International Journal of Applied Mathematics

Volume 34 No. 4 2021, 745-760
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v34i4.11

BOUNDEDNESS OF COMMUTATORS OF AN OSCILLATORY INTEGRAL OPERATORS IN VARIABLE EXPONENT MORREY SPACES

Ali M. Musayev

Azerbaijan State Oil and Industry University
Tazadlig Av. 20, AZ 1601, Baku, AZERBAIJAN

Abstract: In this paper first we prove Calderón-Zygmund-type integral inequalities for oscillatory integral operators and their commutators in the modified Morrey spaces with variable exponent $L^{p(\cdot),\lambda}(\Omega)$, where $\Omega \subset R^n$ are unbounded sets.

After that we prove the boundedness of these operators on the spaces $L^{p(\cdot),\lambda}(\Omega)$.

AMS Subject Classification: 42B20, 42B25, 42B35
Key Words: oscillatory integral operators; Morrey space with variable exponent

1. Introduction

The Morrey spaces in the classical version were introduced in [27] in relation to studying of partial differential equations and the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of the classical Morrey spaces, we refer the readers to [13, 14, 27]. These spaces were widely investigated during the last decades, including the study of classical operators of harmonic analysis, maximal, singular, and potential operators and their generalizations were studied.

The variable exponent analysis is a popular topic which attract many researchers. This topic is mainly focused on the Lebesgue and Sobelev spaces with variable order of integrability and operator theory in these spaces. The

Received: February 18, 2021

study of these spaces has been stimulated by problems of various fields, influenced by many applications, for instance, mechanics of the continuum medium, elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions, see [8], [29], [34]. Recently some optimal control problem in variable Sobolev spaces is studied in [4]. In particular, various results on non-weighted and weighted boundedness in Lebesgue spaces with variable exponents $p(x)$ have been proved for maximal, singular and fractional type operators, we refer to surveying papers [10] and [31].

The variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(\Omega)$, were introduced and studied in [2] in the Euclidean setting in case of bounded sets. The boundedness of the maximal operator in variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the log-condition on $p(\cdot)$, $\lambda(\cdot)$ was proved in [2]. Hästö in [19] used his new "local-to-global" approach to extend the result of [2] on the maximal operator to the case of the whole space R^n. The boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces was proved in [21].

In the case of constant p and λ, the results on the boundedness of potential operators and classical Calderón-Zygmund singular operators go back to [1] and [28], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [7].

A distribution kernel $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \leq C|x-y|^{-n} \quad \text{for all} \quad x \neq y,$$

$$|K(x,y) - K(x,z)| \leq C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if} \quad |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leq C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if} \quad |x-y| > 2|x-\xi|.$$

The Calderón-Zygmund type singular operator and the oscillatory integral operator are defined by

$$Tf(x) = \int_{\Omega} K(x,y)f(y)dy, \tag{1}$$

$$Sf(x) = \int_{\Omega} e^{P(x,y)}K(x,y)f(y)dy, \tag{2}$$

where $P(x,y)$ is a real valued polynomial defined on $\Omega \times \Omega$. Lu and Zhang [26] used L^2-boundedness of T to get L^p-boundedness of S with $1 < p < \infty$.

The commutator generated by the operator S for a given measurable function b is formally defined by

$$[S,b]f = S(bf) - bS(f).$$

Let

$$T^*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$$

be the maximal singular operator, where $T_\varepsilon f(x)$ is the usual truncation

$$T_\varepsilon f(x) = \int_{\{y\in\Omega:|x-y|\geq\varepsilon\}} K(x,y)f(y)dy.$$

In this paper first we prove Calderón-Zygmund-type integral inequalities for oscillatory integral operators and their commutators in the modified Morrey spaces with variable exponent $L^{p(\cdot),\lambda}(\Omega)$, where $\Omega \subset R^n$ are unbounded sets. After that we prove the boundedness of these operators on the spaces $L^{p(\cdot),\lambda}(\Omega)$.

