International Journal of Applied Mathematics

Volume 34 No. 4 2021, 681-692

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

doi: http://dx.doi.org/10.12732/ijam.v34i4.7

ON COMPUTATION OF EIGENFUNCTIONS OF COMPOSITE TYPE EQUATIONS WITH REGULAR BOUNDARY VALUE CONDITIONS

Nurlan S. Imanbaev[1,2 §], Yeleu Kurmysh[2]

[1]Institute of Mathematics and Mathematical Modeling
Str. Pushkin 125, 050010 – Almaty, KAZAKHSTAN

[2]Department of Mathematics
South Kazakhstan State Pedagogical University
Str. Baitursynov 13, 160000 – Shymkent, KAZAKHSTAN

Abstract: In this paper, we consider the question on computation of eigenvalues and eigenfunctions of a third-order composite type equation in a rectangular region D of the space $W_2^3(0,1)$ satisfying the following boundary conditions

$$u|_{\partial D} = 0, \quad u_x(0,y) = u_x(1,y), \quad u_y(x,0) = u_y(x,1),$$

where $D = \{x, y : 0 < x < 1, 0 < y < 1\}$. All eigenvalues and eigenfunctions of the considered spectral problem are found, and the adjoint operator is constructed.

AMS Subject Classification: 35M10; 35M20

Key Words: composite type equation; regular, periodic boundary value conditions; eigenvalues; eigenfunctions, adjoint operator; characteristic determinant; zeros of entire functions

1. Introduction

An extensive literature is devoted to study initial-boundary value problems for various types of third-order partial differential equations. We note from several later publications a series of works where the conjugation problem for a third-order equation with multiple characteristics, with an alternating function at the

Received: February 3, 2021

§Correspondence author

highest derivative was studied in [15], and the question of solvability of some nonlocal problems for a loaded third-order equation was investigated in [25]. In [21], questions of solvability of a nonlocal problem for a hyperbolic equation with degenerate integral conditions were considered.

Solution of initial-boundary value problems for partial differential equations by the Fourier method is almost always reduced to a problem on eigenvalues of some ordinary differential operators [3], [4], [13], [18], [19], [23].

A number of spectral boundary value problems for the composite type equation

$$\frac{\partial}{\partial x}\left(u_{xx}+u_{yy}\right)+\lambda u=0,$$

were investigated in [3], [4], where all eigenvalues and system of root vectors were found.

The present paper is devoted to finding the eigenvalues and eigenfunctions of one boundary value problem for the third order partial differential equation

$$Lu\equiv u_{xxx}+u_{yyy}+\lambda u=0, \tag{1}$$

which is also a composite type equation [4], where the complex number λ is a spectral parameter.

2. Formulation of the problem

In a rectangular region D of the space $W_2^3\left(0,1\right)$ find eigenvalues and eigenfunctions of the equation (1) satisfying the following boundary conditions

$$u|_{\partial D}=0,\quad u_x\left(0,y\right)=u_x\left(1,y\right),\quad u_y\left(x,0\right)=u_y\left(x,1\right), \tag{2}$$

where $D=\{x,y:0<x<1,0<y<1\}$.

3. Distribution of eigenvalues of the problem (1)-(2)

Looking for a solution of the problem (1) - (2) by the Fourier method in the form

$$u\left(x,y\right)=X\left(x\right)\cdot Y\left(y\right),$$

we come to the following spectral problems in the space $W_2^3\left(0,1\right)$ for ordinary differential operators:

$$L_0X\equiv X'''+\mu X=0,\ X\left(0\right)=X\left(1\right)=0,\ X'\left(0\right)=X'\left(1\right), \tag{3}$$

$$L_1 Y \equiv Y''' + \nu Y = 0,\ Y(0) = Y(1) = 0,\ Y'(0) = Y'(1), \tag{4}$$

moreover, $\lambda = \mu + \nu$.

