\ m = 1,2,3, \\ \sigma_{i3}\left(w^m\right) = 0 \quad \text{on} \quad \mathbb{R}^2 \backslash \bar{D}(0,1);\ i = 1,2,3, \\ w^m(y) \to 0 \quad \text{as} \quad |y| \longrightarrow +\infty,\ y_3 > 0, \end{cases} \tag{8}$$

where $e_m = (\delta_{1m}, \delta_{2m}, \delta_{3m})\,; m = 1,2,3,$ $D(0,1)$ is the unit disc centred at the origin, and $\sigma\left(w^m\right) = \left(\sigma_{ij}\left(w^m\right)\right)_{i,j=1,2,3}$ with $\sigma_{ij}\left(w^m\right) = \lambda_0 e_{ll}\left(w^m\right)\delta_{ij} + 2\mu_0 e_{ij}\left(w^m\right)$.

The solution w^m of (8) is given (see [16]), for $j, m = 1,2,3,$ through

$$w_j^m(y) = \int_{D(0,1)} q_m((\zeta,\xi)) G_{mj}\left(y_1 - \zeta, y_2 - \xi, y_3\right) d\zeta d\xi, \forall y \in \mathbb{R}^{3+}, \tag{9}$$

where

$$q_m(\zeta,\xi) = \begin{cases} \mathbf{1}_{D(0,1)}(\zeta,\xi) \frac{3\mu_0(\lambda_0+\mu_0)}{\pi(2\lambda_0+3\mu_0)\sqrt{1-\zeta^2-\xi^2}} & \text{for } m = 1,2, \\ \mathbf{1}_{D(0,1)}(\zeta,\xi) \frac{2\mu_0(\lambda_0+\mu_0)}{\pi(\lambda_0+2\mu_0)\sqrt{1-\zeta^2-\xi^2}} & \text{for } m = 3, \end{cases} \tag{10}$$

$\mathbf{1}_{D(0,1)}$ being the characteristic function of $D(0,1)$, and $\left(G_{ij}\right)_{i,j=1,2,3}$ is the Green symmetric tensor for the half-space $y_3 > 0$ occupied by an isotropic and homogenous elastic material of Lame's constants λ_0 and μ_0. This tensor is given (see [28]) by

$$\begin{cases} G_{11}(y) = \frac{1}{4\pi\mu_0(\lambda_0+\mu_0)} \left\{ \frac{(\lambda_0+2\mu_0)r+(\lambda_0+\mu_0)y_3}{r(r+y_3)} \right. \\ \qquad\qquad \left. + y_1^2 \frac{r(\lambda_0 r+y_3(\lambda_0+\mu_0))+y_3^2(\lambda_0+\mu_0)}{r^3(r+y_3)^2} \right\}, \\ G_{12}(y) = \frac{1}{4\pi\mu_0(\lambda_0+\mu_0)} \left\{ \frac{r(\lambda_0 r+y_3(\lambda_0+\mu_0))+y_3^2(\lambda_0+\mu_0)}{r^3(r+y_3)^2} \right\} y_1 y_2, \\ G_{13}(y) = \frac{1}{4\pi\mu_0(\lambda_0+\mu_0)} \left\{ \frac{y_1 y_3(\lambda_0+\mu_0)}{r^3} - \frac{\mu_0 y_1}{r(r+y_3)} \right\}, \\ G_{22}(y) = \frac{1}{4\pi\mu_0(\lambda_0+\mu_0)} \left\{ \frac{(\lambda_0+2\mu_0)r+(\lambda_0+\mu_0)y_3}{r(r+y_3)} \right. \\ \qquad\qquad \left. + y_2^2 \frac{r(\lambda_0 r+y_3(\lambda_0+\mu_0))+y_3^2(\lambda_0+\mu_0)}{r^3(r+y_3)^2} \right\}, \\ G_{23}(y) = \frac{1}{4\pi\mu_0(\lambda_0+\mu_0)} \left\{ \frac{y_2 y_3(\lambda_0+\mu_0)}{r^3} - \frac{\mu_0 y_2}{r(r+y_3)} \right\}, \\ G_{33}(y) = \frac{1}{4\pi\mu_0(\lambda_0+\mu_0)} \left\{ \frac{y_3^2(\lambda_0+\mu_0)}{r^3} + \frac{(\lambda_0+2\mu_0)}{r} \right\}, \end{cases} \tag{11}$$

where $r = |y| = \left(y_1^2 + y_2^2 + y_3^2\right)^{\frac{1}{2}}$.