We use the following notations: R^n is the n-dimensional Euclidean space, $\Omega \subset R^n$ is an open set, $\chi_E(x)$ is the characteristic function of a set $E \subseteq R^n$, $B(x,r) = \{y \in R^n : |x-y| < r\})$, $\widetilde{B}(x,r) = B(x,r) \cap \Omega$, by c,C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line.

2. Preliminaries on variable exponent weighted Lebesgue and Morrey spaces

Let Ω be an unbounded set in R^n and $p(\cdot)$ be a measurable function on Ω with values in $(1,\infty)$. We mainly suppose that

$$1 < p_- \leq p(x) \leq p_+ < \infty, \tag{3}$$

where $p_- := ess\inf_{x\in\Omega} p(x)$, $p_+ := ess\sup_{x\in\Omega} p(x)$. By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 : \ I_{p(\cdot)}\left(\frac{f}{\eta}\right) \leq 1\right\},$$

this is a Banach function space. By $p'(\cdot)=\frac{p(x)}{p(x)-1}$, $x\in\Omega$, we denote the conjugate exponent. In the one-dimensional case $n=1$ we deal with the interval $(0,\infty]$ and the standart Lebesgue measure (see e.g. [11]). We refer to [5] for definition and fundamental properties of Banach function spaces. The space $L^{p(\cdot)}(\Omega)$ coincides with the space

$$\left\{f(x):\left|\int_\Omega f(y)g(y)dy\right|<\infty \quad \text{for all} \quad g\in L^{p'(\cdot)}(\Omega)\right\} \tag{4}$$

up to the equivalence of the norms

$$\|f\|_{L^{p(\cdot)}(\Omega)}\approx \sup_{\|g\|_{L^{p'(\cdot)}}\le 1}\left|\int_\Omega f(y)g(y)dy\right| \tag{5}$$

see [23, Theorem 2.3], or [30, Theorem 3.5].

Let $\mathcal{P}(\Omega)$ be the set of bounded measurable functions $p:\Omega\to[1,\infty)$;

$\mathcal{P}^{log}(\Omega)$ be the set of exponents $p\in\mathcal{P}(\Omega)$ satisfying the local log-condition

$$|p(x)-p(y)|\le\frac{A}{-\ln|x-y|},\ \ |x-y|\le\frac{1}{2}\ \ x,y\in\Omega, \tag{6}$$

where $A=A(p)>0$ does not depend on x,y.

By $\mathcal{A}^{log}(\Omega)$ we denote the set of bounded exponents $p:\Omega\to R^n$ satisfying the condition (6); by $P^{log}(\Omega)$ we denote the set of exponents $p\in\mathcal{P}^{log}(\Omega)$ with $1<p_-\le p_+<\infty$; for Ω which may be unbounded, by $\mathcal{P}_\infty(\Omega)$, $\mathcal{P}^{log}_\infty(\Omega)$, $P^{log}_\infty(\Omega)$, $\mathcal{A}^{log}_\infty(\Omega)$ we denote the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded)

$$|p(x)-p(\infty)|\le\frac{A_\infty}{\ln(2+|x|)},\ \ x\in R^n, \tag{7}$$

where $p_\infty=\lim\limits_{x\to\infty}p(x)>1$.

We will also make use of the estimate provided by the following lemma (see [8], Corollary 4.5.9),

$$\|\chi_{\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)}\le Cr^{\theta_p(x,r)},\qquad x\in\Omega,\ p\in P^{log}_\infty(\Omega), \tag{8}$$

where $\theta_p(x,r)=\begin{cases}\frac{n}{p(x)},\ r\le 1,\\ \frac{n}{p(\infty)},\ r\ge 1\end{cases}$.

