

COMPARISON OF APPORTIONMENT METHODS IN BULGARIAN ELECTIONS

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Abstract: We compare two apportionment methods used in Bulgarian electoral practice in 12 parliamentary elections in the period 1991-2021: the D'Hondt method and the Hare-Niemeyer method. Extreme properties of D'Hondt method are also analyzed for the Bulgarian electoral system consisting of 240 parliamentary seats and 4% barrier. The effect of rounding errors on automatic voting computations is also demonstrated.

AMS Subject Classification: 91B12

Key Words: apportionment methods, Bulgarian parliamentary elections

1. D'Hondt vs. Hare-Niemeyer

Elections in Bulgaria

After the Liberation from Ottoman Empire (1879) elections in Bulgaria had been carried out by proportional, plurality and mixed type electoral systems. The proportional elections up to 2007 had been governed by the D'Hondt method (DM) known in USA as the Jefferson method. After 2007 the Hare-Niemeyer method (HM) has been used. This method is known in USA as the Hamilton method.

The Bulgarian electoral system after 1991 is proportional with one exception in 2009 when 209 MP-s had been elected by a proportional system and 31 MP-s by a plurality system. The standard DM and HM had been used. The present electoral system is characterized by the number $S = 240$ of MP-s and the barrier

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$b = 0.04$. It is a bi-proportional system based on HM. The transfer to HM was due to the regional disproportion effects that occur in 2005 elections [2].

Notations

We use the following notations: $\mathbb{N}$ – the set of positive integers; $\mathbb{N}_0$ – the set of non-negative integers; $\mathbb{R}$ – the set of real numbers; $\mathbb{R}_0$ – the set of non-negative real numbers; A^n – the Cartesian product of n copies of the set A ; $\text{sum}(\mathbf{x}) = x_1 + x_2 + \dots + x_n$ – the sum of the elements of the vector $\mathbf{x} = [x_1, x_2, \dots, x_n] \in \mathbb{R}^n$; $|\mathbf{x}| = [|x_1|, |x_2|, \dots, |x_n|] \in \mathbb{R}_0^n$ – the absolute value of the vector $\mathbf{x}$; $\|\mathbf{x}\|_1 = \text{sum}(|\mathbf{x}|)$ – the 1-norm of $\mathbf{x}$; $\|\mathbf{x}\|_\infty = \max\{|x_i| : i = 1, 2, \dots, n\}$ – the infinity-norm of $\mathbf{x}$; $\mathbf{1}_n = [1, 1, \dots, 1] \in \mathbb{N}^n$; $\text{fix}(A)$ – the integer rounded value of the array A towards zero, e.g. $\text{fix}(\pi) = 3$; $\text{flo}(A)$ – the machine rounded value of the array A ; eps – the machine epsilon; $\preceq$ – the partial element-wise order relation in $\mathbb{R}^n$.

Proportional electoral systems

Suppose that n parties $P_1, P_2, \dots, P_n$ with votes $v_1, v_2, \dots, v_n$ pass the barrier $b \in (0, 1)$. In particular this means that $v_i \geq bV$, where $V = \text{sum}(\mathbf{v})$ and $\mathbf{v} = [v_1, v_2, \dots, v_n] \in \mathbb{N}^n$ is the vector of votes. Each proportional electoral system (PES) has a random choice mechanism in case when e.g. two parties with equal votes compete for a single seat. We suppose that the parties are preliminary ordered by ties so that the seat is given to the party with less index. A PES Π is a rule which transforms the pair $(S, \mathbf{v}) \in \mathbb{N} \times \mathbb{N}^n$ into the vector

$$\mathbf{s} = [s_1, s_2, \dots, s_n] = \Pi(S, \mathbf{v}) \in \mathbb{N}_0^n, \text{sum}(\mathbf{s}) = S$$

satisfying the approximate equalities

$$\frac{v_i}{s_i} \simeq \frac{V}{S} \quad (i = 1, 2, \dots, n)$$

and such that $\Pi(S, m\mathbf{v}) = \Pi(S, \mathbf{v})$ for any $m \in \mathbb{N}$. The function Π may be extended to $\mathbb{N} \times \mathbb{R}_+^n$, i.e. $\mathbf{v}$ may not be an integer vector.