The boundary value conditions in (3), (4) refer to regular boundary value conditions by J.D. Birkhoff [2]. An important result, established by J.D. Birkhoff, was to estimate resolvent of a regular differential operator and to establish asymptotics of the spectrum. In the monography by M.A. Naimark [20], a subclass of regular boundary value conditions is distinguished, where it is noted that for an odd order of the equation all strongly regular conditions are regular. In (3), (4) the boundary value conditions are periodic boundary value conditions.

Let us solve the problem (3) (the problem (4) is solved similarly). The general solution of the equation (3) has the form

$$X(x) = C_1 e^{2ax} + \left(C_2 \cdot \cos\sqrt{3}ax + C_3 \cdot \sin\sqrt{3}ax\right) \cdot e^{-ax}, \tag{5}$$

where C_1, C_2, C_3 are arbitrary constants,

$$a = \frac{\sqrt[3]{-\mu}}{2} \neq 0. \tag{6}$$

According to results of [7], [8], characteristic determinant of the spectral problems (3), (4) is an entire analytic function of a variable a, which coincides with the exponential type quasi-polynomial with commensurable exponents:

$$\begin{gathered}\Delta(a) = \left(\sqrt{3}+3i\right) e^{\left(1+i\sqrt{3}\right)a} + \left(\sqrt{3}-3i\right) e^{\left(1-i\sqrt{3}\right)a} \\ + \left(\sqrt{3}+3i\right) e^{-\left(1+i\sqrt{3}\right)a} + \left(\sqrt{3}-3i\right) e^{-\left(1-i\sqrt{3}\right)a} \\ -2\sqrt{3}e^{2a} - 2\sqrt{3}e^{-2a}.\end{gathered} \tag{7}$$

In [7], [8], a conjugate indicator diagram of the function $\Delta(a)$ is constructed, which is a regular hexagon on the complex plane a, the sides consisting of segments:

$$\left[\overline{1-i\sqrt{3}};\ \overline{-1-i\sqrt{3}}\right],\ \left[\overline{-1+i\sqrt{3}};\ \overline{1+i\sqrt{3}}\right],\ \left[\overline{-2};\ \overline{-1-i\sqrt{3}}\right],$$

$$\left[\overline{-1+i\sqrt{3}};\ \overline{-2}\right],\ \left[\overline{1+i\sqrt{3}};\ \overline{2}\right],\ \left[\overline{2};\ \overline{1-i\sqrt{3}}\right],$$

where the bar means complex conjugation and the length of each segment is equal to $d = 2$ which means commensurability of the exponents of each series. The rays, which are perpendicular to the indicator diagram, are called critical. According to [15], there are exactly six critical rays on the plane a, that is,

$\arg\sqrt[3]{a} = \frac{\pi}{6} + \frac{\pi n}{3}$,
$n = 0, 1, 2, 3, 4, 5$. Along the critical rays, zeros of each corresponding series $\Delta(a)$ from (7) are located, and in [7], [8] all the zeros of $\Delta(a)$ are found explicitly:

- segment $\left[\overline{2}; \overline{1 - i\sqrt{3}}\right]$, 1-st series of zeros:

$$a_{k1} = \frac{2ik\pi}{-1 + i\sqrt{3}} + \frac{\ln\left|\frac{2\sqrt{3}}{\sqrt{3}+3i}\right| + iArg\left(\frac{2\sqrt{3}}{\sqrt{3}+3i}\right)}{-1 + i\sqrt{3}}, \quad k = 1, 2, 3, ...,$$

where $\ln\left|\frac{2\sqrt{3}}{\sqrt{3}+3i}\right| + iArg\left(\frac{2\sqrt{3}}{\sqrt{3}+3i}\right) = const$;

- segment $\left[\overline{-1 - i\sqrt{3}}; \overline{1 - i\sqrt{3}}\right]$, 2-nd series of zeros:

$$a_{k2} = \frac{ik\pi}{1 + i\sqrt{3}} + \frac{const}{2\left(1 + i\sqrt{3}\right)}, \quad k = 1, 2, 3, ...,$$