A locally integrable function $\omega:\Omega\to(0,\infty)$ is called a weight. We say that $\omega\in A_p(\Omega)$, $1<p<\infty$, if there is a constant $C>0$ such that

$$\left(\frac{1}{|\widetilde{B}(x,t)|}\int_{\widetilde{B}(x,t)}\omega(x)dx\right)\left(\frac{1}{|\widetilde{B}(x,t)|}\int_{\widetilde{B}(x,t)}\omega^{1-p'}(x)dx\right)^{p-1}\le C,$$

where $1/p + 1/p' = 1$. We say that $\omega \in A_1(\Omega)$ if there is a constant $C > 0$ such that $M\omega(x) \leq C\omega(x)$ almost everywhere.

The extrapolation theorems (Lemma 1 and Lemma 2 below) are originally due to Cruz-Uribe, Fiorenza, Martell and Pérez [6]. Here we use the form in [8], see Theorem 7.2.1 and Theorem 7.2.3 in [8].

Lemma 1. ([8]) *Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose that for some fixed $0 < p_0 < \infty$, every $(f, g) \in \mathcal{F}$ and every $\omega \in A_1$,*

$$\int_\Omega |f(x)|^{p_0}\omega(x)dx \leq C_0 \int \Omega |g(x)|^{p_0}\omega(x)dx.$$

Let $p(\cdot) \in P(\Omega)$ with $p_0 \leq p_-$. If maximal operator is bounded on $L^{\left(\frac{p(\cdot)}{p_0}\right)'}(\Omega)$, then there exists a constant $C > 0$ such that for all $(f, g) \in \mathcal{F}$,

$$\|f\|_{L^{p(\cdot)}(\Omega)} \leq C\|g\|_{L^{p(\cdot)}(\Omega)}.$$

Lemma 2. ([8]) *Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose that for some fixed $0 < p_0 < q_0 < \infty$, every $(f, g) \in \mathcal{F}$ and every $\Omega \in A_1$,*

$$\left(\int_\Omega |f(x)|^{q_0}\omega(x)dx\right)^{\frac{1}{q_0}} \leq C_0 \left(\int_\Omega |g(x)|^{p_0}\omega^{\frac{p_0}{q_0}}(x)dx\right)^{\frac{1}{p_0}}.$$

Let $p(\cdot) \in P(\Omega)$ with $p_0 \leq p_-$ and $\frac{1}{p_0} - \frac{1}{q_0} < \frac{1}{p_+}$, and define $q(x)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}.$$

If maximal operator is bounded on $L^{\left(\frac{q(\cdot)}{q_0}\right)'}(\Omega)$, then there exists a constant $C > 0$ such that for all $(f, g) \in \mathcal{F}$,

$$\|f\|_{L^{q(\cdot)}(\Omega)} \leq C\|g\|_{L^{p(\cdot)}(\Omega)}.$$

Singular operators within the framework of the spaces with variable exponents were studied in [9]. From Theorem 4.8 and Remark 4.6 of [9] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω, but valid for an arbitrary open set Ω under the corresponding condition in $p(x)$ at infinity.

Theorem 1. ([9, Theorem 4.8]) *Let $\Omega \subset R^n$ be a unbounded open set and $p \in P^{log}(\Omega)$. Then the singular integral operator T is bounded in $L^{p(\cdot)}(\Omega)$.*

Let $\lambda(x)$ be a measurable function on Ω with values in $[0, n]$. The variable Morrey space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ is defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{\frac{\lambda(x)}{p(x)}-\theta_p(x,t)} \|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

respectively.

Definition 2. We define the $BMO(\Omega)$ space as the set of all locally integrable functions f with finite norm

$$\|f\|_{BMO} = \sup_{x\in\Omega,\, r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}|dy,$$

or

$$\|f\|_{BMO} = \inf_{C} \sup_{x\in\Omega,\, r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y) - C|dy <,$$

where $f_{\widetilde{B}(x,t)}(x) = |\widetilde{B}(x,t)|^{-1} \int_{\widetilde{B}(x,t)} f(y)dy$.