Observe that the solution of problem (8) posed in the half-space $\mathbb{R}^3$:

$$\mathbb{R}^{3-} = \left\{y = (y_1, y_2, y_3) \in \mathbb{R}^3;\ y_3 < 0\right\},$$

can be also expressed in terms of formula (9) with the Green symmetric tensor (11) for the half-space $y_3 < 0$. We still denote $w^m(y); m = 1,2,3$, this solution. Now, since

$$\int_{\mathbb{R}^{3\pm}} \sigma_{ij}\left(w^m\right) e_{ij}\left(w^l\right) dy = \delta_{ml} \int_{D(0,1)} q_m\left(y_1, y_2\right) dy_1 dy_2,$$

we deduce, using (10), the following formula

$$\int_{\mathbb{R}^{3\pm}} \sigma_{ij}\left(w^m\right) e_{ij}\left(w^l\right) dy = \begin{cases} \frac{6\mu_0}{3+\kappa_0} & \text{if } m,l = 1,2, \\ 0 & \text{if } m \neq l, \\ \frac{4\mu_0}{1+\kappa_0} & \text{if } m = l = 3, \end{cases} \tag{12}$$

where $\kappa_0 = \frac{\lambda_0 + 3\mu_0}{\lambda_0 + \mu_0}$.

2.1.2. Second step: test-functions

Let $h \in \mathbb{N}^*; k \in \{1, 2, \ldots, N(h)\}$, and $j \in \{1, \ldots, n_k(h)\}$. We define the rotation matrix $R\left(x^*_{j,k}\right) = \left(\alpha_{ml}\left(x^*_{j,k}\right)\right)_{m,l=1,2,3}$ through

$$\begin{cases} \alpha_{11} = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos\theta_3, & \alpha_{23} = \cos\theta_1 \sin\theta_3, \\ \alpha_{12} = \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2 \cos\theta_3, & \alpha_{31} = \sin\theta_2 \sin\theta_3, \\ \alpha_{13} = \sin\theta_1 \sin\theta_3, & \alpha_{32} = -\cos\theta_2 \sin\theta_3, \\ \alpha_{21} = -\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2 \cos\theta_3, & \alpha_{33} = \cos\theta_3, \\ \alpha_{22} = -\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \cos\theta_3, & \end{cases} \tag{13}$$

where $\theta_m = \theta_m\left(x^*_{j,k}\right)$; $m = 1,2,3$, are the Euler angles which define a new system of coordinates of origin the point $x^*_{j,k}$. The orientation is preserved by the transformation $R\left(x^*_{j,k}\right)$.

Let $s_h = (r_h)^\epsilon; \epsilon \in \left(\frac{d+1}{2d}, 1\right)$, and the unit normal $\nu^h_{j,k}$ on the contact zone $Z_{j,k}(h)$, with

$$\nu^h_{j,k} = R\left(x^*_{j,k}\right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta_1\left(x^*_{j,k}\right) \sin\theta_3\left(x^*_{j,k}\right) \\ \cos\theta_1\left(x^*_{j,k}\right) \sin\theta_3\left(x^*_{j,k}\right) \\ \cos\theta_3\left(x^*_{j,k}\right) \end{pmatrix}.$$