- segment $\left[\overline{-1 + i\sqrt{3}}; \ \overline{1 + i\sqrt{3}}\right]$, 3-rd series of zeros:

$$a_{k3} = ik\pi + const, \quad k = 1, 2, 3, ...,$$

- segment $\left[\overline{-2}; \overline{-1 - i\sqrt{3}}\right]$, 4-th series of zeros:

$$a_{k4} = \frac{2ik\pi}{1 + i\sqrt{3}} + \frac{const}{1 + i\sqrt{3}}, \quad k = 1, 2, 3, ...,$$

- segment $\left[\overline{-2}; \overline{-1 - i\sqrt{3}}\right]$, 5-th series of zeros:

$$a_{k5} = -\frac{2ik\pi}{1 + i\sqrt{3}} - \frac{const}{1 + i\sqrt{3}}, \quad k = 1, 2, 3, ...,$$

- segment $\left[\overline{-1 + i\sqrt{3}}; \overline{-2}\right]$, 6-th series of zeros:

$$a_{k6} = \frac{2ik\pi}{1 - i\sqrt{3}} + \frac{const}{1 - i\sqrt{3}}, \quad k = 1, 2, 3, ...,$$

Therefore, according to (6), for spectral boundary value problems (3) and (4) we have 6 series of the corresponding eigenvalues of the operator L_0:

$$\mu_{k1} = -\left(\frac{4ik\pi}{-1 + i\sqrt{3}} + \frac{const}{-1 + i\sqrt{3}}\right)^3, \quad k = 1, 2, 3, ...$$

$$\mu_{k2} = -\left(\frac{2ik\pi}{1+i\sqrt{3}} + \frac{const}{1+i\sqrt{3}}\right)^3, \quad k = 1, 2, 3, ...$$

$$\mu_{k3} = -(2ik\pi + const)^3, \quad k = 1, 2, 3, ...$$

$$\mu_{k4} = -\left(\frac{2ik\pi}{1+i\sqrt{3}} + \frac{const}{1+i\sqrt{3}}\right)^3, \quad k = 1,\ 2,\ 3, ...$$

$$\mu_{k5} = \left(\frac{2ik\pi}{1+i\sqrt{3}} + \frac{const}{1+i\sqrt{3}}\right)^3, \quad k = 1, 2, 3, ...$$

$$\mu_{k6} = -\left(\frac{4ik\pi}{1-i\sqrt{3}} + \frac{const}{1-i\sqrt{3}}\right)^3, \quad k = 1, 2, 3, ...,$$

and the operator L_1:

$$\nu_{l1} = -\left(\frac{4il\pi}{-1+i\sqrt{3}} + \frac{const}{-1+i\sqrt{3}}\right)^3, \quad l = 1, 2, 3, ...$$

$$\nu_{l2} = -\left(\frac{2il\pi}{1+i\sqrt{3}} + \frac{const}{1+i\sqrt{3}}\right)^3, \quad l = 1, 2, 3, ...$$

$$\nu_{l3} = -(2il\pi + const)^3, \quad l = 1, 2, 3, ...$$

$$\nu_{l4} = -\left(\frac{2il\pi}{1+i\sqrt{3}} + \frac{const}{1+i\sqrt{3}}\right)^3, \quad l = 1, 2, 3, ...$$

$$\nu_{l5} = \left(\frac{2il\pi}{1+i\sqrt{3}} + \frac{const}{1+i\sqrt{3}}\right)^3, \quad l = 1, 2, 3, ...$$

$$\nu_{l6} = -\left(\frac{4il\pi}{1-i\sqrt{3}} + \frac{const}{1-i\sqrt{3}}\right)^3, \quad l = 1, 2, 3, ...$$

Hence, it is easy to justify the following

Lemma 1. *Let (7) be a characteristic polynomial of the problems (3), (4), and (6) hold. Then zeros of the entire analytic function $\Delta(a)$ in (7) are eigenvalues of the operators L_0 and L_1, which adequately determine eigenvalues of the operator*

$$L : \lambda_{klj} = \pm(\mu_{kj} + \nu_{lj}),$$

where $k = 1, 2, 3, ...,\ l = 1, 2, 3, ...,\ j = \left(\overline{1;6}\right)$ means each series.

