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ON ZERO DIVISOR GRAPH OF MATRIX RING $M_n(\mathbf{Z}_p)$

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Abstract: Consider $M_n(\mathbf{Z}_p)$, the matrix ring of order n over the field $\mathbf{Z}_p$. In this paper, we deduce a relation to find the number of zero divisors in matrix ring $M_n(\mathbf{Z}_p)$. We prove that zero divisor graph of $M_2(\mathbf{Z}_p)$ is a regular directed graph. We also prove that the diameter of zero divisor graph of $M_n(\mathbf{Z}_p)$ is 2.

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Key Words: zero divisors; zero divisor graph; matrix rings; eccentricity

1. Introduction

The connection between graph theory and ring theory was established by Beck [3] in 1988. For a given commutative ring R, Beck defined a undirected graph $G(R)$ with vertex set R, such that distinct vertices x and y are adjacent if and only if $xy = 0$. By definition, $G(R)$ is simple that is without loops, so the nilpotent elements of the ring are not considered. Also every vertex is adjacent to zero vertex, so $G(R)$ is connected with diameter at most 2. Beck's work was mainly concerned with coloring of graphs.

Anderson and Livingston [2] considered the set of zero divisors of R as the vertex set. Let $Z(R)^*$ denote the set of non-zero zero divisors of a ring R. The zero divisor graph of R is a graph with vertex set $Z(R)^*$, such that distinct vertices x and y are adjoint if $xy = 0$.

Consider $\mathbf{Z}$, the ring of integers and $M_2(\mathbf{Z}_p)$, the ring of square 2×2 matrices, with coefficient in $\mathbf{Z}_p$, p a prime. Let $Z(M_2(\mathbf{Z}_p))^*$ denote the set of non-zero

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zero divisors of $M_2(\mathbf{Z}_p)$, d_u and $\epsilon(u)$ denote the degree and the eccentricity of vertex u. In this paper we will study the zero divisor graph of $M_2(\mathbf{Z}_p)$. In section 2, we recall definitions and results related to zero divisor graphs and matrix rings. In section 3, we give a relation to find the number of edges in the zero divisor graph of $M_2(\mathbf{Z}_p)$.

2. Zero divisor graph

Let R be an arbitrary ring. Let $Z(R)$ and $Z(R)^*$ be the set of zero divisors and set of non zero zero divisors respectively. For a non-commutative ring R, an element $a \in R$ is called a *left zero divisor* if there exist a non-zero x such that $a.x = 0$. Similarly, an element $b \in R$ is called a *right zero divisor* if there exist a non-zero y such that $y.b = 0$. Set of left and right zero divisors of R are denoted by $Z_l(R)$ and $Z_r(R)$ respectively. Redmond [11, 12] extended the definition of zero divisor graph to non-commutative rings. The directed zero divisor graph of R is defined with vertex set $Z(R)^*$ such that $x \to y$ is an edge if and only if $xy = 0$.

Božić and Petrović [4] studied the zero divisor graph of matrix rings. They gave a relation between the diameter of the zero-divisor graph of a commutative ring R and that of the matrix ring $M_n(R)$. Birch, Thibodeaux, Tucci [6] studied the zero divisor graph of finite direct product of rings and gave a relation to find the number of edges in the same.

Now a days, lot of work has been done by many researchers on zero divisor graphs. Zero divisor graphs for different rings has been defined and their properties have been discussed. Warfel [5] studied the zero divisor graph of direct product of two commutative rings. Akbari and Mohammadian [8] have made significant contribution in the study of zero divisor graph of non-commutative ring. Anderson and Badawi [1] defined total graph of commutative ring. The total graph $T(\Gamma(R))$ of a commutative ring R is an undirected graph with R as vertex set and any two elements $x, y \in R$, x is adjacent to y if and only if $x + y$ is a zero divisor of R. For more results on zero divisor graphs, readers may refer to [7], [13], [15], [16], [18].