Definition 3. We define the $BMO_{p(\cdot)}(\Omega)$ space as the set of all locally integrable functions f with finite norm

$$\|f\|_{BMO_{p(\cdot)}} = \sup_{x\in\Omega,\, r>0} \frac{\|(f(\cdot) - f_{\widetilde{B}(x,r)})\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}}{\|\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}}.$$

Theorem 4. ([24]) *Let $\Omega \subset R^n$ be an open unbounded set and $p \in P^{log}_{\infty}(\Omega)$. Then the norms $\|\cdot\|_{BMO_{p(\cdot)}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

Before proving the main theorems, we need the following lemma.

Lemma 3. ([18]) *Let $b \in BMO(\Omega)$. Then there is a constant $C > 0$ such that*

$$\left|b_{\widetilde{B}(x,r)} - b_{\widetilde{B}(x,t)}\right| \le C\|b\|_* \ln\frac{t}{r} \quad \text{for} \quad 0 < 2r < t,$$

where C is independent of b, x, r, and t.

We will use the following results on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [15].

Theorem 5. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \le C \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{ess\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

3. Oscillatory integral operators in $L^{p(\cdot),\lambda}(\Omega)$

In this section we prove Calderón-Zygmund-type integral inequalities for oscillatory integral operators S and the commutators generated by S, $[b,S]$, in the modified Morrey spaces with variable exponent $L^{p(\cdot),\lambda}(\Omega)$, where $\Omega \subset R^n$ are unbounded sets. After that we prove the boundedness of the operators S and $[b,S]$ on the spaces $L^{p(\cdot),\lambda}(\Omega)$ by the help of these Calderón-Zygmund-type integral inequalities.

Lemma 4. ([25]) *Let K be a standard Calderón-Zygmund kernel and the Calderón-Zygmund singular integral operator T is of type $(L^2(\Omega), L^2(\Omega))$. Then for any real polynomial $P(x,y)$ and $\Omega \in A_p$ $(1 < p < \infty)$, there exists constants $C > 0$ independent of the coefficients of P such that*

$$\|Sf\|_{L^p_\Omega(\Omega)} \le C\|f\|_{L^p_\Omega(\Omega)}.$$

Theorem 1. *Let $\Omega \subset R^n$ be an open unbounded set and $p \in P_\infty^{log}(\Omega)$. Then the operator S is bounded on the space $L^{p(\cdot)}(\Omega)$.*

Proof. From Lemmas 1 and 4, we obtain that the operator S is bounded on the space $L^{p(\cdot)}(\Omega)$. □

The following theorem gives the Calderón-Zygmund-type integral inequality for oscillatory integral operators S.

Theorem 2. *Let $\Omega \subset R^n$ be an open unbounded set, $p \in P_\infty^{log}(\Omega)$ and $f \in L^{p(\cdot)}(\Omega)$. Then the following inequality*

$$\|Sf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s} \tag{1}$$

holds, where C is independent of f, $x \in \Omega$ and t.

Proof. We represent f as $f = f_1 + f_2$,

$$f_1(y) = f(y)\chi_{\widetilde{B}(x,2t)}(y),\ f_2(y) = f(y)\chi_{\Omega\setminus\widetilde{B}(x,2t)}(y),\ \ t > 0, \tag{2}$$

and have

$$\|Sf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} + \|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By Theorem 1 we obtain

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|Sf_1\|_{L^{p(\cdot)}(\Omega)} \le C\|f_1\|_{L^{p(\cdot)}(\Omega)},$$

so

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le C\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))}.$$

Taking into account the inequality

$$\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s},$$

we get

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{3}$$

To estimate $\|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}$, we observe that

$$|Sf_2(z)| \le C \int_{\Omega\setminus B(x,2t)} \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \le t$, $|z-y| \ge 2t$ imply $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$. Therefore we get

$$|Sf_2(z)| \le C \int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{-n} |f(y)| dy,$$

