### International Journal of Applied Mathematics

Volume 34 No. 4 2021, 607-619

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

doi: http://dx.doi.org/10.12732/ijam.v34i4.1

# ESTIMATES FOR FRACTIONAL POWERS AND LOGARITHM OF OPERATORS WITH HILBERT-SCHMIDT RESOLVENTS AND PERTURBATION RESULTS

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**Abstract:** Let A be a closed operator on a separable Hilbert space with the spectrum in the open right half-plane and a bounded Hermitian component, and let the resolvent of A be a Hilbert-Schmidt operator. The paper deals with the function

 $h_{\mu}(A) = \int_{0}^{\infty} (A + tI)^{-1} d\mu(t),$ 

where  $\mu$  is a nondecreasing function and I is the unit operator. We establish norm estimates and perturbations results for  $h_{\mu}(A)$ . As particular cases the fractional powers and logarithm of A are considered.

AMS Subject Classification: 47A56, 47A63, 47A55

**Key Words:** Hilbert space; fractional power of operators; operator logarithm Hirsch type functions; perturbations

#### 1. Introduction and statement of the main result

Let  $\mathcal{H}$  be a separable complex Hilbert space with a scalar product  $\langle .,. \rangle$ , the norm  $\|.\| = \sqrt{\langle .,. \rangle}$  and unit operator I. For a linear operator T, D(T) is the domain,  $\sigma(T)$  denotes the spectrum,  $R_z(T) = (T - zI)^{-1}$  ( $z \notin \sigma(T)$ ) is the resolvent,  $T^*$  is the adjoint operator. If T is bounded, then  $\|T\|$  is its operator norm. By  $\mathcal{S}_p$  ( $1 \leq p < \infty$ ) we denote the Schatten - von Neumann ideal of

Received: June 21, 2021 © 2021 Academic Publications

compact operators K with the finite norm  $N_p(K) := [\operatorname{trace} (KK^*)^{p/2}]^{1/p}$ . So  $S_2$  is the Hilbert-Schmidt ideal.

Throughout the paper, A and  $\tilde{A}$  are closed linear operators on  $\mathcal{H}$ . The paper is devoted to the norm estimates for operator valued functions of the form

 $h_{\mu}(A) = \int_{0}^{\infty} (A + tI)^{-1} d\mu(t),$ 

where  $\mu(t)$  is a nondecreasing function defined on  $[0,\infty)$  and satisfying the condition

 $\int_0^\infty \frac{d\mu(t)}{t+1} < \infty.$ 

We also derive norm estimates for the difference  $h_{\mu}(A) - h_{\mu}(\tilde{A})$ .

The considered functions belong to the class of so called Hirsch functions [9]. The important examples here are the operator fraction powers and operator logarithm. The theories of the fraction powers and logarithm are well developed. For the classical results, see [8, 9]. The recent investigations can be found in [1, 2, 4, 5], [10]-[15] and the references given therein.

At the same time, to the best of our knowledge, the norm estimates and perturbation results for these operator functions are derived mainly in the case of self-adjoint operators, cf. [3]. Below we do not require that A and  $\tilde{A}$  are self-adjoint.

Put  $A_I := (A - A^*)/2i$  and  $A_R(A + A^*)/2$  and assume that

$$\beta(A) := \inf \operatorname{Re} \sigma(A) > 0. \tag{1.1}$$

It is also supposed that

$$D(A^*) = D(A), A_I \text{ is bounded}, A_R \text{ is invertible and } A_R^{-1} \in \mathcal{S}_2.$$
 (1.2)

Denote by  $a_k$  the eigenvalues of A with their multiplicities taken into account and enumerated in the non-decreasing order of their absolute values:  $|a_k| \leq |a_{k+1}|$  (k = 1, 2, ...).

We will check that conditions (1.1), (1.2) yield that

$$\tau(A) := \left[\sum_{k=1}^{\infty} \frac{1}{|a_k|^2}\right]^{1/2} < \infty.$$

Put

$$\phi(x) := \sum_{k=0}^{\infty} \frac{x^k}{\sqrt{k!}} \quad (x \ge 0).$$

Now we are in a position to formulate the main result of this paper.

