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SOME INEQUALITIES FOR AN EXTENDED BETA FUNCTION

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Abstract: Intensive studies aiming to extend the beta function and to establish some properties for these extensions have been recently carried out. In this article, we investigate some inequalities for a special extension of the beta function. Based on some integral inequalities, we establish several inequalities involving this extended beta function that generalize some results already discussed in the literature.

AMS Subject Classification: 33B15; 33B99

Key Words: extension of extended Beta function; extended Beta function; integral inequalities

1. Introduction

The well-known beta function, also called the Euler's integral of the first kind, is defined for x, y > 0 by

$$B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$
 (1)

The basic properties of this function as well as its application in various

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contexts can be found in the literature, see for example, [1, 2, 6].

In [3] Chaudhry et al. introduced an extension of the beta function as follows: for any x, y > 0 and $p \ge 0$, they defined

$$B(x,y;p) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left(\frac{-p}{t(1-t)}\right) dt.$$
 (2)

Other representations of this extended beta function and its connections with some other special functions are discussed in [3] and [8]. The importance of this last type of functions is highlighted in [4] by some of their applications. It is worth mentioning to pay attention that the extension of the beta function presented in the paper [3] is different from that given in [4].

The preceding extension process was continued by adding other parameters to the beta function. For instance, a second extension of B(x, y) was introduced by Choi et al. in [7] as follows: for any x, y > 0 and $p, q \ge 0$, they defined

$$\mathcal{B}(x,y;p,q) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) dt.$$
 (3)

If p = 0, then (2) coincides with (1). If p = q then (3) is reduced to (2). Making the change of variables t = 1 - u in (3), it is not hard to check that the following relationship

$$\mathcal{B}(x, y; p, q) = \mathcal{B}(y, x; q, p) \tag{4}$$

holds for any x,y>0 and $p,q\geq 0$. By the change of variables $t=\cos^2\theta$ we get the following integral representation

$$\mathcal{B}(x,y;p,q) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta \exp\left(-p\sec^2\theta - q\csc^2\theta\right) d\theta.$$
 (5)

Some algebraic properties of $\mathcal{B}(x,y;p,q)$ as well as various applications can be found in [7]. In particular, the extended Gauss hypergeometric function, the extended confluent hypergeometric function and the beta distribution were investigated in [7].

The fundamental goal of this paper is to investigate some inequalities involving the extended beta function $\mathcal{B}(x,y;p,q)$. Our inequalities obtained here are of course generalizations of those discussed for B(x,y) and B(x,y;p) in [6] and [11], respectively.

2. Inequalities via Chebychev's inequality

In the ongoing section we present some inequalities for $\mathcal{B}(x,y;p.q)$ by using the so-called Chebychev's integral inequality which we will recall below. We first need to state the following definition.

Definition 1. Let f and g be two real functions defined on a nonempty interval I of \mathbb{R} . We say that f and g are synchronous (resp. asynchronous) on I, if they are monotonic in the same sense (resp. in the opposite sense) on I. That is, the following

$$(f(x) - f(y))(g(x) - g(y)) \ge (\le) 0$$

holds for any $x, y \in I$.

This class of functions plays an important place in mathematical analysis. As an example, we mention the following result, known in the literature as the Chebychev inequality [9, 10], which will be needed later.

Lemma 2. Let I be a nonempty interval of \mathbb{R} and let $f,g,h:I \longrightarrow \mathbb{R}$ be such that $h(x) \geq 0$ for all $x \in I$. We assume that h,hfg,hf and hg are integrable on I. If f and g are synchronous (resp. asynchronous) on I then the following inequality holds:

$$\int_{I} h(t) dt \times \int_{I} h(t)f(t)g(t) dt \ge (\le) \int_{I} h(t)f(t) dt \int_{I} h(t)g(t) dt.$$
 (6)

The following lemma will be also needed in the sequel.

Lemma 3. Let $\varphi:[0,1] \longrightarrow \mathbb{R}$ be the function defined by

$$\forall t \in [0,1] \qquad \varphi(t) = (at+b)(c(1-t)+d),$$

where a, b, c and d are four given real numbers. Then the following assertions hold:

(i) If $ac \ge 0$ and $\min \left(b(c+d), d(a+b) \right) \ge 0$ then $\varphi(t) \ge 0$ for all $t \in [0,1]$.

(ii) If
$$ac \le 0$$
 and $\max \left(b(c+d), d(a+b)\right) \le 0$ then $\varphi(t) \le 0$ for all $t \in [0,1]$.

Proof. (i) First assume that $ac \neq 0$. Then $\varphi(t)$ is a quadratic function in t. Computing the derivative φ' of φ and solving the equation $\varphi'(t) = 0$ in \mathbb{R} we

find as unique root $t_0 \in \mathbb{R}$ such that

$$t_0 = \frac{1}{2} \left(1 + \frac{d}{c} - \frac{b}{a} \right), \quad \varphi(t_0) = \frac{(ac + ad + bc)^2}{4ac}.$$

Now, we should envisage the two cases $t_0 \in [0,1]$ and $t_0 \notin [0,1]$. Assume that ac > 0.

- If $t_0 \notin [0,1]$, then φ is monotonic on [0,1] and we have $\inf_{t \in [0,1]} \varphi(t) = \min(\varphi(0), \varphi(1))$. Since $\varphi(0) = b(c+d)$ and $\varphi(1) = d(a+b)$ then the condition $\min\left(b(c+d), d(a+b)\right) \geq 0$ implies that $\varphi(t) \geq 0$ for any $t \in [0,1]$.
- If $t_0 \in [0, 1]$ then, by ac > 0, $\varphi(t_0) \ge 0$ is a maximal value of φ on [0, 1] and hence, $\min(\varphi(0), \varphi(1)) \ge 0$ implies that $\varphi(t) \ge 0$ for any $t \in [0, 1]$.