To estimate Sf_2, we first prove the following auxiliary inequality

$$\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy$$
$$\leq Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{4}$$

With this aim we choose $\delta > 0$ and proceed as follows

$$\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy$$

$$\leq \delta \int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n+\delta}|f(y)|dy \int_{|x-y|}^\infty s^{-\delta-1}ds$$

$$\leq C \int_t^\infty s^{-n}\frac{ds}{s} \int_{\{y\in\Omega: 2t\leq|x-y|\leq s\}} |f(y)|dy$$

$$\leq C \int_t^\infty s^{-n} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \|\chi_{\widetilde{B}(x,s)}\|_{L^{p'(\cdot)}(\Omega)} \frac{ds}{s}$$

$$\leq C \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{5}$$

Hence by inequality (5), we get

$$\|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \leq C\|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}$$

$$= Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{6}$$

From (3) and (6) we arrive at (1). □

In the following theorem we prove the boundedness of the operators S on the spaces $L^{p(\cdot),\lambda}(\Omega)$.

Theorem 3. *Let $\Omega \subset R^n$ be an open unbounded set, $p \in P_\infty^{log}(\Omega)$ and $0 \leq \lambda(x) < n$. Then the singular integral operator S is bounded from the space $L^{p(\cdot),\lambda}(\Omega)$ to the space $L^{p(\cdot),\lambda}(\Omega)$.*

Proof. Let $f \in L^{p(\cdot),\lambda}(\Omega)$. Then the following equality

$$\|Sf\|_{L^{p(\cdot),\lambda}(\Omega)} = \sup_{x\in\Omega,\, t>0} [t]_1^{\frac{\lambda(x)}{p(x)}} t^{-\theta_p(x,t)} \|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)} \tag{7}$$

holds. We estimate $\|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}$ in (7) by means of Theorems 2, 5 and obtain

$$\begin{aligned}
&\|Sf\|_{L^{p(\cdot),\lambda}(\Omega)} \\
&\le C \sup_{x\in\Omega,\, t>0} [t]_1^{\frac{\lambda(x)}{p(x)}} t^{-\theta_p(x,t)} t^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s} \\
&\le C \sup_{x\in\Omega,\, t>0} [t]_1^{\frac{\lambda(x)}{p(x)}} t^{-\theta_p(x,t)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} = C\|f\|_{\mathcal{L}^{p(\cdot),\lambda}(\Omega)}.
\end{aligned}$$

□

The following theorem gives the Calderón-Zygmund-type integral inequality for the commutators $[b, S]$.

Theorem 4. *Let $\Omega \subset R^n$ be an open unbounded set, $p \in P_\infty^{log}(\Omega)$ and $b \in BMO(\Omega)$. Then the following inequality*

$$\begin{aligned}
&\|[b,S]f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \\
&\le C\|b\|_* \|t^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \left(1 + \ln\frac{s}{t}\right) \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}
\end{aligned} \tag{8}$$

holds for every $f \in L^{p(\cdot)}(\Omega)$, where C does not depend on $f, x \in \Omega$ and t.

Proof. We represent function f as in (2) and have

$$\|[b,S]f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|[b,S]f_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} + \|[b,S]f_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}.$$

Since $[b, S]$ is bounded on the space $L^{p(\cdot)}(\Omega)$, we obtain

$$\begin{aligned}
&\|[b,S]f_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|[b,S]f_1\|_{L^{p(\cdot)}(\Omega)} \\
&\le C\|b\|_*\|f_1\|_{L^{p(\cdot)}(\Omega)} = C\|b\|_*\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))},
\end{aligned} \tag{9}$$

where C does not depend on f. From (9) we easily obtain

$$\|[b,S]f_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}$$

$$\leq C\|b\|_* t^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \left(1+\ln\frac{s}{t}\right) \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s} \tag{10}$$