There are three desirable properties that a PES should have [1]. The first one is the monotonicity property

$$(v_i - v_j)(s_i - s_j) \geq 0 \quad (i, j = 1, 2, \dots, n).$$

The second property is $\Pi(S, \mathbf{v}) \preceq \Pi(S + 1, \mathbf{v})$. The violation of this property is known as Alabama paradox (AP). Higher order AP are considered in [3].

Denote by $\mathbf{f} = [f_1, f_2, \dots, f_n] = S\mathbf{v}/V$ the vector of fractional seats. Then the third property, or the Quota rule, is $\text{fix}(\mathbf{f}) \preceq \mathbf{s} \preceq \text{fix}(\mathbf{f}) + \mathbf{1}_n$.

PES may be defined in terms of extremal problems, e.g. the DM corresponds to the min-max problem

$$J_D = \min \left\{ \frac{v_i}{s_i} : i = 1, 2, \dots, n \right\} \rightarrow \max.$$

In turn, the HM corresponds to the minimization problem

$$J_H = \|\mathbf{s} - \mathbf{f}\|_1 \rightarrow \min.$$

Denote by $\mathbf{s}_D = \Pi_D(S, \mathbf{v})$, $\mathbf{s}_H = \Pi_H(S, \mathbf{v}) \in \mathbb{N}^n$ the vectors of seats produced by the DM and HM, respectively. Both methods satisfy the monotonicity property. The DM does not admit the Alabama paradox but may violate the Quota rule. The HM admits the Alabama paradox but satisfy the Quota rule.

The D'Hondt method

Algorithmically the DM is realized as follows. The parties are ordered so that $v_1 \geq v_2 \geq \dots \geq v_n$ and the quantities $d_{i,j} = v_j/i$ are computed for $i = 1, 2, \dots, S$ and $j = 1, 2, \dots, n$. These quantities are ordered as $d_{i_1, j_1} \geq d_{i_2, j_2} \geq \dots \geq d_{i_S, j_S}$ and marked. If there are two equal quantities $d_{i,j} = d_{i,k}$ with $j < k$ then $d_{i,j}$ is ordered before $d_{i,k}$. If there are two equal quantities $d_{i,k} = d_{j,k}$ with $i < j$ then $d_{i,k}$ is ordered before $d_{j,k}$. Each party P_j obtains s_j seats, where s_j is the number of marked quantities $d_{i,j}$.

The Hare-Niemeyer method

The HM acts as follows. Denote by $\mathbf{r} = \mathbf{f} - \text{fix}(\mathbf{f})$ the vector of fractional remainders. The parties are numbered so that $r_1 \geq r_2 \geq \dots \geq r_n$. If there are parties with equal fractional remainders then the ordering is done by ties. Each party P_i obtains $\text{fix}(f_i)$ seats. If $\mathbf{r} = \mathbf{0}$ then the process is over. If $\text{sum}(\mathbf{r}) = r \geq 1$ then there are $r < n$ more seats to be distributed. They are given one by one to parties $P_1, P_2, \dots, P_r$, i.e. $s_j = \text{fix}(f_j) + 1$ for $j = 1, 2, \dots, r$.

2. Alabama free method

The AP is a disadvantage when HM is used to determine the number of members of Congress from different American states proportionally to the population but

it is not a problem in case of proportional elections. Anyway, PES that obey the Quota rule and avoid AP had been proposed. Such a PES with function $\Pi(M, \mathbf{v})$ is presented below. Note an AP may occur only if $n \geq 3$ and $S \geq 3$.