Remark 2. In the case when $a = \frac{\sqrt[3]{-\mu}}{2} = 0$, the general solution (3), representing in the form $X(x) = ax^2 + bx + c$ and satisfying the boundary value conditions in (3), we have $X(x) = 0$, that is, $\mu_0 = 0$ is not an eigenvalue of the operator L_0. Similarly, $\nu_0 = 0$ is a regular point of the operator L_1. So $\lambda_0 = 0$ is not an eigenvalue of the operator L.

The connection between the zeros of quasi-polynomials and spectral problems is reflected in [1], [6], [14], [16], [17], [24].

The works [11], [12], [26] are devoted to investigation of zeros of entire functions having an integral representation, connected by spectral problems of a third-order differential operator with nonlocal boundary value conditions.

In [9], [10], [22], characteristic determinant of the spectral problem for the Sturm-Liouville operator with perturbed regular boundary value conditions was calculated, which is an entire analytic function of the spectral parameter, where stability of the basis property of systems of root functions was studied, and [5] multiple solutions of a nonhomogeneous Sturm-Liouville equation with nonlocal boundary conditions.

4. Calculation of eigenfunctions of the problem (1)-(2)

The fundamental difference of this paper from [7], [8] and [11], [12] is the determination of eigenfunctions of the operators L_0 and L_1. We present a scheme for calculating the eigenfunctions of the operator L_0 (eigenfunctions of the operator L_1 are calculated similarly).

Substituting in order the found eigenvalues of the operator L_0 of each series from Section 3 into (5) and satisfying the problem (3), we obtain the corresponding eigenfunctions of the operator L_0 of each series. We formulate this fact in the following form.

Lemma 3. *If all items of Lemma 1 are satisfied, which are defined all eigenvalues of the operator L_0 of each series in Section 3, then the system of eigenfunctions of the operator L_0 will be*

$$X_{k1}(x) = C_1 e^{2p_{k1}x}\varphi_{k1}(x) + \left(C_2\psi_{k1}(x) + C_3\tau_{k1}(x)\right)e^{p_{k1}x},$$

$$X_{k2}(x) = C_1 e^{2p_{k2}x}\varphi_{k2}(x) + \left(C_2\psi_{k2}(x) + C_3\tau_{k2}(x)\right)e^{p_{k2}x},$$

$$X_{k3}(x) = C_1\varphi_{k3}(x) + C_2\psi_{k3}(x) + C_3\tau_{k3}(x),$$

$$X_{k4}(x) = C_1 e^{2p_{k4}x}\varphi_{k4}(x) + \left(C_2\psi_{k4}(x) + C_3\tau_{k4}(x)\right)e^{p_{k4}x},$$

$$X_{k5}(x) = C_1 e^{2p_{k5}x}\varphi_{k5}(x) + (C_2\psi_{k5}(x) - C_3\tau_{k5}(x))\, e^{p_{k5}x},$$

$$X_{k6}(x) = C_1 e^{2p_{k6}x}\varphi_{k6}(x) + (C_2\psi_{k6}(x) + C_3\tau_{k6}(x))\, e^{p_{k6}x},$$

where

$$\varphi_{k1}(x) = \cos 2q_{k1}x - i\sin 2q_{k1}x,$$

$$\begin{gathered}\psi_{k1}(x) = \cos q_{k1}x \cdot \cos\sqrt{3}p_{k1}x \cdot ch\sqrt{3}q_{k1}x \\ - \sin\sqrt{3}p_{k1}x \cdot sh\sqrt{3}q_{k1}x \cdot \sin q_{k1}x \\ -i(\cos\sqrt{3}p_{k1}x \cdot ch\sqrt{3}q_{k1}x \cdot \sin q_{k1}x \\ + \cos q_{k1}x \cdot \sin\sqrt{3}p_{k1}x \cdot sh\sqrt{3}q_{k1}x),\end{gathered}$$