If ac=0 then φ is an affine function. In this case, it is easy to check that $\varphi(t) \geq 0$ provided that the condition $\min \Big(b(c+d),d(a+b)\Big) \geq 0$, when ac=0, is satisfied. The first part of the lemma is proved.

(ii) The case $ac \leq 0$ can be stated in a similar manner and we then conclude the second part of the lemma. The details are simple and therefore omitted here for the reader.

Now, we are in the position to state our first main result which reads as follows.

Theorem 4. Let x, x', y, y' > 0 and p, p', q, q' > 0. Then the two following assertions hold:

(i) If
$$(x'-x)(y-y') \ge 0$$
 and min $((p'-p)(y-y'+q-q'), (q'-q)(x-x'+p-p')) \ge 0$ then we have

$$\mathcal{B}(x, y; p, q) \times \mathcal{B}(x', y'; p', q') \ge \mathcal{B}(x, y'; p, q') \times \mathcal{B}(x', y; p', q) \tag{7}$$

(ii) If
$$(x'-x)(y-y') \le 0$$
 and $\max \left((p'-p)(y-y'+q-q'), (q'-q)(x-x'+p-p') \right) \le 0$ then the inequality (7) is reversed.

Proof. (i) Let us consider the following three functions defined for any $t \in (0,1)$ by

$$f(t) = t^{x'-x} \exp\left(\frac{p-p'}{t}\right), \ g(t) = (1-t)^{y'-y} \exp\left(\frac{q-q'}{1-t}\right),$$
$$h(t) = t^{x-1} (1-t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right).$$

Let us remark first that h is positive and integrable on (0; 1). In other hand, we have for all $t \in (0, 1)$

$$f'(t) = \left[(x' - x)t + p' - p \right] t^{x' - x - 2} \exp\left(\frac{p - p'}{t}\right)$$

and

$$g'(t) = \left[(y - y')(1 - t) + q - q' \right] (1 - t)^{y' - y - 2} \exp\left(\frac{q - q'}{1 - t}\right).$$

According to Lemma 3, if the two following conditions

$$(x'-x)(y-y') \ge 0$$
 and $\min ((p'-p)(y-y'+q-q'), (q'-q)(x-x'+p-p')) \ge 0$

are fulfilled, then the functions f and g are synchronous on (0,1). This, with (6), yields

$$\int_0^1 h(t) \ dt \times \int_0^1 h(t) f(t) g(t) \ dt \ge (\le) \int_0^1 f(t) h(t) \ dt \int_0^1 g(t) h(t) \ dt.$$

This, with the explicit expressions of f(t), g(t) and h(t), gives

$$\int_{0}^{1} t^{x-1} (1-t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) dt$$

$$\times \int_{0}^{1} t^{x'-1} (1-t)^{y'-1} \exp\left(-\frac{p'}{t} - \frac{q'}{1-t}\right) dt$$

$$\geq (\leq) \int_{0}^{1} t^{x-1} (1-t)^{y'-1} \exp\left(-\frac{p}{t} - \frac{q'}{1-t}\right) dt$$

$$\times \int_{0}^{1} t^{x'-1} (1-t)^{y-1} \exp\left(-\frac{p'}{t} - \frac{q}{1-t}\right) dt.$$

Whence (7) follows.

(ii) It is similar to (i) by utilizing Lemma 3, (ii). We omit the routine details here. $\hfill\Box$

The previous theorem has many consequences. In particular, we cite the following corollary.

Corollary 5. Let x, y > 0 and $p, q \ge 0$. Assume that $(q-p)(y-x+q-p) \ge 0$. Then we have

$$\left(\mathcal{B}(x,y;p,q)\right)^2 \ge B(x,x;p) \ B(y,y;q). \tag{8}$$

In particular, the inequalities

$$\left(B(x,y;p)\right)^2 \ge B(x,x;p) \ B(y,y;p),$$

$$\left(\mathcal{B}(x,x;p,q)\right)^2 \ge B(x,x;p) \ B(x,x;q),$$

hold true for any x, y > 0 and $p, q \ge 0$.

Proof. We apply the previous theorem with x'=y,y'=x,p'=q and q'=p. We have $(x'-x)(y-y')=(y-x)^2\geq 0$ and

$$\min \left((p'-p)(y-y'+q-q'), (q'-q)(x-x'+p-p') \right) = (q-p)(y-x+q-p).$$

Then (8) follows from Theorem 4 with the help of (4). The two other inequalities follow from (8) when considering the cases p = q and x = y, respectively.

Remark 6. If p = p' = q = q' = 0 then Theorem 4 and Corollary 5 are reduced to [6, Theorem 1] and [6, Corollary 1], respectively. If p = p' and q = q', then Theorem 4 coincides with [11, Theorem 1].

We now state another main result which concerns some inequalities involving the two previous extended beta functions, namely B(x, y; p) and $\mathcal{B}(x, y; p, q)$.

Theorem 7. For any x, y > 0 and $p, q \ge 0$ we have the following inequalities

$$\mathcal{B}(x, y; p, q) \times \mathcal{B}(x, y; q, p) \ge B(x, y; p) \times B(x, y; q) \tag{9}$$

and

$$\mathcal{B}(x, y; p, q) \times \mathcal{B}(y, x; p, q) \ge B(x, y; p) \times B(x, y; q). \tag{10}$$

Proof. To establish (9), we apply (6) with the functions f, g and h defined on (0,1) by $f(t) = \exp\left(\frac{q-p}{t}\right), g(t) = \exp\left(\frac{p-q}{1-t}\right)$ and $h(t) = t^{x-1}(1-t)^{y-1}\exp\left(-\frac{q}{t} - \frac{p}{1-t}\right)$.