Let the pair $(S, \mathbf{v})$ be given. We shall construct a sequence of seat distributions $\mathbf{s}^{(M)} = [s_1^{(M)}, s_2^{(M)}, \dots, s_n^{(M)}] = \Pi(M, \mathbf{v}) \in \mathbb{N}^n$ such that $\text{sum}(\mathbf{s}^{(M)}) = M$. The new method satisfies the Quota rule for all $M \geq 3$ and avoids AP. It is described as follows. Denoting $\mathbf{h}^{(M)} = [h_1^{(M)}, h_2^{(M)}, \dots, h_n^{(M)}] = \Pi_H(M, \mathbf{v})$ we have the following two possibilities:

1. The inequality $\mathbf{h}^{(M)} \preceq \mathbf{h}^{(M+1)}$ is fulfilled for $M = 3, 4, \dots, S$. We set $\mathbf{s}^{(M)} = \mathbf{h}^{(M)}$ and $\Pi = \Pi_H$.
2. There exists a smallest integer $M < S$ such that the inequality $\mathbf{h}^{(M)} \preceq \mathbf{h}^{(M+1)}$ is violated. We set $\Pi(K, \mathbf{v}) = \Pi_H(K, \mathbf{v})$ for $K \leq M$. Denote $\mathbf{f} = M\mathbf{v}/V$ and $\mathbf{r} = [r_1, r_2, \dots, r_n] = \mathbf{f} - \text{fix}(\mathbf{f})$.

In case 2 the set $\mathcal{N} = \{1, 2, \dots, n\}$ is represented as the union of two sets $\mathcal{J}$ and $\mathcal{N} \setminus \mathcal{J}$, such that $h_i^{(M+1)} \geq h_i^{(M)}$ for $i \in \mathcal{J}$ and $h_i^{(M+1)} = h_i^{(M)} - 1$ for $i \in \mathcal{N} \setminus \mathcal{J}$. Let $r_k = \max\{r_j : j \in \mathcal{J}\}$, where if necessary the index k may be determined by ties. Now the distribution

$$\Pi(M+1, \mathbf{v}) = [s_1^{(M+1)}, s_2^{(M+1)}, \dots, s_n^{(M+1)}]$$

is given by $s_i^{(M+1)} = h_i^{(M)}$ for $i \in \mathcal{N} \setminus \{k\}$ and $s_k^{(M+1)} = h_k^{(M)} + 1$. Next we set $\Pi(K, \mathbf{v}) = \Pi_H(K, \mathbf{v})$ for $K \geq M+2$ until the next AP occurs (if any).

We stress that the vector $\Pi(M+1, \mathbf{v})$ differs from the vector $\Pi(M, \mathbf{v})$ only in the element with index k .

Let for example $\mathbf{v} = [35, 133, 132]$. Then $\Pi_H(3, \mathbf{v}) = [1, 1, 1]$, $\Pi_H(4, \mathbf{v}) = [0, 2, 2]$ and AP occurs for the pair $(3, \mathbf{v})$. We have $r = [0.35, 0.33, 0.32]$, $\mathcal{J} = \{2, 3\}$ and $k = 2$, $r_2 = 0.33$. Hence $\Pi(4, \mathbf{v}) = [1, 2, 1]$. Further on $\Pi(M, \mathbf{v}) = \Pi_H(M, \mathbf{v})$ for $M = 5, 6, \dots, 12$ and $\Pi_H(12, \mathbf{v}) = [2, 5, 5]$, $\Pi_H(13, \mathbf{v}) = [1, 6, 6]$ is the second AP. Here again $\mathcal{J} = \{2, 3\}$ and we obtain $\Pi(13, \mathbf{v}) = [2, 6, 5]$. AP for $M = 3 + 9p$, $p \geq 2$, are treated similarly.