**Theorem 1.** Let conditions (1.1), (1.2) hold. Then

$$||h_{\mu}(A)|| \le \phi(2||A_I||\tau(A))J(A,\mu),$$
 (1.3)

where

$$J(A,\mu) := \int_0^\infty \frac{d\mu(t)}{t + \beta(A)}.$$

The proof of this theorem is divided into a series of lemmas, which are presented in the next two sections.

If A is selfadjoint, then  $\phi(2||A_I||\tau(A)) = \phi(0) = 1$  and by Theorem 1  $||h_\mu(A)|| \leq J(A,\mu)$ . Note that by the Schwarz inequality,

$$\sum_{k=0}^{\infty} \frac{x^k}{(k!)^{1/2}} = \sum_{k=0}^{\infty} \frac{(\sqrt{2}x)^k}{(\sqrt{2})^k (k!)^{1/2}} \le (\sum_{j=0}^{\infty} \frac{2^j x^{2j}}{j!} \sum_{k=0}^{\infty} \frac{1}{2^k})^{1/2} = \sqrt{2}e^{x^2}.$$

Thus

$$\phi(x) \le \sqrt{2}e^{x^2} \quad (x \ge 0).$$
 (1.4)

In addition,

$$\phi(x) = 1 + x \sum_{k=1}^{\infty} \frac{x^{k-1}}{\sqrt{k!}} \le 1 + x \sum_{k=1}^{\infty} \frac{x^{k-1}}{\sqrt{(k-1)!}} = 1 + x\phi(x)$$

and consequently,

$$\phi(x) \le 1 + \sqrt{2x}e^{x^2} \quad (x \ge 0). \tag{1.5}$$

For  $x \le 1 - 1/\sqrt{2}$ , (1.5) is sharper than (1.4).

Theorem 1, (1.4) and (1.5) imply

$$||h_{\mu}(A)|| \le \sqrt{2}J(A,\mu)\exp[4||A_I||^2\tau^2(A)]$$
 (1.6)

and

$$||h_{\mu}(A)|| \le J(A,\mu)(1 + 2\sqrt{2}||A_I||\tau(A)\exp[4||A_I||^2\tau^2(A)]). \tag{1.7}$$

#### 2. Existence of the Schur basis

Recall the Keldysh theorem, cf. [7, Theorem V. 8.1].

**Theorem 2.** Let  $B = K_0(I + K_1)$ , where  $K_0 = K_0^* \in \mathcal{S}_r$  for some  $r \in [1, \infty)$  and  $K_1$  is compact. In addition, let from Bf = 0  $(f \in \mathcal{H})$  it follows that f = 0. Then B has a complete in  $\mathcal{H}$  system of the root vectors.

Note that due to (1.1), A is boundedly invertible.

**Lemma 3.** Under conditions (1.1) and (1.2) the operator  $A^{-1}$  has a complete system of root vectors and  $A^{-1} \in \mathcal{S}_2$ .

*Proof.* Since  $A=A_R+iA_I=(I+iA_IA_R^{-1})A_R$ , we have  $I+iA_IA_R^{-1}=AA_R^{-1}$ . But

$$(AA_R^{-1})^{-1} = A_R A^{-1} = (A - iA_I)A^{-1} = I - iA_I A^{-1}.$$

Since  $A_I$  is bounded, operator  $I + iA_R^{-1}A_I$  is boundedly invertible. We have

$$A^{-1} = A_R^{-1} (I + iA_I A_R^{-1})^{-1} \in \mathcal{S}_2.$$

In addition,  $(I + iA_IA_R^{-1})^{-1} = I + K_2$ , where

$$K_2 = (I + iA_I A_R^{-1})^{-1} - I = -iA_I A_R^{-1} (I + iA_I A_R^{-1})^{-1} \in \mathcal{S}_2.$$

So  $A^{-1} = A_R^{-1}(I + K_2)$ , and from  $A^{-1}f = 0$   $(f \in \mathcal{H})$  it follows that f = 0. Now the Keldysh theorem implies the required result.

**Lemma 4.** Let an operator A on  $\mathcal{H}$  have a compact resolvent, and for some  $b \notin \sigma(A)$ ,  $(A - bI)^{-1}$  have a complete system of root vectors. Then there is an orthogonal normal (Schur) basis  $\{e_k\}_{k=1}^{\infty}$ , in which A is representable by a triangular matrix  $(a_{jk})_{1 \leq j \leq k \leq \infty}$ :

$$Ae_k = \sum_{j=1}^k a_{jk}e_j \text{ and } \langle Ae_k, e_k \rangle = a_k \ (k = 1, 2, ...),$$
 (2.1)

where  $a_k$  (k = 1, 2, ...) are the eigenvalues of A.