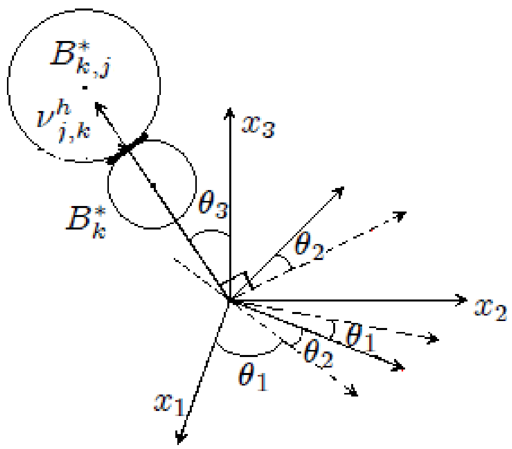

Figure 1: Local system of coordinates of origin the point $x^*_{k,j}$

We define

$$\begin{aligned} &B^+\left(x^*_{j,k}, s_h\right) = B\left(x^*_{j,k}, s_h\right) \cap \left\{x \cdot \nu^h_{j,k} > 0\right\}, \\ &B^-\left(x^*_{j,k}, s_h\right) = B\left(x^*_{j,k}, s_h\right) \cap \left\{x \cdot \nu^h_{j,k} < 0\right\}, \\ &B^\pm_h = \bigcup_{k=1}^{N(h)} \bigcup_{j=1}^{n_k(h)} B^\pm\left(x^*_{j,k}, s_h\right), \\ &B_h = B^+_h \cup B^-_h, \end{aligned} \tag{14}$$

where $B\left(x^*_{j,k}, s_h\right)$ is the ball of radius S_h centred at $x^*_{j,k}$. We define the truncation function $\varphi^h_{j,k}; h \in \mathbb{N}^*, k \in \{1, 2, \ldots, N(h)\}, j \in \{1, \ldots, n_k(h)\}$, through

$$\varphi^h_{j,k}(x) = \begin{cases} 1 & \text{in } B\left(x^*_{j,k}, \frac{s_h}{2}\right), \\ 0 & \text{in } \Omega \backslash B\left(x^*_{j,k}, s_h\right). \end{cases} \tag{15}$$

We build the local functions $w^m_h, m = 1, 2, 3$, defined on $B\left(x^*_{j,k}, s_h\right)$; $h \in \mathbb{N}^*$, $k \in \{1, 2, \ldots, N(h)\}, j \in \{1, \ldots, n_k(h)\}$, through

$$w^m_h(x) = \varphi^h_{j,k}(x) R^{-1}\left(x^*_{j,k}\right)\left(w^m_h\left(\frac{R^{-1}\left(x^*_{j,k}\right)\left(x - x^*_{j,k}\right)}{r_h}\right) - e^m\right). \tag{16}$$

.

Proposition 4. *Let $\Phi \in \mathbf{C}_c^1\left(\Omega, \mathbb{R}^3\right)$. If $r_h = c\rho_h^d$, c being a positive constant, then*

$$\lim_{h\to\infty} \int_{\Omega_h} \sigma_{ij}\left(w_h^m \Phi_m\right) e_{ij}\left(w_h^l \Phi_l\right) dx = \frac{2c}{\mathcal{H}(\Lambda)} \int_\Sigma A(x)\Phi(x).\Phi(x) d\mathcal{H}^d(x),$$

where

$$A(x) = R^{-1}(x) \begin{pmatrix} \frac{3\mu_0}{3+\kappa_0} & 0 & 0 \\ 0 & \frac{3\mu_0}{3+\kappa_0} & 0 \\ 0 & 0 & \frac{2\mu_0}{1+\kappa_0} \end{pmatrix} R(x), \tag{17}$$

$R(x)$ being the rotation matrix defined in (13) but here with angles $\theta_m(x)$; $m = 1, 2, 3, x \in \Sigma$.

Proof. Introducing the change of variables $y = \frac{R^{-1}\left(x_{j,k}^*\right)\left(x - x_{j,k}^*\right)}{r_h}$, one can check, using the definition of w_h^m, that

$$\begin{aligned} &\int_{\Omega_h} \sigma_{ij}\left(w_h^m \Phi_m\right) e_{ij}\left(w_h^l \Phi_l\right) dx \\ &= \sum_{k=1}^{N(h)} \sum_{l=1}^{n_k(h)} \int_{B\left(x_{\iota,k}^h, s_h\right)} \sigma_{ij}\left(w_h^m \Phi_m\right) e_{ij}\left(w_h^l \Phi_l\right) dx \\ &= A_h + B_h + O\left(s_h^d\right) \end{aligned} \tag{18}$$