3. Zero divisor of $M_n(\mathbf{Z}_p)$

In this section, we find the number of non-zero zero divisors and number of edges in the zero divisor graph of matrix ring $M_2(\mathbf{Z}_p)$. First we will find the general solution of equation $ax+by \equiv 0(mod\ p)$. For convenience, we write the solution of $ax+by \equiv 0(mod\ p)$ as (x,y).

Lemma 1. *Let $a,b \in \mathbf{Z}$ and p a prime. The general solution of the equation $ax+by \equiv 0(mod\ p)$, $a \neq 0$ is given by*

$$S = \{(mn(mod\ p), n) \mid m = \frac{-b}{a}, n = 0, 1, \ldots, p-1\}.$$

Proof. Let $a \neq 0$. Then we have two cases:

1. If $b=0$, then equation becomes $ax+0y \equiv 0(mod\ p)$, which has p solutions, namely
$$(0,0),(0,1)\ldots,(0,p-1).$$

2. If $b \neq 0$, then we have

$$\begin{aligned} ax+by &\equiv 0\ (mod\ p) \\ \Rightarrow ax &\equiv -by\ (mod\ p) \\ \Rightarrow x &\equiv \frac{-b}{a}y\ (mod\ p) \\ \Rightarrow x &\equiv my\ (mod\ p),\ where\ m = \frac{-b}{a} \in \mathbf{Z}. \end{aligned}$$

The solution set of the above equation is

$$S=\{(0,0),(m(mod\ p),1),(2m(mod\ p),2),...,((p-1)m(mod\ p),(p-1))\},$$

with $\mid S \mid = p$. From above cases, we conclude that the general solution of $ax+by \equiv 0(mod\ p)$, $a \neq 0$ is

$$S = \{(mn(mod\ p), n) \mid m = \frac{-b}{a}, n = 0, 1, \ldots, p-1\}.$$

□

Let $S_n = \{(a,b) \mid a(mn)+bn \equiv 0(mod\ p)\}, \mid S_n \mid = p$. From the above lemma, we observe that for different values of n, the sets S_n are equal.

Theorem 2. *Let $M_2(\mathbf{Z}_p)$ be as usual. Then number of non-zero right zero divisors (or left zero divisors) of any element is either equal to 0 or p^2-1.*

Proof. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2(\mathbf{Z}_p),\ A \neq 0.$

Case 1: If $A.B \not\equiv 0(mod\ p)$, where $0 \neq B \in M_2(\mathbf{Z}_p)$, then the number of non-zero right zero divisor of A is equal to 0.

Case 2: If $A.B \equiv 0(mod\ p)$, where $0 \neq B \in M_2(\mathbf{Z}_p)$, and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. Then

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (mod\ p)$$

$$\Rightarrow a_{11}b_{11} + a_{12}b_{21} \equiv 0(mod\ p) \quad and \tag{1}$$
$$a_{11}b_{12} + a_{12}b_{22} \equiv 0(mod\ p) \quad and \tag{2}$$
$$a_{21}b_{11} + a_{22}b_{21} \equiv 0(mod\ p) \quad and \tag{3}$$
$$a_{21}b_{12} + a_{22}b_{22} \equiv 0(mod\ p). \tag{4}$$

Let $a_{11} \neq 0$, then from Lemma 1 we have p solutions of equation (1) and equation (2). That is, there are p values of (b_{11}, b_{21}) and (b_{12}, b_{22}) which satisfy equation (1) and equation (2). Also for these solutions of equation (1) and equation (2), we get p choices for (a_{21}, a_{22}) which satisfy equation (3) and equation (4). Excluding the case when $b_{11} = b_{12} = b_{21} = b_{22} = 0$, we get $p^2 - 1$ choices for matrix B, such that $A.B \equiv 0(mod\ p)$. □

Corollary 3. *The zero divisor graph of $M_2(\mathbf{Z}_p)$ is a regular directed graph.*

But this result does not hold for a higher order matrix ring. Here we give an example for $M_3(\mathbf{Z}_2)$:

Example 4. Let $M_3(\mathbf{Z}_2)$ be as usual. Then $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has 8 right zero divisors whereas $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has 16 right zero divisors.