from the fact that $\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))}$ is non-decreasing in t, so that $\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))}$ on the right-hand side of (9) is dominated by the right-hand side of (10). To estimate $\|[b,S]f_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}$, we observe that

$$|[b,S]f_2(z)| \leq C \int_{\Omega\setminus B(x,2t)} |b(z)-b(y)| \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \leq t$, $|z-y| \geq 2t$ imply $\frac{1}{2}|z-y| \leq |x-y| \leq \frac{3}{2}|z-y|$. Therefore

$$|[b,S]f_2(z)| \leq C \int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{-n} |b(z)-b(y)|\,|f(y)|dy.$$

To estimate $[b,S]f_2$, first we need to prove the following auxiliary inequality

$$\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|b(z)-b(y)||f(y)|dy$$

$$\leq C\|b\|_* \int_t^\infty s^{-\theta_p(x,s)} \left(1+\ln\frac{s}{t}\right) \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{11}$$

For $z \in \widetilde{B}(x,t)$ we have

$$\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|b(z)-b(y)||f(y)|dy$$
$$\leq \int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dy$$
$$+ \int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|b(z)-b_{\widetilde{B}(x,t)}||f(y)|dy = J_1 + J_2.$$

With this aim we choose $\delta > 0$ and from Theorem 4 and Lemma 3 we obtain

$$J_1 = \int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dy$$

$$\leq \delta \int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n+\delta}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dy \int_{|x-y|}^\infty s^{-\delta-1}ds$$
$$\leq C \int_t^\infty s^{-n-1} \int_{\{y\in\Omega:2t\leq|x-y|\leq s\}} |b(y)-b_{\widetilde{B}(x,t)}||f(y)|dyds$$

$$\leq C\int_t^\infty s^{-n-1}\|b(\cdot)-b_{\widetilde{B}(x,s)}\|_{L^{p'(\cdot)}(\widetilde{B}(x,s))}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}ds$$

$$+C\int_t^\infty s^{-n-1}|b_{\widetilde{B}(x,t)}-b_{\widetilde{B}(x,s)}|\int_{\widetilde{B}(x,s)}|f(y)|dyds$$

$$\leq C\|b\|_*\int_t^\infty s^{-\theta_p(x,s)-n-1}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}ds$$

$$+C\|b\|_*\int_t^\infty s^{-\theta_p(x,s)-n-1}\ln\frac{s}{t}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}ds$$

$$\leq C\|b\|_*\int_t^\infty s^{-\theta_p(x,s)}\left(1+\ln\frac{s}{t}\right)\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}.$$

To estimate J_2, from (4) we have

$$J_2=|b(z)-b_{\widetilde{B}(x,t)}|\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|f(y)|dy$$

$$\leq C|B(x,t)|^{-1}\int_{\widetilde{B}(x,t)}|b(z)-b(y)|dy\int_t^\infty s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}$$

$$\leq CM_b\chi_{B(x,t)}(z)\int_t^\infty s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},$$

where C does not depend on x,t.

Hence by inequality (11), we get

$$\|[b,S]f_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\leq C\|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}\|b\|_*$$

$$\times\int_t^\infty\left(1+\ln\frac{s}{t}\right)s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}$$

$$=\|b\|_*t^{\theta_p(x,t)}\int_t^\infty s^{-\theta_p(x,s)}\left(1+\ln\frac{s}{t}\right)\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}. \tag{12}$$

From (10) and (12) we arrive at (8). □

In the following theorem we prove the boundedness of the operators $[b,S]$ on the spaces $L^{p(\cdot),\lambda}(\Omega)$.