where

$$A_h = \sum_{k=1}^{N(h)} \sum_{\iota=1}^{n_k(h)} r_h \left\{ \begin{array}{c} \left(R\left(x_{\iota,k}^*\right)\Phi\left(x_{\iota,k}^*\right)\right)_m \left(R\left(x_{\iota,k}^*\right)\Phi\left(x_{\iota,k}^*\right)\right)_l \\ \times \int_{R^{-1}\left(x_{\omega,k}^*\right)B^+\left(x_{\omega k}^*, \frac{s_h}{r_r}\right)} \sigma_{ij}\left(w^m\right) e_{ij}\left(w^l\right) dy \end{array} \right\},$$

$$B_h = \sum_{k=1}^{N(h)} \sum_{\iota=1}^{n_k(h)\, k=1} r_h \left\{ \begin{array}{c} \left(R\left(x_{\iota,k}^*\right)\Phi\left(x_{\iota,k}^*\right)\right)_m \left(R\left(x_{\iota,k}^*\right)\Phi\left(x_{\iota,k}^*\right)\right)_l \\ \times \int_{R^{-1}\left(x_{\iota,k}^*\right)B^-\left(x_{\iota,k}^*, \frac{s_h}{r_h}\right)} \sigma_{ij}\left(w^m\right) e_{ij}\left(w^l\right) dy \end{array} \right\}.$$

According to (12), we have

$$\int_{R^{-1}\left(x_{\iota,k}^*\right)B^\pm\left(x_{\iota,k}^*, \frac{s_h}{r_h}\right)} \sigma_{ij}\left(w^m\right) e_{ij}\left(w^l\right) dy \underset{h\to\infty}{\sim} \left\{ \begin{array}{ll} \frac{6\mu_0}{3+\kappa_0} & \text{if } m, l = 1, 2, \\ 0 & \text{if } m \neq l, \\ \frac{4\mu_0}{1+\kappa_0} & \text{if } m = l = 3. \end{array} \right.$$

Since Λ is invariant under some iterative inversions (see for instance [8]), each point of tangency $x_{\iota,n}$ between the touching balls B_k and $B_{\iota,k}$; $k \in$

$\{1,2,\ldots,N(h)\}, \iota \in \{1,\ldots,n_k(h)\}$, is an iterate of some fixed tangent point $x \in \Lambda$. Hence, as $N(h) \sim \mathrm{s}_{h\to\infty}\, \rho_h^{-d}$, (see [8], [9], [10] and [11]), and $r_h = c\rho_h^d$, we have, according to [18, Theorem 6.1],

$$\lim_{h\to\infty} \sum_{k=1}^{N(h)} \sum_{\iota=1}^{n_k(h)} c\rho_h^d \left\{ \left(R\left(x^*_{\iota,k}\right)\Phi\left(x^*_{\iota,k}\right)\right)_m \left(R\left(x^*_{\iota,k}\right)\Phi\left(x^*_{\iota,k}\right)\right)_l \right\}$$

$$= \lim_{h\to\infty} \sum_{k=1}^{N(h)n_k(h)} \sum_{\iota=1}^{c} \frac{c}{N(h)} \left\{ \left(R\left(x_{\iota,k}\right)\Phi\left(x_{\iota,k}\right)\right)_m \left(R\left(x_{\iota,k}\right)\Phi\left(x_{\iota,k}\right)\right)_l \right\}$$

$$= \frac{c}{\mathcal{H}^d(\Lambda)} \int_\Sigma (R(x)\Phi(x))_m (R(x)\Phi(x))_l d\mathcal{H}^d(x).$$

We thus obtain

$$\lim_{h\to\infty} A_h = \lim_{h\to\infty} B_h = \frac{2c}{\mathcal{H}(\Lambda)} \int_\Sigma A(x)\Phi(x).\Phi(x) d\mathcal{H}^d(x)$$

and, using (18),

$$\lim_{h\to\infty} \int_{\Omega_h} \sigma_{ij}\left(w_h^m \Phi_m\right) e_{ij}\left(w_h^l \Phi_l\right) dx = \frac{2c}{\mathcal{H}(\Lambda)} \int_\Sigma A(x)\Phi(x).\Phi(x) d\mathcal{H}^d(x).$$