$$\begin{gathered}\tau_{k1}(x) = \cos q_{k1}x \cdot sin\sqrt{3}p_{k1}x \cdot ch\sqrt{3}q_{k1}x \\ - \sin q_{k1}x \cdot \cos\sqrt{3}p_{k1}x \cdot sh\sqrt{3}q_{k1}x \\ -i(\sin q_{k1}x \cdot \sin\sqrt{3}p_{k1}x \cdot ch\sqrt{3}q_{k1}x \\ + \cos q_{k1}x \cdot \cos\sqrt{3}p_{k1}x \cdot sh\sqrt{3}q_{k1}x),\end{gathered}$$

$$p_{k1} = -\frac{k\pi\sqrt{3}}{2} + \frac{\pi}{4\sqrt{3}},\ q_{k1} = \frac{k\pi}{2} - \frac{\pi}{6},$$

$$\varphi_{k2}(x) = \cos 2q_{k2}x + i\sin 2q_{k2}x,$$

$$\begin{gathered}\psi_{k2}(x) = \cos q_{k2}x \cdot \cos\sqrt{3}p_{k2}x \cdot ch\sqrt{3}q_{k2}x \\ + \sin\sqrt{3}p_{k2}x \cdot sh\sqrt{3}q_{k2}x \cdot \sin q_{k2}x \\ -i(\sin\sqrt{3}p_{k2}x \cdot sh\sqrt{3}q_{k2}x \cdot \cos q_{k2}x \\ - \sin q_{k2}x \cdot \cos\sqrt{3}p_{k2}x \cdot ch\sqrt{3}q_{k2}x),\end{gathered}$$

$$\begin{gathered}\tau_{k2}(x) = \cos q_{k2}x \cdot \sin\sqrt{3}p_{k2}x \cdot ch\sqrt{3}q_{k2}x \\ + \sin q_{k2}x \cdot \cos\sqrt{3}p_{k2}x \cdot sh\sqrt{3}q_{k2}x \\ -i(\cos q_{k2}x \cdot \cos\sqrt{3}p_{k2}x \cdot sh\sqrt{3}q_{k2}x \\ + \sin q_{k2}x \cdot \sin\sqrt{3}p_{k2}x \cdot ch\sqrt{3}q_{k2}x),\end{gathered}$$

$$p_{k2} = -\frac{k\pi\sqrt{3}}{2} - \frac{\pi}{4},\ q_{k2} = \frac{k\pi}{2} - \frac{\pi}{12},$$

$$\varphi_{k3}(x) = \cos 2q_{k3}x - i\sin 2q_{k3}x,$$

$$\psi_{k3}(x) = \cos q_{k3}x \cdot ch\sqrt{3}q_{k3}x - ich\sqrt{3}q_{k3}x \cdot sinq_{k3}x,$$

$$\tau_{k3}(x) = \sin q_{k3}x \cdot sh\sqrt{3}q_{k3}x - i\sin q_{k3}x \cdot ch\sqrt{3}q_{k3}x,$$

$$q_{k3} = \frac{k\pi}{2} - \frac{\pi}{3},$$

$$\varphi_{k4}(x) = cos2q_{k4}x + isin2q_{k4}x,$$

$$\psi_{k4}(x) = \cos q_{k4}x \cdot \cos\sqrt{3}p_{k4}x \cdot ch\sqrt{3}q_{k4}x$$
$$+\sin\sqrt{3}p_{k4}x \cdot sh\sqrt{3}q_{k4}x \cdot \sin q_{k4}x$$
$$-i(\sin\sqrt{3}p_{k4}x \cdot sh\sqrt{3}q_{k4}x \cdot \cos q_{k4}x -$$
$$-\sin q_{k4}x \cdot \cos\sqrt{3}p_{k4}x \cdot ch\sqrt{3}q_{k4}x),$$