It can be easily verified that f and g are synchronous on (0,1) and h satisfies the required conditions in Lemma 6. Writing explicitly (6) we immediately get (9). Otherwise, (10) follows from (9) with the help of (4). The proof is finished.

Remark 8. Taking x = y in (9) we get again the last inequality of Corollary 5.

We have the following main result as well.

Theorem 9. Let x, y > 0 and $p, q \ge 0$. Assume that $(x-1)(y-1) \le (\ge) 0$. Then the following inequality

$$\mathcal{B}(x+k,y+k;p,q) \times \mathcal{B}(k+1,k+1;p,q) \ge (\le)\mathcal{B}(x+k,k+1;p,q) \times \mathcal{B}(k+1,y+k;p,q)$$
 (11)

holds true for any real number $k \geq 0$.

Proof. For all $t \in (0,1)$, we set

$$f(t) = t^{x-1}$$
, $g(t) = (1-t)^{y-1}$, $h(t) = t^k (1-t)^k \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right)$.

Obviously, $f'(t) = (x-1)t^{x-2}$ and $g'(t) = -(y-1)(1-t)^{y-2}$ and so f and g are synchronous (resp. asynchronous) on (0,1) provided that $(x-1)(y-1) \leq (\geq)0$. Further, h is positive and integrable on (0,1). Using the Chebyshev inequality (6) with the previous functions f, g and h, we derive the inequality (11).

The following remark is of interest.

Remark 10. Assume that $(x-1)(y-1) \le (\ge)0$. If in (11) we take k=0 we then obtain the following inequality

$$\mathcal{B}(x,y;p,q) \times \mathcal{B}(1,1;p,q) \ge (\le)\mathcal{B}(x,1;p,q) \times \mathcal{B}(y,1;q,p). \tag{12}$$

In particular, if moreover p = q = 0 then (12) implies that $B(x, y) \ge (\le) \frac{1}{xy}$, which was established by Dragomir et al. in [6, Theorem 3].

It is worth mentioning that if $(x-1)(y-1) \leq 0$ then (12) implies that

$$\mathcal{B}(x, y; p, q) \ge \mathcal{B}(x, 1; p, q) \times \mathcal{B}(y, 1; q, p).$$

3. Inequalities via Hölder's inequality

In this section we will invest the classical Hölder's inequality to establish some results related to the extended beta function $\mathcal{B}(x, y; p, q)$. The main result of this section is recited as follows.

Theorem 11. $\mathcal{B}(x,y;p,q)$, as a function in four variables, is logarithmically convex on $(0,\infty)^2 \times [0,\infty)^2$. That is, for any x,y,x',y'>0; $p,q,p',q'\geq 0$ and $a,b\geq 0$ with a+b=1, we have

$$\mathcal{B}\left[a(x,y;p,q) + b(x',y';p',q')\right] \le \left(\mathcal{B}(x,y;p,q)\right)^a \times \left(\mathcal{B}(x',y';p',q')\right)^b. \tag{13}$$

Proof. For the sake of simplicity, we set

$$\Psi =: \mathcal{B} \Big[a(x, y; p, q) + b(x', y'; p', q') \Big].$$

We have

$$\begin{split} \Psi &= \mathcal{B} \Big[(ax + bx', ay + by'; ap + bp', aq + bq') \Big] \\ &= \int_0^1 t^{ax + bx' - 1} (1 - t)^{ay + by' - 1} \exp\left(-\frac{ap + bp'}{t} - \frac{aq + bq'}{1 - t} \right) dt \\ &= \int_0^1 t^{ax + bx' - a - b} (1 - t)^{ay + by' - a - b} \exp\left(-\frac{ap + bp'}{t} - \frac{aq + bq'}{1 - t} \right) dt \\ &= \int_0^1 t^{a(x - 1) + b(x' - 1)} (1 - t)^{a(y - 1) + b(y' - 1)} \\ &\times \exp\left(-\frac{ap + bp'}{t} - \frac{aq + bq'}{1 - t} \right) dt \\ &= \int_0^1 \left(t^{x - 1} (1 - t)^{y - 1} \exp\left(-\frac{p}{t} - \frac{q}{1 - t} \right) \right)^a \\ &\times \left(t^{x' - 1} (1 - t)^{y' - 1} \exp\left(-\frac{p'}{t} - \frac{q'}{1 - t} \right) \right)^b dt. \end{split}$$

Let $n = \frac{1}{a}$ and $m = \frac{1}{b}$ for which $\frac{1}{n} + \frac{1}{m} = 1$. According to the Hölder integral inequality applied for the following functions

$$f(t) = \left(t^{x-1}(1-t)^{y-1}\exp\left(-\frac{p}{t} - \frac{q}{1-t}\right)\right)^a$$

and

$$g(t) = \left(t^{x'-1}(1-t)^{y'-1}\exp\left(-\frac{p'}{t} - \frac{q'}{1-t}\right)\right)^b,$$

we obtain

$$\Psi \le \left(\int_0^1 \left(f(t)\right)^n\right)^{\frac{1}{n}} \times \left(\int_0^1 \left(g(t)\right)^m\right)^{\frac{1}{m}}.$$

Hence the inequality (13) follows.

Remark 12. As it is well known, every logarithmically convex function is convex and hence $\mathcal{B}(x, y; p, q)$ is convex. This can be also deduced from (13) when using the standard Young's inequality.

The previous result has many consequences. The following corollary is immediate.