□

2.1.3. Third step: Γ-convergence

Let $u \in \bigcap_{k\in\mathbb{N}} \mathbf{C}^1\left(\overline{B_k}, \mathbb{R}^3\right), u = 0$ on $\partial\Omega$. We denote by $r_\Sigma\left([u]_\Sigma\right)$ a image of $[u]_\Sigma$ through the continuous map r_Σ from $\boldsymbol{B}^2_\alpha\left(\Sigma, \mathbb{R}^3\right)$ into $\mathbf{H}^1\left(\Omega\backslash\Sigma, \mathbb{R}^3\right)$. We define the sequence $\left(u_0^h\right)_h$ as follows:

$$u_0^h = \begin{cases} u\left(1 - \varphi^h_{j,k}\right) + w_h^m r_\Sigma\left(\frac{1}{2}\left[u_m\right]_\Sigma\right) & \text{in } B^+\left(x^h_{j,k}, s_h\right) \cap \Omega, \\ u\left(1 - \varphi^h_{j,k}\right) + w_h^m r_\Sigma\left(-\frac{1}{2}\left[u_m\right]_\Sigma\right) & \text{in } B^-\left(x^h_{j,k}, s_h\right) \cap \Omega, \\ u & \text{in } \Omega\backslash B_h, \end{cases} \tag{19}$$

where $B^\pm\left(x^*_{j,k}, s_h\right)$, B_h are defined in (14), and $\varphi^h_{j,k}$ the test-function w_h^m are, respectively, defined in (15) and (16).

Then, using the same method as in [1] , we conclude that the sequence $\left(F_{2,h}\right)_h$ Γ-converges, with respect to the topology τ_2 to the functional $F_{2,c}$ defined in (2).

3. Conclusion

Our asymptotic analysis showed that the contact on the fractal interface between the granular materials can be modeled by a 3-dimensional macroscopic contact law which describes the functional energy of field studied. The results obtained can be interpreted as part of the process of erosion of a gouge during the nucleation phase of an earthquake where the concentration of stresses on the interface inside the gouge exceeds its capacities causing a rupture or even a generation of new fractures. Also, the results here has other implications, such as the modeling of the behavior of composite materials or the study of certain industrial processes (see for example [20] and [21]).

Concerning a next study on the problem, we hope to extend it to the propagation of elastic waves in an Apollonian stack of discs or spheres with for essential motivation applications to seismic.

References

[1] Y. Abouelhanoune, M. El Jarroudi, Interfacial contact model in a dense network of elastic materials, *Functional Analysis and Its Applications*, **55** (2021), 1–14.

[2] D.R. Adams, N.G. Meyers, Bessel potentials. Inclusion relations among classes of exceptional sets, *Indiana Univ. Math. Mech.*, **22**, No 9 (1973), 873-900.

[3] S.V. Anishchik, N.N. Medvedev, Three-dimensional Apollonian packing as a model for dense granular systems, *Phys. Rev. Lett.*, **75** (1995), 4314-4317.

[4] H. Attouch, *Variational Convergence for Functions and Operators*, Appl. Math. Ser., Pitman, London (1984).

[5] R.M. Baram, H.J. Herrmann, N. Rivier, Space-filling bearings in three dimensions, *Phys. Rev. Lett.*, **92**, No 4 (2004), Art. 044301.

[6] R.M. Baram, H.J. Hermann, Self-similar space-filling packings in three dimensions fractals, *Phys. Rev. Lett.*, **12**, No 3 (2004), 293-301.

[7] Y. Ben-Zion, C.G. Sammis, Characterization of fault zones, *Pure and Applied Geophysics*, **160** (2003), 677-715.

[8] M. Borkovec,W. De Paris, R. Peikert, The fractal dimension of the Apollonian sphere packing, *Fractals*, **2**, No 4 (1994), 521-526.

[9] D.W. Boyd, The residual set dimension of Apollonian packing, *Mathematika*, **20** (1973), 170-174.

[10] D.W. Boyd, An algorithm for generating the sphere coordinates in a three-dimensional osculatory packing, *Math. Comp*, **27**, No 122 (1973), 369-377.

[11] D.W. Boyd, The sequence of radii of the Apollonian packing, *Math. Comp.*, **39**, No 159 (1982), 249-254.

[12] A. Brillard, M. El Jarroudi, Fractal relaxed problems in elasticity, *Integral Methods in Science and Engineering*, **1** (2010), 75-84.