$$\tau_{k4}(x) = \cos q_{k4}x \cdot sin\sqrt{3}p_{k4}x \cdot ch\sqrt{3}q_{k4}x$$
$$+\sin q_{k4}x \cdot \cos\sqrt{3}p_{k4}x \cdot sh\sqrt{3}q_{k4}x$$
$$-i(\cos q_{k4}x \cdot \cos\sqrt{3}p_{k4}x \cdot sh\sqrt{3}q_{k4}x$$
$$+\sin q_{k4}x \cdot \sin\sqrt{3}p_{k4}x \cdot ch\sqrt{3}q_{k4}x),$$

$$p_{k4} = -\frac{k\pi\sqrt{3}}{2} - \frac{\pi}{12},\ q_{k4} = \frac{k\pi}{2} + \frac{\pi}{12},$$

$$\varphi_{k5}(x) = \cos 2q_{k5}x + i\sin 2q_{k5}x,$$

$$\psi_{k5}(x) = \cos q_{k5}x \cdot \cos\sqrt{3}p_{k5}x \cdot ch\sqrt{3}q_{k5}x$$
$$+\sin\sqrt{3}p_{k5}x \cdot sinq_{k5}x \cdot sh\sqrt{3}q_{k5}x$$
$$+i(\sin q_{k5}x \cdot \cos\sqrt{3}p_{k5}x \cdot ch\sqrt{3}q_{k5}x$$
$$-\cos q_{k5}x \cdot sin\sqrt{3}p_{k5}x \cdot ch\sqrt{3}q_{k5}x),$$

$$\tau_{k5}(x) = \cos q_{k5}x \cdot \sin\sqrt{3}p_{k5}x \cdot ch\sqrt{3}q_{k5}x$$
$$+\sin q_{k5}x \cdot \cos\sqrt{3}p_{k5}x \cdot sh\sqrt{3}q_{k5}x$$
$$+i(\sin q_{k5}x \cdot \sin\sqrt{3}p_{k5}x \cdot ch\sqrt{3}q_{k5}x$$
$$-\cos q_{k5}x \cdot \cos\sqrt{3}p_{k5}x \cdot sh\sqrt{3}q_{k5}x),$$

$$p_{k5} = -\frac{k\pi\sqrt{3}}{2} + \frac{\pi}{12},\ q_{k5} = \frac{k\pi}{2} + \frac{\pi\sqrt{3}}{12},$$

$$\varphi_{k6}(x) = \cos 2q_{k6}x + i\sin 2q_{k6}x,$$

$$\psi_{k6}(x) = \cos q_{k6}x \cdot \cos\sqrt{3}p_{k6}x \cdot ch\sqrt{3}q_{k6}x$$
$$+\sin\sqrt{3}p_{k6}x \cdot sh\sqrt{3}q_{k6}x \cdot \sin q_{k6}x$$
$$+i(\sin\sqrt{3}p_{k6}x \cdot sh\sqrt{3}q_{k6}x \cdot \cos q_{k6}x$$
$$-\sin q_{k6}x \cdot \cos\sqrt{3}p_{k6}x \cdot ch\sqrt{3}q_{k6}x),$$

$$\tau_{k6}(x) = \cos q_{k6}x \cdot sin\sqrt{3}p_{k6}x \cdot ch\sqrt{3}q_{k6}x$$
$$+\sin q_{k6}x \cdot cos\sqrt{3}p_{k6}x \cdot sh\sqrt{3}q_{k6}x$$
$$+i(\sin q_{k6}x \cdot \sin\sqrt{3}p_{k6}x \cdot ch\sqrt{3}q_{k6}x$$
$$-\cos q_{k6}x \cdot \cos\sqrt{3}p_{k6}x \cdot sh\sqrt{3}q_{k6}x),$$

$$p_{k6} = \frac{k\pi}{2} - \frac{\pi}{12},\ q_{k6} = \frac{k\pi\sqrt{3}}{2} - \frac{\pi\sqrt{3}}{3},\quad k = 1, 2, 3, ...$$

Repeating all the process of obtaining formulas of eigenfunctions for the operator L_0 for the problem (4), we obtain a similar system of eigenfunctions $Y_{lj}(y)$, $l = 1, 2, 3, ...,$ $j = \left(\overline{1;6}\right)$ of the operator L_1 of each series.