Corollary 13. For any x, y, x', y' > 0 and $p, q, p', q' \ge 0$, the next inequality holds true

$$\left(\mathcal{B}\left(\frac{x+x'}{2}, \frac{y+y'}{2}; \frac{p+p'}{2}, \frac{q+q'}{2}\right)\right)^2 \leq \mathcal{B}(x, y; p, q) \times \mathcal{B}(x', y'; p', q').$$

The second corollary is as follows.

Corollary 14. Let x, y > 0, $m, n \ge 0$ and let k, k' be two real numbers such that $|k| \le m$ and $|k'| \le n$. Then we have

$$\left(\mathcal{B}(x,y;m,n)\right)^{2} \le \mathcal{B}(x,y;m-k,n-k') \times \mathcal{B}(x,y;m+k,n+k'). \tag{14}$$

Proof. Making in (13) the following choices

$$a = b = 1/2; x' = x, y' = y; p = m - k, p' = m + k; q = n - k', q' = n + k',$$
 we immediately get (14).

Now, let us observe the following remark.

Remark 15. If in (14) we choose k = m and k' = n, we get

$$\left(\mathcal{B}(x,y;m,n)\right)^2 \le B(x,y) \times \mathcal{B}(x,y;2m,2n).$$

By a mathematical induction we can deduce that, for any integer $r \geq 0$, we have

$$\left(\mathcal{B}(x,y;m,n)\right)^{2^r} \le \left(B(x,y)\right)^{2^r-1} \times \mathcal{B}(x,y;2^rm,2^rn).$$

4. Inequalities via Grüss inequality

In 1935, Grüss proved an integral inequality which gives an approximation for the integral of a product of two functions in terms of the product of integrals of the two functions, see [9, 10] for instance. This inequality reads as follows.

Lemma 16. Let f and g be two functions defined and integrable over (a,b). Assume that, for all $x \in (a,b)$, we have

$$\phi \le f(x) \le \Phi$$
 and $\gamma \le g(x) \le \Gamma$,

where ϕ, Φ, γ and Γ are fixed real numbers. Then,

$$\left| \frac{1}{b-a} \int_{a}^{b} f(t)g(t) dt - \frac{1}{(b-a)^{2}} \int_{a}^{b} f(t) dt \times \int_{a}^{b} g(t) dt \right|$$

$$\leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma). \quad (15)$$

Further, the constant 1/4 is the best possible.

The following lemma gives some approximations that will be needed in the sequel.

Lemma 17. Let a and b be two positive real numbers. We consider the functions $f_{a,b}$, $g_{a,b}$ and $h_{a,b}$ defined on (0,1) by,

$$f_{a,b}(t) = \exp\left(-\frac{a}{t} - \frac{b}{1-t}\right), \ g_{a,b}(t) = t^a(1-t)^b, \ h_{a,b}(t) = \cos^a t \sin^b t.$$

Then we have,
$$\sup_{t \in (0,1)} f_{a,b}(t) = \exp\left(-(\sqrt{a} + \sqrt{b})^2\right), \sup_{t \in (0,1)} g_{a,b}(t) = \frac{a^a b^b}{(a+b)^{a+b}}$$

and $\sup_{t \in (0,1)} h_{a,b}(t) = \frac{a^{\frac{a}{2}} b^{\frac{b}{2}}}{(a+b)^{\frac{a+b}{2}}}.$

Proof. By studying the variations of $f_{a,b}$, $g_{a,b}$ and $h_{a,b}$ on (0,1), it can be shown that their maximums are attained, respectively, at t_0 , t_1 and t_2 such that

$$t_0 = \frac{\sqrt{a}}{\sqrt{a} + \sqrt{b}}, \ t_1 = \frac{a}{a+b}, \ t_2 = \arccos\left(\frac{\sqrt{a}}{\sqrt{a+b}}\right).$$

Computing $f_{a,b}(t_0)$, $g_{a,b}(t_1)$ and $h_{a,b}(t_2)$ we get the desired results after simple algebraic operations.

The first main result of this section reads as follows.

Theorem 18. Let x, y, x', y' > 0 and $p, q, p', q' \ge 0$. Then the following inequality holds:

$$\left| \mathcal{B}(x+x'+1,y+y'+1;p+p',q+q') - \mathcal{B}(x+1,y+1,p,q) \right| \times \left| \mathcal{B}(x'+1,y'+1;p',q') \right| \le \frac{1}{4} U_{x,y} U_{x',y'} V_{p,q} V_{p',q'},$$

where we set

$$U_{x,y} =: \frac{x^x y^y}{(x+y)^{x+y}} \text{ and } V_{p,q} =: \exp\left(-(\sqrt{p} + \sqrt{q})^2\right).$$
 (16)

Proof. For all $t \in (0,1)$ we define the following functions

$$f(t) = t^{x}(1-t)^{y} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right)$$

and

$$g(t) = t^{x'} (1-t)^{y'} \exp\left(-\frac{p'}{t} - \frac{q'}{1-t}\right).$$

It is obvious that f and g are defined and integrable on (0,1). By Lemma 17 we have, for any $t \in (0,1)$,

$$0 \le f(t) \le \frac{x^x y^y}{(x+y)^{x+y}} \exp\left(-(\sqrt{p} + \sqrt{q})^2\right) =: U_{x,y} V_{p,q}$$

and

$$0 \le g(t) \le \frac{(x')^{x'}(y')^{y'}}{(x'+y')^{x'+y'}} \exp\left(-(\sqrt{p'}+\sqrt{q'})^2\right) =: U_{x',y'} V_{p',q'}.$$

Substituting these in (15), with the previous explicit expressions of f and g, we get the desired result.

Remark 19. We note that the previous theorem is a generalization of [6, Theorem 8].

From Theorem 18 we immediately deduce the following corollary.