Theorem 5. *Let $Z(M_2(\mathbf{Z}_p))^*$ be the set of non-zero zero divisors of $M_2(\mathbf{Z}_p)$. Then cardinality of $Z(M_2(\mathbf{Z}_p))^*$ is equal to $(p-1)(p+1)^2$.*

Proof. The only zero divisors of $M_2(\mathbf{Z}_p)$ are of the form:

$$S_1 = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \Big| a, b \in \mathbf{Z}_p, \text{ if } a = 0 \text{ then } b \neq 0 \text{ and if } b = 0 \text{ then } a \neq 0 \right\},$$

$$S_2 = \left\{ \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \Big| a, b \in \mathbf{Z}_p, \text{ if } a = 0 \text{ then } b \neq 0 \text{ and if } b = 0 \text{ then } a \neq 0 \right\},$$

$$S_3 = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \Big| a, b \in \mathbf{Z}_p, \ a \neq 0 \text{ and } b \neq 0 \right\},$$

$$S_4 = \left\{ \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \Big| a, b \in \mathbf{Z}_p, \ a \neq 0 \text{ and } b \neq 0 \right\},$$

$$S_5 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big| a, b, c, d \in \mathbf{Z}_p \setminus 0 \right\}.$$

We note the following:

S_i	S_1	S_2	S_3	S_4	S_5
Cardinality of S_i	$p^2 - 1$	$p^2 - 1$	$(p-1)^2$	$(p-1)^2$	$(p-1)^3$

Clearly, the cardinalities of set S_1, S_2, S_3 and S_4 are correct. For S_5, let $a, b \in \mathbf{Z}_p \setminus 0$. From Theorem 5, we see that for every (a, b) we get $p - 1$ choices (excluding (0,0)) for(c,d). So we have $(p-1)^2$ choices for (a, b), and for each choice of (a, b), we have $(p-1)$ choices for (c, d). So, the number of elements in $S_5 = (p-1)^3$.

Therefore, the total number of zero divisors of $M_2(\mathbf{Z}_p)$ is

$$\begin{aligned} & p^2 - 1 + p^2 - 1 + (p-1)^2 + (p-1)^2 + (p-1)^3 \\ & = 2(p^2 - 1) + 2(p-1)^2 + (p-1)^3 \\ & = 2(p-1)(p+1) + (p-1)^2(2 + p - 1) \\ & = 2(P-1)(p+1) + (p-1)^2(p+1) \\ & = (p-1)(p+1)(2 + p - 1) \\ & = (p-1)(p+1)^2. \end{aligned}$$

□

Example 6. Consider $M_2(\mathbf{Z}_2)$. Then

$$S_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\},$$

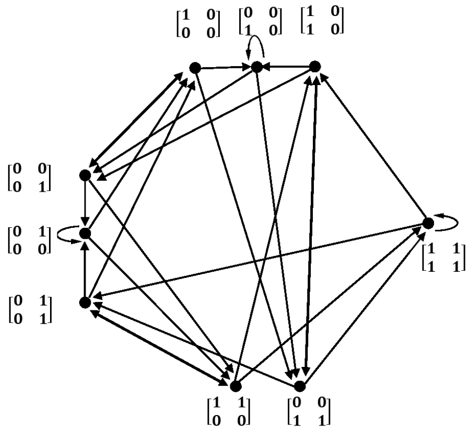

Figure 1

$$S_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\},$$

$$S_3 = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\},$$

$$S_4 = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\},$$

$$S_5 = \phi.$$

So the total number of zero divisors is $9 = (2-1)(2+1)^2$.

Example 7. Consider $M_2(\mathbf{Z}_3)$. Then

$$S_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \right\},$$

$$S_2 = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \right\},$$

$$S_3 = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \right\},$$

$$S_4 = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \right\},$$

$$S_5 = \left\{ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \right\}.$$

Therefore, the total number of zero divisors is $32 = (3-1)^2(3+1)^2$.