Thus, we come to the main result of this work:

Theorem 4. *The system of eigenfunctions of the operator L, that is, of the spectral problem (1) - (2), corresponding to its eigenvalues $\lambda_{klj} = \pm\left(\mu_{kj} + \nu_{lj}\right)$, from Section 3, Lemma 1, according to $u(x, y) = X(x) \cdot Y(y)$, has the form*

$$u_{klj}(x, y) = X_{kj}(x) \cdot Y_{lj}(y), k = 1, 2, 3, ..., \ l = 1, 2, 3, ..., \ j = \left(\overline{1;6}\right),$$

where $X_{kj}(x)$ and $Y_{lj}(y)$ are defined by the formulas of Lemma 2.

5. Conjugate problems

$L_0X \equiv l_0(X) = X'''(x)$. By applying integration by parts, we get the Lagrange formula:

$$\int\limits_0^1 l_0(X)\,\overline{v(x)}dx + \int\limits_0^1 X(x)\,\overline{l_0^*(v)}dx = X''(1)\,\overline{v(1)} - X''(0)\,\overline{v(0)}$$
$$-\left[\overline{v'(0)} - \overline{v'(1)}\right] \cdot X'(0) + X(1) \cdot \overline{v''(1)} - X(0) \cdot \overline{v''(0)}.$$

Here $l_0^*(v)$ is a conjugate differential expression:

$$l_0^*(v) = -v'''(x), \quad 0 < x < 1. \tag{8}$$

Consequently, the operator L_0^*, which is conjugate to the operator L_0 is given by the differential expression (8) and boundary value conditions:

$$v(1) = v(0) = 0, \quad v'(0) - v'(1) = 0. \tag{9}$$

Similarly, for the operator L_1 there is a conjugate operator

$$L_1Y \equiv l_1(Y) = Y'''(y), \quad L_1^* : l_1^*(v) = -v'''(y), \ 0 < y < 1,$$

with the boundary value conditions (9).

Therefore, in the region D, conjugate problem to the problem (1) - (2) will be

$$L^*V = V_{xxx} + V_{yyy} - \lambda V = 0,$$

satisfying the boundary value conditions:

$$V|_{\partial D} = 0,\ V_x(1, y) = V_x(0, y),\ V_y(x, 0) = V_y(x, 1).$$

Acknowledgements

This research is funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No AP09260752)

References

[1] R. Bellman and K. Cook, *Differential-Difference Equations*, Academic Press, New York (1963).

[2] G.D. Birkhoff, On the asymptotic chapacter of the certain linear differential equations containing a parameter, *Transactions of the American Mathematical Society*, **9**, No 2 (1908), 219-231.

[3] T.D. Dzhuraev, B.V. Loginov and I.A. Malyugina, Calculation of eigenvalues and eigenfunctions of some differential operators of the third and fourth orders, In: *Differ. Equations. Math. Phyz. and Their Appl.*, Fan, Tashkent, (1989), 24-36 (in Russian).

[4] T.D. Dzhuraev and Ya. Popelek, On classification and reduction to the canonical form of third-order partial differential equations, *Differ. Equations*, **27**, No 10 (1991), 1734-1745.

[5] A.M.A. El-Sayed, M.Sh. Mohamed and R.E.M. Embia, On the multiple solutions of a nonhomogeneous Sturm-Liouville equation with nonlocal boundary conditions, *International Journal of Applied Mathematics*, **32**, No 1 (2019), 35-44; doi: 10.12732/ijam.v32i1.3.

[6] O.H. Hald, Discontinuous inverse eigen value problems, *Communications on Pure Applied Mathematics*, **37**, No 5 (1984), 539-577.

[7] N.S. Imanbaev, Distribution of eigen values of a third-order differential operator with strongly regular non local boundary conditions, In: *AIP Conf. Proc.*, **1997**, Art. No 020027 (2018), 1-5; doi: 10.1063/1.5049021.

[8] N.S. Imanbaev, On zeros of a quasi-polynomial of exponential type connected with a regular third-order differential operator, *Mathematical Journal*, **18**, No 2 (2018), 124-132.