Corollary 20. Let x, y > 0 and $p, q \ge 0$. Then one has

$$\left| \mathcal{B}(2x+1,2y+1;2p,2q) - \left(\mathcal{B}(x+1,y+1;p,q) \right)^2 \right| \le \frac{1}{4} U_{x,y}^2 V_{p,q}^2.$$

The following result may be stated as well.

Theorem 21. Let $x, y \ge 1$ and $p, q \ge 0$. Then the following inequality holds:

$$\left| \mathcal{B}(x, y; p, q) - \mathcal{B}(x, 1; p, 0) \times \mathcal{B}(y, 1; q, 0) \right| \le \frac{1}{4} e^{-(p+q)}.$$
 (17)

Proof. Consider the functions f and g defined on (0,1) by

$$f(t) = t^{x-1} \exp\left(-\frac{p}{t}\right)$$
 and $g(t) = (1-t)^{y-1} \exp\left(-\frac{q}{1-t}\right)$.

Since $x, y \ge 1$ then for any $t \in (0, 1)$, we have that

$$0 \le f(t) \le e^{-p} \quad \text{and} \quad 0 \le g(t) \le e^{-q}.$$

According to (15) we then obtain

$$\left| \mathcal{B}(x,y;p,q) - \int_0^1 f(t) \, dt \, \int_0^1 g(t) \, dt \, \right| \le \frac{1}{4} e^{-(p+q)}. \tag{18}$$

It is obvious that

$$\int_0^1 f(t) \ dt =: \int_0^1 t^{x-1} \exp\left(-\frac{p}{t}\right) dt =: \mathcal{B}(x, 1; p, 0)$$

and by a change of variables u = 1 - t, one has

$$\int_0^1 g(t) dt =: \int_0^1 (1-t)^{y-1} \exp\left(-\frac{q}{1-t}\right) dt =: \mathcal{B}(y,1;q,0).$$

Substituting these in (18) we get (17), so completing the proof.

Remark 22. (i) The inequality (17) is an extension of [6, Theorem 9]. That is, if p = q = 0 then (17) is reduced to:

$$\forall x, y \ge 1 \qquad \left| B(x, y) - \frac{1}{xy} \right| \le \frac{1}{4}. \tag{19}$$

(ii) By (3) with Lemma 17 we easily deduce that the inequality

$$\mathcal{B}(x, y; p, q) \le B(x, y) \exp\left(-(\sqrt{p} + \sqrt{q})^2\right)$$

holds for any x, y > 0 and $p, q \ge 0$. This, when combined with (19), implies that the inequality

$$\mathcal{B}(x, y; p, q) \le \left(\frac{1}{xy} + \frac{1}{4}\right) \exp\left(-(\sqrt{p} + \sqrt{q})^2\right)$$

holds true for any $x, y \ge 1$ and $p, q \ge 0$.

We have the following result as well.

Theorem 23. Let x, y, x', y' > 0 and $p, q, p', q' \ge 0$. The following inequality holds

$$\left| \mathcal{B}\left(x + x' + \frac{1}{2}, y + y' + \frac{1}{2}; p + p', q + q'\right) - \frac{1}{\pi} \mathcal{B}(x + \frac{1}{2}, y + \frac{1}{2}; p, q) \times \mathcal{B}(x' + \frac{1}{2}, y' + \frac{1}{2}; p', q') \right| \le \frac{\pi}{4} U_{x,y} U_{x',y'} V_{p,q} V_{p',q'},$$
(20)

where $U_{x,y}$ and $V_{p,q}$, as $U_{x',y'}$ and $V_{p',q'}$, are defined by (16).

Proof. Let us rewrite (5), for the sake of clearness,

$$\mathcal{B}(x,y;p,q) = 2 \int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta \exp\left(-p\sec^2\theta - q\csc^2\theta\right) d\theta.$$

Consider the following functions defined on $(0, \frac{\pi}{2})$ by

$$f(\theta) = \cos^{2x} \theta \sin^{2y} \theta \exp(-p \sec^2 \theta - q \csc^2 \theta)$$

and

$$g(\theta) = \cos^{2x'} \theta \sin^{2y'} \theta \exp(-p' \sec^2 \theta - q' \csc^2 \theta)$$
.

By Lemma 17 we can write,

$$0 \le f(\theta) \le \frac{x^x y^y}{(x+y)^{x+y}} \exp\left(-(\sqrt{p} + \sqrt{q})^2\right) =: U_{x,y} V_{p,q}$$

and

$$0 \le g(\theta) \le \frac{(x')^{x'}(y')^{y'}}{(x'+y')^{x'+y'}} \exp\left(-(\sqrt{p'}+\sqrt{q'})^2\right) =: U_{x',y'} V_{p',q'}.$$

Applying the Grüss inequality (15) we get

$$\left| \frac{2}{\pi} \int_0^{\frac{\pi}{2}} f(\theta) g(\theta) \ d\theta - \left(\frac{2}{\pi}\right)^2 \int_0^{\frac{\pi}{2}} f(\theta) \ d\theta \times \int_0^{\frac{\pi}{2}} g(\theta) \ d\theta \right| \\
\leq \frac{1}{4} U_{x,y} \ U_{x',y'} \ V_{p,q} \ V_{p',q'}.$$

This latter inequality is equivalent to the following one

$$\left| \frac{1}{\pi} \mathcal{B} \left(x + x' + \frac{1}{2}, y + y' + \frac{1}{2}; p + p', q + q' \right) - \frac{1}{\pi^2} \mathcal{B} \left(x + \frac{1}{2}, y + \frac{1}{2}; p, q \right) \times \mathcal{B} \left(x' + \frac{1}{2}, y' + \frac{1}{2}; p', q' \right) \right| \le \frac{1}{4} U_{x,y} \ U_{x',y'} \ V_{p,q} \ V_{p',q'},$$

and hence (20). The proof is finished.