*Proof.* Put M = A - bI. Due to [7, Lemma I.4.1], there is an orthonormal basis  $\{e_k\}_{k=1}^{\infty}$ , in which  $M^{-1}$  is represented by a triangular matrix  $(d_{jk})_{1 \leq j \leq k \leq \infty}$ :

$$M^{-1}e_k = \sum_{j=1}^k d_{jk}e_j \quad (k = 1, 2, ...).$$

Due to the spectrum mapping theorem,  $d_{kk} = 1/(a_k - b)$  are the eigenvalues of  $M^{-1}$ . Denote

$$P_k = \sum_{j=1}^k \langle ., e_j \rangle e_j \ (k = 1, 2, ...); P_0 = 0.$$

Then

$$M^{-1}P_k = P_k M^{-1}P_k \ (k = 1, 2, ...).$$
 (2.2)

Besides.

$$\Delta P_k M^{-1} \Delta P_k = \frac{1}{a_k - b} \Delta P_k \ (\Delta P_k = P_k - P_{k-1}; \ k = 1, 2, \dots).$$
 (2.3)

Let us check that

$$MP_k f = P_k M P_k f \ (k = 1, 2, ...; \ f \in D(A)).$$
 (2.4)

Indeed,  $M^{-1}P_k$  is an invertible  $k \times k$  matrix, and therefore,  $M^{-1}P_k\mathcal{H}$  is dense in  $P_k\mathcal{H}$ . Since  $\Delta P_jP_k=0$  for j>k, we have  $0=\Delta P_jMM^{-1}P_k=\Delta P_jMP_kM^{-1}P_k$ . Hence  $\Delta P_jMf=0$  for any  $f\in P_k\mathcal{H}$ . So (2.4) is valid and  $\Delta P_jMe_k=0$  for k< j. Hence,  $\Delta P_jM=\Delta P_jM(I-P_{j-1})$ , and

$$\Delta P_j = \Delta P_j \Delta P_j = \Delta P_j M M^{-1} \Delta P_j = \Delta P_j M (I - P_{j-1}) P_j M^{-1} \Delta P_j$$
$$= \Delta P_j M \Delta P_j M^{-1} \Delta P_j = \Delta P_j M \Delta P_j d_{jj}.$$

So

$$\Delta P_j M \Delta P_j = \frac{1}{d_{jj}} \Delta P_j = (a_j - b) \Delta P_j. \tag{2.5}$$

From (2.4) and (2.5) it follows

$$AP_k f = P_k A P_k f \ (k = 1, 2, ...; \ f \in D(A))$$
 (2.6)

and

$$\Delta P_i A \Delta P_i = a_i \Delta P_i \quad (j = 1, 2, \dots). \tag{2.7}$$

This implies the required result.

From Lemmas 3 and 4 it follows

Corollary 5. Let conditions (1.1) and (1.2) hold. Then there is an orthogonal normal basis  $\{e_k\}_{k=1}^{\infty}$ , such that (2.1) is valid, and therefore, A = S + W, where  $Se_k = a_k e_k$  (k = 1, 2, ...) (the diagonal part of A) and W is defined by

$$We_k = \sum_{j=1}^{k-1} e_j a_{jk} \quad (k = 2, 3, ...),$$

Therefore,  $\sigma(S) = \sigma(A) = \{a_k\}$  and by the Weyl inequalities [7, Section II.3]

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^2} \le N_2^2(A^{-1}),$$

and consequently,  $S^{-1} \in \mathcal{S}_2$ .

#### 3. Proof of Theorem 1

Put

$$S_I := (S - S^*)/2i = \sum_{j=1}^{\infty} (\text{Im } a_j) \Delta P_j$$

and  $W_I = A_I - S_I = (W - W^*)/2i$ . Since  $P_{k-1}WP_k = WP_k$ , it is not hard to check that  $\langle We_k, e_k \rangle = 0$ . In addition,  $\langle Ae_k, e_k \rangle = \langle Se_k, e_k \rangle = a_k$ , and we have  $\langle A_Ie_k, e_k \rangle = \langle S_Ie_k, e_k \rangle = \text{Im } a_k$ . So  $|\text{Im } a_k| \leq ||A_I||$  and  $||S_I|| = \sup_k |\text{Im } a_k| \leq ||A_I||$ .