[9] N.S. Imanbaev, On stability of the basis property of root vectors system of the Sturm-Liouville operator with an integral perturbation of conditions in nonstrongly regular Samarskii-Ionkin type problems, *International*

Journal of Differential Equations, **2015** (2015), Art. No 641481, 1-6; doi: 10.1155/2015/641481.

[10] N.S. Imanbaev, Stability of the basis property of eigenvalue systems of Sturm-Liouville operators with integral perturbation of the boundary condition, *Electronic Journal of Differential Equations*, **2016** (2016), Art. No 87, 1-8.

[11] N.S. Imanbaev and B.E. Kanguzhin, On zeros of entire functions having an integral representation, *News of the National Academy of Sciences of the Republic of Kazakhstan, Ser. Phys.-Math.*, No 3 (1995), 47-52 (in Russian).

[12] N.S. Imanbaev, B.E. Kanguzhin, and B.T. Kalimbetov, On zeros the characteristic determinant of the spectral problem for a third-order differential operator on a segment with nonlocal boundary conditions, *Advances in Difference Equations*, **2013** (2013), No 110; doi: 10.1186/1687-1847-2013-110.

[13] T.Sh. Kalmenov, On spectrum of the Tricomi problem for the Lavrent'ev-Bitsadze equation, *Differ. Equations*, **13**, No 8 (1977), 1418-1425.

[14] B.E. Kanguzhin and M.A. Sadybekov, *Differential Operators on a Segment. Distribution of Eigenvalues*, Gylym, Shymkent (1996) (in Russian).

[15] A.I. Kozhanov and S.V. Potapova, Conjugation problem for a third-order equation with multiple characteristics, with an alternating function at the highest derivative, *Vestnik NGU, Ser. Math., Mech., Informat.*, **15**, No 2 (2015), 51-59 (in Russian).

[16] A.F. Leont'ev, *Entire Functions and Exponential Problems*, Nauka, Moscow (1983) (in Russian).

[17] V.B. Lidskiy, V.A. Sadovnichy, Regularized sums of roots of one class of entire functions, *Functional Analysis*, **1**, No 2 (1967), 52-59.

[18] E.I. Moiseev, *Mixed Type Equations With a Spectral Parameter*, MGU, Moscow (1988) (in Russian).

[19] A.M. Nakhushev, *Problems with Shift for Partial Differential Equations*, Nauka, Moscow (2006) (in Russian).

[20] M.A. Naimark, *Linear Differential Operators*, Nauka, Moscow (1969) (in Russian).

[21] L.S. Pulkina and V.A. Kirichek, Solvability of a nonlocal problem for a hyperbolic equation with degenerate integral conditions, *Vestnik Sam. gos. Tehn. Univer., Ser. Fiz.-Math. Nauki*, **23**, No 2 (2019), 229-245 (in Rissian).

[22] M.A.Sadybekov and N.S. Imanbaev, Characteristic determinant of a boundary value problem, which does not have the basis property, *Eurasian Mathematical Journal*, **8**, No 2 (2017), 40-46.

[23] M.A. Sadybekov and E.M. Orynbasarov, Basis property of system of root functions of a boundary value problem with shift for the Lavrent'ev-Bitsadze equation, *Dokl. A.N. USSR*, **324**, No 6 (1992), 1152-1154.

[24] A.M. Sedletsky, When all zeros of an entire function of exponential type lie in a curvilinear half-plane (necessary condition), *Matem. Sbornik*, **186**, No 9 (1995), 125-134 (in Russian).

[25] O.S. Zikirov and D.K. Kholikov, Solvability of some nonlocal problems for a loaded third-order equation, *Sib. Elektron. Matem. Izv.*, **17** (2020), 77-88; doi: 10.33048/semi.2020.17.007.

[26] A.A. Shkalikov, On the basis property of eigenfunctions of ordinary differential operators with integral boundary value conditions, *Vestnik MGU, Ser. Math. Mech.*, No 6 (1982), 12-21 (in Russian).