[13] R. Capitanelli, M.R. Lancia, M.A. Vivaldi, Insulating layers of fractal type, *Differ. Integ. Equs*, **26**, N 9/10 (2013), 1055-1076.

[14] T. Carleman, Sur la rèsolution de certaines équations intègrales, *Arkiv Math., Astron., Fysik*, **16** (1922).

[15] G. Dal Maso, *An Introduction to* Γ*-Convergence*, Ser. PNLDEA 8, Springer (1993).

[16] M. El Jarroudi, Asymptotic analysis of contact problems between an elastic material and thin-rigid plates, *Appl. Anal.*, **89**, No 5 (2010), 693-715.

[17] M. El Jarroudi, A. Brillard, Asymptotic behaviour of contact problems between two elastic materials through a fractal interface, *J. Math. Pures Appl.*, **89** (2008), 505-521.

[18] K. Falconer, *Techniques in Fractal Geometry*, J. Wiley and Sons, Chichester (1997).

[19] H.J. Herrmann, G. Mantica, D. Bessis, Space-filling bearings, *Phys. Rev. Lett*, **65**, No 26 (1990), 3223-3226.

[20] H. Hertz, On the contact of elastic solids, *J. Rein. Angew. Math.*, **92** (1882), 156-171.

[21] P.W. Jones, Quasiconformal mappings and extendability of functions in Sobolev spaces, *Acta Math.*, **147** (1981), 71-88.

[22] A. Jonsson, H. Wallin, Boundary value problems and Brownian motion on fractals, *Chaos Solitons Frac.*, **8**, No 2 (1997), 191-205.

[23] J.C. Kim, K.H. Auh, D.M. Martin, Multi-level particle packing model of ceramic agglomerates, *Model. Simul. Mater. Sci. Eng.*, **8** (2000), 159-168.

[24] V.A. Kondratiev, O.A. Oleinik, On Kornís inequalities, *C.R. Acad. Sci. Paris, Ser. I*, **308** (1989), 483-487.

[25] M.R. Lancia, A transmission problem with a fractal interface, *Z. Anal. Anwend.*, **21** (2002), 113-133.

[26] L. Landau, E. Lifchitz, *Thèorie de l'élasticité*, Editions Mir, Moscow (1967).

[27] C. Marone, C.B. Raleigh, C.H. Scholz, Frictional behavior and constitutive modeling of simulated fault gouge, *J. Geoph. Res.*, **95**, No B5 (1990), 7007-7025.

[28] R.D. Mauldin, M. Urbanski, Dimension and measures for a curvilinear Sierpinski gasket or Apollonian packing, *Adv. Math.*, **136** (1998), 26-38.

[29] N.G. Meyers, Integral inequalities of Poincaré and Wirtinger type, *Arch. Rat. Mech. Analysis*, **68** (1978), 113-120.

[30] U. Mosco, M.A. Vivaldi, Fractal reinforcement of elastic membranes. *Arch. Rat. Mech. Analysis*, **194**, (2009), 49-74.

[31] U. Mosco, M.A. Vivaldi, Thin fractal Obers, *Math. Meth. Appl. Sci.*, **36** (2013), 2048-2068.

[32] N.I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Groningen (1963).

[33] S. Roux, A. Hansen, H.J. Herrmann, A model for gouge deformation: implication for remanent magnetization, *Geoph. Res. Lett.*, **20**, No 14 (1993), 1499-1502.

[34] V.S. Rychkov, Linear extension operators for restrictions of function spaces to irregular open sets, *Stud. Math.*, **140**, No 2 (2000), 141-162.

[35] C.G. Sammis, R.L. Biegel, Fractals, fault-gouge, and friction, *Pure and Applied Geophysics*, **131**, Nos 1/2 (1989), 255-271.

[36] C.H. Scholz, *The Mechanics of Earthquakes and Faulting*, Cambridge University Press, London (2002).

[37] S.J Steacy, C.G. Sammis, An automaton for fractal patterns of fragmentation, *Nature*, **353** (1991), 250-252.

[38] D. Sullivan, Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinan groups, *Acta. Math.*, **153** (1984), 259-277.

[39] D.M. Walker, A.Tordesillas, Topological evolution in dense granular materials: A complex networks perspective, *Int. J. Solids Str.*, **47** (2010), 624-639.