Theorem 5. *Let $\Omega\subset R^n$ be an open unbounded set, $p\in P_\infty^{log}(\Omega)$, $b\in BMO(\Omega)$ and $0\leq\lambda(x)<n$. Then the singular integral operator $[b,S]$ is bounded from the space $L^{p(\cdot),\lambda}(\Omega)$ to the space $L^{p(\cdot),\lambda}(\Omega)$.*

Proof. Let $f \in L^{p(\cdot),\lambda}(\Omega)$. We have

$$\|[b,S]f\|_{L^{p(\cdot),\lambda}(\Omega)} = \sup_{x\in\Omega,\ t>0} [t]_1^{\frac{\lambda(x)}{p(x)}} t^{-\theta_p(x,t)} \|[b,S]f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By Theorems 5 and 4 we obtain

$$\begin{aligned}
&\|[b,S]f\|_{L^{p(\cdot),\lambda}(\Omega)} \\
&\le C\|b\|_* \sup_{x\in\Omega,\ t>0} [t]_1^{\frac{\lambda(x)}{p(x)}} t^{-\theta_p(x,t)} t^{\theta_p(x,t)} \\
&\int_t^\infty s^{-\theta_p(x,s)} \left(1+\ln\frac{s}{t}\right) \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s} \\
&\le C\|b\|_* \sup_{x\in\Omega,\ t>0} [t]_1^{\frac{\lambda(x)}{p(x)}} t^{-\theta_p(x,t)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} = C\|b\|_* \|f\|_{L^{p(\cdot),\lambda}(\Omega)}
\end{aligned}$$

which completes the proof. □

References

[1] D.R. Adams, A note on Riesz potentials, *Duke Math.*, **42** (1975), 765-778.

[2] A. Almeida, J.J. Hasanov, S.G. Samko, Maximal and potential operators in variable exponent Morrey spaces, *Georgian Math. Journal*, **15**, No 2 (2008), 1-15.

[3] Z.O. Azizova, Hardy operators in the local "complementary" generalized variable exponent weighted Morrey spaces, *International Journal of Applied Mathematics*, **33**, No 4 (2020), 675-684; DOI: 10.12732/ijam.v33i4.11.

[4] R.A. Bandaliyev, V.S. Guliyev, I.G. Mamedov, A.B. Sadigov, The optimal control problem in the processes described by the Goursat problem for a hyperbolic equation in variable exponent Sobolev spaces with dominating mixed derivatives, *J. Comput. Appl. Math.*, **305** (2016), 11-17.

[5] C. Bennett, R. Sharpley, *Interpolation of Operators,* Ser. Pure and Applied Mathematics, 129, Academic Press, Inc., Boston, MA, 1988.

[6] D. Cruz-Uribe, A. Fiorenza, J.M. Martell, C. Perez, The boundedness of classical operators on variable L^p spaces, *Ann. Acad. Scient. Fennicae, Math.*, **31** (2006), 239-264.

[7] F. Chiarenza, M. Frasca, Morrey spaces and Hardy–Littlewood maximal function, *Rend. Math.*, **7** (1987), 273-279.

[8] L. Diening, P. Harjulehto, Hästö, and M. Ružička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer-Verlag, Lecture Notes in Mathematics, Vol. 2017, Berlin, 2011.

[9] L. Diening and M. Rúźićka, Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics, *J. Reine Angew. Math.*, **563** (2003), 197-220.

[10] L. Diening, P. Hasto, A. Nekvinda, Open problems in variable exponent Lebesgue and Sobolev spaces, In: "*Function Spaces, Differential Operators and Nonlinear Analysis*" (Proc. Conference held in Milovy, Bohemian-Moravian Uplands, May 28-June 2, 2004), Math. Inst. Acad. Sci. Czech Republic (2005), Praha.

[11] D.E. Edmunds, J. Lang, A. Nekvinda, On $L_{p(x)}$ norms, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, **455** (1999), 219-225.

[12] L. Ephremidze, V. Kokilashvili, S. Samko, Fractional, maximal and singular operators in variable exponent Lorentz spaces, *Frac. Calc. Appl. Anal.*, **11**, No 4 (2008), 407-420.