We discuss the calculation of the number of edges in $M_2(\mathbf{Z}_p)$ as follows:

Theorem 8. *Let $M_2(\mathbf{Z}_p)$ be as usual. Then zero divisor graph of $M_2(\mathbf{Z}_p)$ has $(p-1)^2(p+1)^3$ edges.*

Proof. From Theorem 5, we know that there are $(p-1)(p+1)^2$ zero divisors in $M_2(\mathbf{Z}_p)$. Also there are p^2-1 right zero divisors (or left zero divisors) of every zero divisor (including idempotent elements), that is each element has (p^2-1) directed edges. Therefore, total number of edges are,

$$\begin{aligned}
&= (p^2-1)(p^2-1) + (p^2-1)(p^2-1) + (p^2-1)(p-1)^2 + \\
&\quad (p^2-1)(p-1)^2 + (p^2-1)(p-1)^3 \\
&= 2(p^2-1)^2 + 2(p^2-1)(p-1)^2 + (p^2-1)(p-1)^3 \\
&= (p^2-1)[2(p^2-1) + 2(p-1)^2 + (p-1)^3] \\
&= (p^2-1)(p-1)[2(p+1) + 2(p-1) + (p-1)^2] \\
&= (p^2-1)(p-1)[4p + p^2 + 1 - 2p] \\
&= (p-1)(p+1)(p-1)(p+1)^2 \\
&= (p-1)^2(p+1)^3.
\end{aligned}$$

□

Recall that general linear group, $GL_n(R)$ of order n is the set of $n \times n$ invertible matrices with entries in R. If R is finite field with p elements, then