If in (20) we take x = x', y = y', p = p' and q = q', we immediately obtain the following corollary.

Corollary 24. Let x, y > 0 and $p, q \ge 0$. Then we have

$$\left| \pi \mathcal{B} \left(2x + \frac{1}{2}, 2y + \frac{1}{2}; 2p, 2q \right) - \left(\mathcal{B} \left(x + \frac{1}{2}, y + \frac{1}{2}; p, q \right) \right)^{2} \right|$$

$$\leq \frac{\pi^{2}}{4} U_{x,y}^{2} V_{p,q}^{2}.$$

$$(21)$$

In order to give more inequalities involving $\mathcal{B}(x, y; p, q)$ we need to state the following lemma [5, 6] which gives a weighted version of the Grüss inequality (15).

Lemma 25. Let f and g be two functions defined and integrable over (a,b) and satisfying that

$$\phi \le f(x) \le \Phi$$
 and $\gamma \le g(x) \le \Gamma$

for all $x \in (a,b)$, where ϕ, Φ, γ and Γ are fixed real constants. If h is a non-negative function defined and integrable on (a,b), then we have the following inequality

$$\left| \int_a^b h(t) \ dt \times \int_a^b f(t)g(t)h(t) \ dt - \int_a^b f(t)h(t) \ dt \times \int_a^b g(t)h(t) \ dt \right| \\ \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma) \left(\int_a^b h(t) \ dt \right)^2.$$

Now, the following result may be stated.

Theorem 26. Let x, x', y, y' > 0, $p, p', p'', q, q', q'' \ge 0$ and x'', y'' > -1. Then the following inequality holds,

$$\left| \mathcal{B}(x''+1,y''+1;p'',q'') \times \mathcal{B}(x+x'+x''+1,y+y'+y''+1;p+p'+p'',q+q'+q'') \right|
- \mathcal{B}(x+x''+1,y+y''+1;p+p'',q+q'') \times \mathcal{B}(x'+x''+1,y'+y''+1;p'+p'',q'+q'') \right|
\leq \frac{1}{4} U_{x,y} U_{x',y'} V_{p,q} V_{p',q'} \mathcal{B}^{2}(x''+1,y''+1;p'',q''),$$

where $U_{x,y}$ and $V_{p,q}$, as $U_{x',y'}$ and $V_{p',q'}$, are defined by (16).

Proof. Consider the following functions defined on (0,1) by

$$f(t) = t^{x}(1-t)^{y} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right),$$

$$g(t) = t^{x'}(1-t)^{y'} \exp\left(-\frac{p'}{t} - \frac{q'}{1-t}\right),$$

$$h(t) = t^{x''}(1-t)^{y''} \exp\left(-\frac{p''}{t} - \frac{q''}{1-t}\right).$$

Applying Lemma 25 to the previous functions, and with the help of Lemma 17, we immediately get the desired inequality.

The following result holds as well.

Theorem 27. Let x, y > 0 and $p, q \ge 0$. Then the following inequality

$$\begin{split} \left| \mathcal{B}(x,y;p,q) \times \mathcal{B}(x+x',y+y';p+p',q+q') \right. \\ \left. - \mathcal{B}(x+x',y;p+p',q) \times \mathcal{B}(x,y+y';p,q+q') \right| \\ \leq \frac{1}{4} e^{-(p'+q')} \mathcal{B}^2(x,y;p,q) \end{split}$$

holds true for any x', y', p', q' such that x + x' > 0, y + y' > 0, p + p' > 0 and q + q' > 0.

Proof. Let us consider the functions defined on (0,1) by $f(t) = t^{x'} \exp\left(-\frac{p'}{t}\right)$, $g(t) = (1-t)^{y'} \exp\left(-\frac{q'}{1-t}\right)$ and $h(t) = t^{x-1}(1-t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right)$.

It is clear that for all $t \in (0,1)$ we have

$$0 \le f(t) \le e^{-p'}$$
 and $0 \le g(t) \le e^{-q'}$.

By similar way as previous, we obtain the desired result when applying Lemma 25 with the above functions. The details are immediate and therefore omitted here. \Box

Remark 28. (i) Theorem 26 and Theorem 27 are generalizations of Theorem 18 and Theorem 21, respectively.

(ii) If in Theorem 26 and Theorem 27 we take $p=q=0,\ p'=q'=0$ and p''=q''=0, we get [6, Proposition 1] and [6, Proposition 2], respectively.

5. Inequalities via Ostrowski's inequality

In this section we will apply the Ostrowski inequality [9, 10], recalled below, in the aim to establish more inequalities involving the extended beta function $\mathcal{B}(x, y; p, q)$.

Lemma 29. Let f be a differentiable function on (a,b). Assume that, for all $x \in (a,b)$, we have $|f'(x)| \leq M$ for some fixed real number M > 0. Then,

for any $x \in (a, b)$, we have

$$\left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) dt \right| \le \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2} \right)^{2}}{(b-a)^{2}} \right) (b-a)M. \tag{22}$$

Further the constant 1/4 is the best possible.

As previously, $U_{x,y}$ and $V_{p,q}$ are defined by (16). Our main result in this section reads as follows.

Theorem 30. Let $p, q \ge 0$ and $t \in (0, 1)$. Then the inequality

$$\left| \mathcal{B}(x,y;p,q) - t^{x-1} (1-t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) \right|$$

$$\leq \left\{ \max\left(x-1,y-1\right) U_{x-2,y-2} + \max(p,q) U_{x-3,y-3} \right\} \left(\frac{1}{4} + \left(t - \frac{1}{2}\right)^{2}\right) V_{p,q}$$
(23)

holds for any x, y > 3. For p = q = 0, the inequality

$$\left| \mathcal{B}(x,y) - t^{x-1} (1-t)^{y-1} \right| \le \left\{ \max \left(x - 1, y - 1 \right) U_{x-2,y-2} \right\} \left(\frac{1}{4} + \left(t - \frac{1}{2} \right)^2 \right)$$

holds true for any x, y > 2.