3. Impact of rounding errors

The pair $(S, \mathbf{v})$ is an extreme vote distribution relative to the Quota rule if the quantity $\|\Pi_H(S, \mathbf{v}) - \Pi_D(S, \mathbf{v})\|_\infty$ is maximum. It may be shown that the pair

$(240, \mathbf{v}^0)$, where $\mathbf{v}^0 = [16, \mathbf{1}_9] \in \mathbb{N}^{10}$, is extreme. We have $\mathbf{d}^0 = \Pi_D(240, \mathbf{v}^0) = [159, 9 \times \mathbf{1}_9]$, $\mathbf{h}^0 = \Pi_H(240, \mathbf{v}^0) = [154, 10 \times \mathbf{1}_5, 9 \times \mathbf{1}_4]$ and $\|\mathbf{d}^0 - \mathbf{h}^0\|_\infty = 5$. Other extreme distributions are considered in [3].

Numerical computations in binary double precision machine arithmetic are usually done with relative errors of order less than 10^{-15} . In turn, voting calculations include data of order less than 10^8 . So it should be expected that rounding has no effect on voting calculations. This expectation is wrong as the computational practice of the authors had shown. The reason for this phenomenon soon became clear and is demonstrated below.

Let $\mathbf{x} = [x_1, x_2, \dots, x_n] \in \mathbb{R}^n$ and $\mathbf{max}: \mathbb{R}^n \rightarrow \mathbb{R} \times \mathbb{N}$ be a function such that $\mathbf{max}(\mathbf{x}) = [\mathbf{x}_{\max}, p]$, where $\mathbf{x}_{\max} = x_p$ is the maximum element of $\mathbf{x}$. If there are more than one maximum elements of $\mathbf{x}$ then the code $\mathbf{max}$ returns the index p of the left most maximum element.

Consider the machine solution of the problem with data $(240, \mathbf{v}^0)$. Using the code $\mathbf{max}$ from MATLAB for computing $\mathbf{h}^0$ in double precision floating-point binary arithmetic with $\text{eps} = 2^{-52} \simeq 2.2204 \times 10^{-16}$ we get $\text{flo}(\mathbf{h}^0) = [153, 10 \times \mathbf{1}_6, 9 \times \mathbf{1}_3]$. Thus two elements of $\mathbf{h}^0$, namely the first and the seventh, are computed wrongly. The reason is that the vector of fractional seats $\mathbf{f} = [153.6, 9.6 \times \mathbf{1}_9]$ is rounded to $\text{flo}(\mathbf{f}) = \mathbf{f} - 1.6 \text{eps} [16, \mathbf{1}_9]$. Here the exact value of all fractional remainders r_i is 0.6 but the rounded value of r_1 is less than the rounded value of the remainders $r_2, r_3, \dots, r_{10}$. Thus the computed value of the first element of $\mathbf{h}^0$ is 153 instead of 154.

To avoid the effect of rounding errors on voting calculations one should work with the vector $V\mathbf{r} = S\mathbf{v} - V\text{fix}(\mathbf{f})$ of integer remainders instead with the vector $\mathbf{r}$ of fractional remainders.

4. Comparison of DM and HM

In Tables 1–12 we give results from the implementation of HM and DM to the data from Bulgarian elections for 12 Parliaments since 1991. The method actually used is marked by *.

In the elections for 40th National Assembly only 209 MP-s had been elected by HM (see Table 6) while the rest 31 MP-s had been elected by a plurality system with relative majority.

A total of $2,849 = 11 \times 240 + 209$ MP-s had been elected by PES in Bulgaria since 1991. The difference of number of MP-s elected by both apportionment methods is 18, or 0.6%.

Party	Votes	%	HM	DM*
UDF	1,903,567	34.4	110	110
BSP	1,836,050	33.1	106	106
MRF	418,168	7.6	24	24

Table 1: 36 National Assembly (13.10.1991)

Party	Votes	%	HM	DM*
BSP	2,258,212	43.4	124	125
UDF	1,254,465	24.1	69	69
PUn	338,427	6.5	19	18
MRF	282,711	5.4	15	15
BBB	245,951	4.7	13	13

Table 2: 37 National Assembly (18.12.1994)

Party	Votes	%	HM	DM*
UDF	2,258,212	43.4	124	125
BSP	1,254,465	24.1	69	69
PUn	338,427	6.5	19	18
MRF	282,711	5.4	15	15
BBB	245,951	4.7	13	13