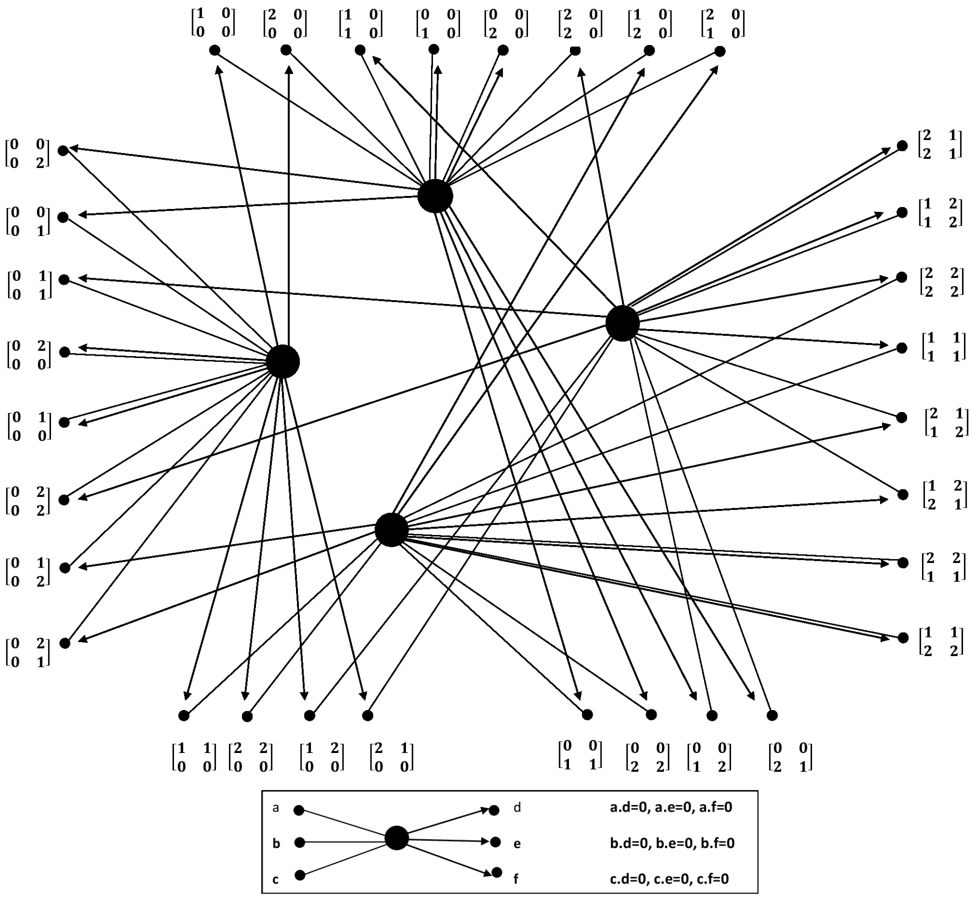

Figure 2: Zero divisor graph of $M_2(\mathbf{Z}_3)$

order of $GL_n(p)$ is given by

$$\prod_{i=0}^{n-1}(p^n - p^k) \quad = \quad (p^n - 1)(p^n - p)(p^n - p^2)\dots(p^n - p^{n-1}). \tag{5}$$

Now we give a relation to find the number of zero divisors of $M_n(\mathbf{Z}_p)$.

Theorem 9. *Number of zero divisors of $M_n(\mathbf{Z}_p)$ is given by $p^{n^2} - \prod_{i=0}^{n-1}(p^n - p^k)$.*

Proof. The number of elements in ring $M_n(\mathbf{Z}_p)$ is equal to p^{n^2}. Also, a non zero element of a ring is either a zero divisor or a unit. From equation (5), number of unit in $M_n(\mathbf{Z}_p)$ is equals to $\prod_{i=0}^{n-1}(p^n - p^k)$. So, number of zero divisors in $M_n(\mathbf{Z}_p)$ are

$$p^{n^2} - \prod_{i=0}^{n-1}(p^n - p^k).$$

□

Next, we will discuss the endomorphism ring of the group G. *End(G)* of group endomorphisms of a group G is a ring with respect to the addition of maps, $(f + g)(x) = f(x) + g(x)$ for all $x \in G$, and the product given by map composition, $(fg)(x) = fog(x) = f(g(x))$ for all $x \in G$. The endomorphism ring of the group $A = \bigoplus_{i=1}^{n} A_i$ isomorphic to the ring of all $n \times n$ matrices $[\alpha_{ji}]$, where $\alpha_{ji} \in Hom(A_i, A_j)$, with the ordinary matrix addition and multiplication [10].

In the following theorem, we will show that the diameter of zero divisor graph of $M_n(\mathbf{Z}_p)$ is 2.

Theorem 10. *The zero divisor graph of the ring $M_n(K)$ has minimal distance 2, where K is a field.*

Proof. We know that $M_n(K) \cong End_K(V)$, V an n-dimensional K-vector space. Let x and y be the non-zero zero divisors in $End_K(V)$. Then x and y are neither injective nor surjective. That is, $V/im(y)$ and $Ker(x)$ are non zero spaces. Let $\bar{z}$ be any non zero K-linear map from $V/im(y)$ to $Ker(x)$; this induces a non zero endomorphism z of V. Since $Ker(x) \subset V$, z is again not bijective, hence a zero divisor. From this construction it is clear that xz and zy are zero. □

Corollary 11. *The eccentricity of vertices of* $\Gamma(M_n(\mathbf{Z}_p)$ *is* 2.

Eccentric connectivity index is introduced by Sharma et al. [14] and is defined as

$$\xi(G) = \sum_{v \in V(G)} d_v \epsilon(v).$$

Farooq and Malik [9] defined the *total eccentricity connectivity* as

$$\zeta(G) = \sum_{v \in V(G)} \epsilon(v).$$

Now, we compute the eccentricity connectivity index of $\Gamma(M_n(\mathbf{Z}_p)$.

Theorem 12. *The eccentricity connectivity index of* $\Gamma(M_n(\mathbf{Z}_p)$ *is* $2(p-1)^2(p+1)^3$.