Proof. For all $t \in (0,1)$ we set

$$\varphi(t) = t^{x-1}(1-t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right).$$

Elementary computation leads to

$$\varphi'(t) = t^{x-3}(1-t)^{y-3} \left[(x-1)t(1-t)^2 - (y-1)t^2(1-t) + p(1-t)^2 - qt^2 \right] \times \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right).$$

It is not hard to check that we have

$$\left| \varphi'(t) \right| \le \left\{ t^{x-2} (1-t)^{y-2} \left| (x-1)(1-t) - (y-1)t \right| + t^{x-3} (1-t)^{y-3} \left| (1-t)^2 p - t^2 q \right| \right\} \exp\left(-\frac{p}{t} - \frac{q}{1-t} \right).$$

One can easily see that

$$\max_{t \in (0,1)} |(x-1)(1-t) - (y-1)t| = \max(x-1, y-1)$$

and

$$\max_{t \in (0,1)} \left| (1-t)^2 p - t^2 q \right| = \max(p,q).$$

According to Lemma 17 and then to the Ostrowski inequality (22), we get the desired inequalities, thus completing the proof. \Box

The previous theorem has many consequences. In particular, taking $t = \cos^2 \theta$ with $\theta \in (0, \pi/2)$ we obtain the following corollary.

Corollary 31. For any x, y > 3, p, q > 0 and $\theta \in (0, \pi/2)$ we have the following estimation

$$\begin{aligned} & \left| \mathcal{B}(x, y; p, q) - \cos^{2x-2}\theta \sin^{2y-2}\theta \exp\left(-p\sec^{2}\theta - q\csc^{2}\theta\right) \right| \\ & \leq \frac{1}{4} \left\{ \max\left(x - 1, y - 1\right) U_{x - 2, y - 2} + \max(p, q) U_{x - 3, y - 3} \right\} \left(1 + \cos^{2}(2\theta)\right) V_{p, q}. \end{aligned}$$

If p = q = 0, then for any x, y > 2 we have

$$\left| \mathcal{B}(x,y) - \cos^{2x-2}\theta \sin^{2y-2}\theta \right| \\
\leq \frac{1}{4} \left\{ \max(x-1,y-1) \ U_{x-2,y-2} \right\} \left(1 + \cos^2(2\theta) \right).$$

Taking t = 1/2 in (23), or $\theta = \pi/4$ in the above corollary, we immediately obtain the following result.

Corollary 32. For any x, y > 3 and $p, q \ge 0$ one has

$$\left| \mathcal{B}(x,y;p,q) - \frac{\exp(-2p - 2q)}{2^{x+y-2}} \right| \le U_{x-2,y-2} + \max(p,q) U_{x-3,y-3} \right\} V_{p,q}.$$

If p = q = 0, the inequality

$$\left| \mathcal{B}(x,y) - \frac{1}{2^{x+y-2}} \right| \le \frac{1}{4} \max(x-1,y-1) \ U_{x-2,y-2}$$

holds for any x, y > 2.

Remark 33. If p = q = 0, then Theorem 30 and Corollary 32 coincide with [6, Theorem 14] and [6, Corollary 8], respectively.

We have the following result as well.

Theorem 34. Let x, y > 2, $p, q \ge 0$ and $\theta \in (0, \frac{\pi}{2})$. Then we have

$$\left| \frac{1}{\pi} \mathcal{B}(x, y; p, q) - \cos^{2x-1} \theta \sin^{2y-1} \theta \exp\left(-p \sec^{2} \theta - q \csc^{2} \theta\right) \right|
\leq \frac{\pi}{2} \left(\frac{1}{4} + \frac{4(\theta - \frac{\pi}{4})^{2}}{\pi^{2}} \right) M_{x, y, p, q} V_{p, q}, \quad (24)$$

where $M_{x,y,p,q} = \left(\max(2x-1,2y-1) U_{x-1,y-1} + 2 \max(p,q) U_{x-2,y-2} \right)$ and $U_{x,y}$ and $V_{p,q}$ are defined by (16).

If p = q = 0, then the inequality

$$\left| \frac{1}{\pi} \mathcal{B}(x,y) - \cos^{2x-1} \theta \sin^{2y-1} \theta \right|$$

$$\leq \frac{\pi}{2} \left(\frac{1}{4} + \frac{4(\theta - \frac{\pi}{4})^2}{\pi^2} \right) \max(2x - 1, 2y - 1) U_{x-1,y-1}$$

holds true for all x, y > 1.

Proof. First, we mention that following (5) we have

$$\mathcal{B}(x, y; p, q) = 2 \int_0^{\pi/2} \Phi_{x, y; p, q}(\theta) \ d\theta,$$

where, for fixed x, y; p, q, we set

$$\Phi_{x,y;p,q}(\theta) =: \Phi(\theta) =: \cos^{2x-1}\theta \sin^{2y-1}\theta \exp\left(-p\sec^2\theta - q\csc^2\theta\right).$$

We have, for all $\theta \in (0, \frac{\pi}{2})$,

$$\frac{d}{d\theta}\Phi(\theta) = -\exp\left(-p\sec^2\theta - q\csc^2\theta\right) \times \left[\cos^{2x-2}\theta\sin^{2y-2}\theta\left(-(2x-1)\sin^2\theta + (2y-1)\cos^2\theta\right) + \cos^{2x-4}\theta\sin^{2y-4}\theta\left(2q\cos^4\theta - 2p\sin^4\theta\right)\right].$$