Table 3: 38 National Assembly (19.04.1997)

Party	Votes	%	HM	DM*
NSR	1,952,513	42.7	120	120
UDF	830,338	18.2	51	51
BSP	783,372	17.1	48	48
MRF	340,395	7.4	21	21

Table 4: 39 National Assembly (17.06.2001)

Party	Votes	%	HM	DM*
BSP	1,129,196	30.9	82	82
NMSR	725,338	19.9	52	53
MRF	467,400	12.8	34	34
Att	296,848	8.1	21	21
UDF	280,323	7.7	20	20
DFB	234,788	6.4	17	17
BNU	189,268	5.2	14	13

Table 5: 40 National Assembly (17.06.2005)

Party	Votes	%	HM*	DM
CDP	1,678,641	39.7	90	91
BSP	748,147	17.7	40	40
MRF	610,521	14.4	33	33
Att	395,733	9.4	21	21
BCo	285,662	6.7	15	15
OLJ	174,582	4.1	10	9
Total			209	209

Table 6: 41 National Assembly (05.07.2009)

Party	Votes	%	HM*	DM
CDP	1,081,605	39.7	97	97
BSP	942,541	17.7	84	84
MRF	400,466	14.4	36	36
Att	258,481	9.4	23	23

Table 7: 42 National Assembly (12.05.2013)

Party	Votes	%	HM*	DM
CDP	1,072,491	32.7	84	85
BSP	505,527	15.4	39	40
MRF	487,134	14.8	38	38
RBl	291,806	8.9	23	23
PFr	239,101	7.3	19	19
BCe	186,938	5.7	15	14
Att	148,262	4.5	11	11
ABR	136,223	4.1	11	10

Table 8: 43 National Assembly (05.10.2014)

Party	Votes	%	HM*	DM
CDP	1,147,491	33.5	95	96
BSP	955,490	27.9	80	80
UPa	318,513	9.3	27	26
MRF	315,976	9.2	26	26
Wil	145,637	4.3	12	12

Table 9: 44 National Assembly (26.03.2017)

Party	Votes	%	HM*	DM
CDB	837,707	26.2	75	76
TSP	565,014	17.7	51	51
BSP	480,146	15.0	43	43
MRF	336,306	10.5	30	30
DBu	302,280	9.4	27	27
SMa	150,940	4.7	14	13

Table 10: 45 National Assembly (04.04.2021)

Party	Votes	%	HM*	DM
TSP	657,829	24.1	65	65
CDB	642,165	23.5	63	63
BSP	365,695	13.4	36	36
DBu	345,331	12.6	34	34
MRF	292,514	10.7	29	29
SMa	136,885	5.0	13	13

Table 11: 46 National Assembly (11.07.2021)

Party	Votes	%	HM*	DM
CCh	673,170	25.7	67	67
CDB	596,456	22.7	59	60
MRF	341,000	13.0	34	34
BSP	267,817	10.2	26	26
TSP	249,743	9.5	25	25
DBu	166,968	6.4	16	16
Rev	127,568	4.9	13	12

Table 12: 47 National Assembly (14.11.2021)

Abbreviation	Party/Coalition
ABR	Alternative for Bulgarian Revival
Att	Attack
BBB	Bulgarian Business Block
BCe	Bulgaria without Censorship
BCo	Blue Coalition
BNU	Bulgarian National Union
BSP	Bulgarian Socialist Party
CCh	Continuing Changes
CDB	Citizen for European Development of Bulgaria
DBu	Democratic Bulgaria
DFB	Democrats for Free Bulgaria
MRF	Movement for Rights and Freedoms
NSR	National Movement for Stability and Rise
OLJ	Order, Law and Justice
PFr	Patriotic Front
PUn	People's Union
RBl	Reformation Block
Rev	Revival
SMA	Stand up. Mafia out
TSP	There is Such a People
UDF	Union of Democratic Forces

Table 13: Abbreviations of parties and coalitions

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