Proof. From Theorem 2 and Corollary 11, we have

$$\begin{aligned}\xi(\Gamma(M_n(\mathbf{Z}_p))) = & \sum_{v \in V(\Gamma(M_n(\mathbf{Z}_p)))} d_v \varepsilon(v) \\ =& 2(p^2-1)(p-1)(p+1)^2 \\ =& 2(p-1)^2(p+1)^3.\end{aligned}$$

□

Corollary 13. *The total eccentricity index of* $\Gamma(M_n(\mathbf{Z}_p))$ *is given by*

$$\zeta(\Gamma(M_n(\mathbf{Z}_p))) = 2(p-1)(p+1)^2$$

4. Conclusion

In this paper, we study the zero divisors in $M_n(\mathbf{Z}_p)$. We find a relation for the number of zero divisors in $M_n(\mathbf{Z}_p)$. We derived a relation for the number of edges in $M_2(\mathbf{Z}_p)$. We proved that $M_2(\mathbf{Z}_p)$ is a regular directed graph. We also proved that diameter of zero divisor graph of $M_n(\mathbf{Z}_p)$ is 2.

References

[1] D. F. Anderson and A. Badawi, The total graph of a commutative ring, *J. Algebra*, **320** (2008), 2706–2719.

[2] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra*, **217** (1999), 434–447.

[3] I. Beck, Coloring of commutative rings, *J. Algebra*, **116** (1988), 208–226.

[4] I. Božić and Z. Petrović, Zero-divisor graphs of matrices over commutative rings, *Comm. Algebra*, **37** (2009), 1186–1192.

[5] J. Warfel, Zero-divisor graphs for direct products of commutative rings, *Wabash College Technical Reports*, **2** (2004).

[6] L. Birch, J. Thibodeaux and R. Tucci, Zero divisor graphs of finite direct products of finite rings, *Comm. Algebra*, **42** (2014), 3852–3860.

[7] M. Axtell and J. Stickles, Zero-divisor graph of idealizations,*J. Pure Appl. Algebra*, **204** (2006), 235–243.

[8] S. Akbari and A. Mohammadian, On the zero-divisor graph of non-commutative ring, *J. Algebra*, **296** (2006), 462–479.

[9] R. Farooq and M. A. Malik, On some eccentricity based topological indices of nanostar dendrimers, *Optoelectronics and Advanced Materials-Rapid Communications*, **9** (2015), 842–849.

[10] P. A. Krylov, A. V. Mikhalev and A. A. Tuganbaev, Endomorphism ring of abelian groups. *J. Math. Sci.*, **110** (2002), 2683–2745.

[11] S. P. Redmond, The zero-divisor graph of a non-commutative ring, *Internat. J. Commutative Rings*, **1** (2002), 203–211.

[12] S. P. Redmond Structure in the zero-divisor graph of a non-commutative ring, *Houston J. Math.*, **30** (2004), 345–355.

[13] S. Safaeeyan, E. Momtahan and M. Baziar, A generalization of the zero-divisor graph for modules, *J. Korean Math. Soc.*, **51** (2014), 87–98.

[14] V. Sharma, R. Goswami, and A. K. Madan, Eccentric connectivity index: a novel highly discriminating topological descriptor for structure-property and structure-activity studies, *J. Chem. Inf. Comp. Sci.*, **37** (1997), 273–282.

[15] P. Singh, and V. K. Bhat, Zero-divisor graphs of finite commutative rings: A survey, *Surv. Math. its Appl.*, **15** (2020), 371–397.

[16] P. Singh, and V. K. Bhat, Adjacency matrix and wiener index of zero divisor graph $\Gamma(\mathbb{Z}_n)$, *J. Appl. Math. Comput.*, **66** (2021), 717–732; doi: 10.1007/s12190-020-01460-2.

[17] P. Singh, and V. K. Bhat, Graph invariants of the line graph of zero divisor graph of Z_n,*J. Appl. Math. Comput.*; doi: 10.1007/s12190-021-01567-0.

[18] X. Ma, D. Wang and J. Zhou, Automorphisms of the zero-divisor graph over 2×2, *J. Korean Math. Soc.*, **53** (2016), 519–532.