By Lemma 17, with the help of (16), we can write

$$\left| \frac{d}{d\theta} \Phi(\theta) \right| \le V_{p,q} \left[U_{x-1,y-1} \max_{\theta \in (0, \frac{\pi}{2})} \left| -(2x-1)\sin^2\theta + (2y-1)\cos^2\theta \right| + 2U_{x-2,y-2} \max_{\theta \in (0, \frac{\pi}{2})} \left| q\cos^4\theta - p\sin^4\theta \right| \right].$$

It is not hard to check that

$$\max_{\theta \in (0, \frac{\pi}{2})} \left| -(2x-1)\sin^2\theta + (2y-1)\cos^2\theta \right| = \max(2x-1, 2y-1)$$

and

$$\max_{\theta \in (0, \frac{\pi}{2})} \left| q \cos^4 \theta - p \sin^4 \theta \right| = \max(p, q).$$

In summary, we have shown that the following inequality

$$\left| \frac{d}{d\theta} \Phi(\theta) \right| \le V_{p,q} \left(\max(2x - 1, 2y - 1) U_{x-1,y-1} + 2 \max(p,q) U_{x-2,y-2} \right)$$

holds true for all $\theta \in (0, \pi/2)$. Now, if we apply the Ostrowski inequality (22) to the function $\theta \longmapsto \Phi(\theta)$ we get (24), so completing the proof.

Setting $\theta = \pi/4$ in (24) we obtain the following result.

Corollary 35. For any x, y > 2 and $p, q \ge 0$ we have

$$\left| \mathcal{B}(x,y;p,q) - \pi \frac{e^{-2(p+q)}}{2^{x+y-1}} \right| \le \frac{\pi^2}{8} \left(\max(2x-1,2y-1) U_{x-1,y-1} + 2 \max(p,q) U_{x-2,y-2} \right) V_{p,q}.$$

For p = q = 0, the inequality

$$\left| B(x,y) - \frac{\pi}{2^{x+y-1}} \right| \le \frac{\pi^2}{8} \max(2x-1,2y-1) U_{x-1,y-1}$$

holds for all x, y > 1.

In order to give more inequalities about $\mathcal{B}(x, y; p, q)$ we need to recall the following lemma, see [6, Corollary 11].

Lemma 36. Let φ be a continuous and differentiable function on (a,b). Assume that φ is integrable on (a,b) and φ' is continuous on (a,b) with $\|\varphi'\| = :$ $\int_a^b |\varphi'(t)| dt < \infty$. Then the following inequality

$$\left| \int_{a}^{b} \varphi(t)dt - \varphi(x) (b-a) \right| \leq \left(\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right) \|\varphi'\|$$

holds for all $x \in (a, b)$.

Now, another main result can be stated as follows.

Theorem 37. Let x, y > 2, $p, q \ge 0$ and $t \in (0,1)$. Then following inequality holds true,

$$\left| \mathcal{B}(x, y; p, q) - t^{x-1} (1 - t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{1 - t}\right) \right|$$

$$\leq \left\{ \max\left(x - 1, y - 1\right) \mathcal{B}(x - 1, y - 1; p, q) + \max(p, q) \mathcal{B}(x - 2, y - 2; p, q) \right\} \left(\frac{1}{2} + \left|t - \frac{1}{2}\right|\right)$$

Further, if p = q = 0, then the inequality

$$\left| \mathcal{B}(x,y) - t^{x-1} (1-t)^{y-1} \right| \le \max \left(x - 1, y - 1 \right) \mathcal{B}(x-1,y-1) \left(\frac{1}{2} + \left| t - \frac{1}{2} \right| \right)$$

holds for any x, y > 1.

Proof. For all $t \in (0,1)$ we set

$$\varphi(t) = t^{x-1}(1-t)^{y-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right).$$

As in the proof of Theorem 30, we have

$$\varphi'(t) = t^{x-3}(1-t)^{y-3} \left[(x-1)t(1-t)^2 - (y-1)t^2(1-t) + p(1-t)^2 - qt^2 \right] \times \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right)$$

and

$$|\varphi'(t)| \le \left[\max(x-1,y-1)t^{x-2}(1-t)^{y-2} + \max(p,q)t^{x-3}(1-t)^{y-3}\right] \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right).$$

Integrating both sides of this latter inequality with respect to $t \in (0,1)$, we find

$$\|\varphi'\| \le \max(x-1, y-1) \mathcal{B}(x-1, y-1; p, q) + \max(p, q) \mathcal{B}(x-2, y-2; p, q) < \infty.$$

So, we can apply Lemma 36 for the function φ on the interval (0,1). This concludes the proof.

Taking t = 1/2 in the previous theorem we obtain the following result.

Corollary 38. For x, y > 2 and $p, q \ge 0$ we have

$$\begin{split} & \left| \mathcal{B}(x, y; p, q) - \frac{\exp(-2p - 2q)}{2^{x + y - 2}} \right| \\ & \leq \frac{1}{2} \Big\{ \max \big(x - 1, y - 1 \big) \, \mathcal{B}(x - 1, y - 1; p, q) + \max(p, q) \, \mathcal{B}(x - 2, y - 2; p, q) \Big\}. \end{split}$$

If p = q = 0, the inequality

$$\left| \mathcal{B}(x,y) - \frac{1}{2^{x+y-2}} \right| \le \frac{1}{2} \max (x-1,y-1) \, \mathcal{B}(x-1,y-1)$$

holds for all x, y > 1.

Remark 39. For p = q = 0, Theorem 37 and Corollary 38 coincide with [6, Theorem 18] and [6, Corollary 13], respectively.

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