[13] G. Di Fazio and M. A. Ragusa, Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients, *J. Funct. Anal.*, **112** (1993), 241-256.

[14] D. Fan, S. Lu and D. Yang, Boundedness of operators in Morrey spaces on homogeneous spaces and its applications, *Acta Math. Sinica (N.S.)*, **14** (1998), 625-634.

[15] V.S. Guliyev, Local generalized Morrey spaces and singular integrals with rough kernel, *Azerb. J. Math.*, **3**, No 2 (2013), 79-94.

[16] V.S. Guliyev, J.J. Hasanov, X.A. Badalov, Maximal and singular integral operators and their commutators on generalized weighted Morrey spaces with variable exponent, *Math. Ineq. Appl.*, **21**, No 1 (2018), 41-61.

[17] V.S.Guliyev, J.J. Hasanov, S.G. Samko, Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces, *Math. Scand.*, **107** (2010), 285-304.

[18] S. Janson, Mean oscillation and commutators of singular integral operators, *Ark. Mat.*, **16** (1978), 263-270.

[19] P. Hästö, Local-to-global results in variable exponent spaces, *Math. Res. Letters*, **15** (2008).

[20] J.J. Hasanov, A.M. Musayev, Oscillatory integral operators and their commutators in modified weighted Morrey spaces with variable exponent, *International Journal of Applied Mathematics* **32**, No 3 (2019), 521-535; DOI: 10.12732/ijam.v32i3.12.

[21] V. Kokilashvili and A. Meskhi, Boundedness of maximal and singular operators in Morrey spaces with variable exponent, *Arm. J. Math. (Electronic)* **1**, No 1 (2008), 18-28.

[22] V. Kokilashvili, On a progress in the theory of integral operators in weighted Banach function spaces, In: "*Function Spaces, Differential Operators and Nonlinear Analysis*" (Proc. of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28 - June 2, 2004), Math. Inst. Acad. Sci. Czech Republic, Praha (2005), 152-175.

[23] O. Kovacik and J. Rakosnik, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.*, **41**, No 116 (1991), 4, 592-618.

[24] Kwok-Pun Ho, Singular integral operators, John-Nirenberg inequalities and Tribel-Lizorkin type spaces on weighted Lebesgue spaces with variable exponents, *Revista De La Union Matematica Argentina*, **57**, No 1 (2016), 85-101.

[25] S. Lu, Y. Ding, and D. Yan, *Singular Integrals and Related Topics*, World Scientific Publishing, Hackensack, NJ, USA, 2007.

[26] S.Z. Lu and Y. Zhang, Criterion on L^p-boundedness for a class of oscillatory singular integrals with rough kernels,*Revista Matematica Iberoamericana*, **8**, No 2 (1992), 201-219.

[27] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, *Trans. Amer. Math. Soc.*, **43** (1938), 126-166.

[28] J. Peetre, On the theory of $L^{p,\lambda}$ spaces, *J. Funct. Anal.*, **4** (1969), 71-87.

[29] M. Ružička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Math. **1748**, Springer, Berlin, 2000.

[30] S.G. Samko, Differentiation and integration of variable order and the spaces $L^{p(x)}$, In: *Proc. of Intern. Conference "Operator Theory and Complex and Hypercomplex Analysis"*, 12-17 December 1994, Mexico City, Mexico, *Contemp. Math.*, **212** (1998), 203-219.

[31] S.G. Samko, On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators, *Integr. Transf. and Spec. Funct.*, **16**, No 5-6 (2005), 461-482.

[32] I.I. Sharapudinov, The topology of the space $L^{p(t)}([0, 1])$, *Mat. Zametki*, **26**, No 3-4 (1979), 613-632.

[33] S.G. Shi, Weighted boundedness for commutators of one class of oscillatory integral operators, *Journal of Beijing Normal University (Natural Science)*, **47** (2011), 344-346.

[34] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Izv. Math.*, **29**, No 1 (1987), 33-66.