Furthermore, note that  $P_{k-1}W^*P_k=0$  and thus  $P_{k-1}W^*e_k=0$ , and

$$||We_k|| = ||P_{k-1}We_k|| = ||P_{k-1}(W - W^*)e_k|| \le ||(W - W^*)e_k||$$
  
=  $2||W_Ie_k||$ .

With  $c_k = \text{Im } a_k$  we have

$$\langle A_I e_k, A_I e_k \rangle = \langle (W_I + c_k) e_k, (W_I + c_k) e_k \rangle = \langle W_I e_k, W_I e_k \rangle$$
$$+ c_k \langle W_I e_k, e_k \rangle + c_k \langle e_k, W_I e_k \rangle.$$

But  $\langle W_I e_k, e_k \rangle = 0$  and therefore  $\langle A_I e_k, A_I e_k \rangle = \rangle W_I e_k, W_I e_k \rangle + c_k^2$ . Consequently,

$$||We_k||^2 \le 4||W_Ie_k||^2 \le 4(||A_Ie_k||^2 - |\operatorname{Im} a_k|^2). \tag{3.1}$$

Hence,

$$N_2^2(WS^{-1}) = \sum_{k=1}^{\infty} \|WS^{-1}e_k\|^2 = \sum_{k=1}^{\infty} \frac{1}{|a_k|^2} \|We_k\|^2$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{|a_k|^2} \sup_{k} \|We_k\|^2 \leq 4 \|A_I\|^2 N_2^2(S^{-1}). \tag{3.2}$$

Lemma 6. Let conditions (1.1) and (1.2) hold. Then

$$\|(A-zI)^{-1}\| \le \frac{1}{d(A,z)}\phi(2\|A_I\|N_2((S-zI)^{-1})) \ (z \notin \sigma(A)),$$

where  $d(A, z) = \inf_k |a_k - z|$ .

*Proof.* Making use of Corollary 5, we obtain  $A - Z = S + W - zI = (I - B_z)(S - zI)$ , where  $B_z := -W(S - Iz)^{-1}$  ( $z \notin \sigma(S) = \sigma(A)$ ). Since W is

bounded and  $(S-z)^{-1} \in \mathcal{S}_{@}$ , we have  $B_z \in \mathcal{S}_2$ . Take into that  $P_{k-1}B_zP_k = B_zP_k$ , and thus  $B_z$  is the triangular compact matrix with the zero diagonal. Therefore it is the limit of nilpotent matrices in the operator norm. Hence, due to [7, Theorem I.4.1],  $B_z$  is quasi-nilpotent, and we can write

$$(I - B_z)^{-1} = \sum_{k=0}^{\infty} B_z^k$$

So,

$$||(Iz - A)^{-1}|| \le ||(S - Iz)^{-1}|| ||(I - B_z)^{-1}|| \le ||(S - Iz)^{-1}|| \sum_{k=0}^{\infty} ||B_z^k||.$$
 (3.3)

Due to [6, Corollary 7.4], for any  $K \in \mathcal{S}_2$ .

$$||K^k|| \le \frac{N_2^k(K)}{(k!)^{1/2}} \quad (k = 1, 2, ...).$$

Making use of this inequality, we get

$$||B_z^k|| \le \frac{N_2^k(B_z)}{(k!)^{1/2}} \quad (k = 1, 2, ...).$$

But by (3.2)

$$\begin{split} N_2^2(B_z) &= N_2^2(W(S-zI)^{-1}) = \sum_{k=1}^{\infty} \|W(S-z)^{-1}e_k\|^2 \\ &= \sum_{k=1}^{\infty} \frac{1}{|a_k - z|^2} \|We_k\|^2 \\ &\leq \sum_{k=1}^{\infty} \frac{1}{|a_k - z|^2} \sup_k \|We_k\|^2 \leq 4 \|A_I\|^2 N_2^2((S-zI)^{-1}). \end{split}$$

Thus

$$||B_z^k|| \le \frac{2||A_I||^k N_2^k ((S-zI)^{-1})}{(k!)^{1/2}} \quad (k=1,2,\ldots).$$

Moreover,

$$||(S - zI)^{-1}|| = \frac{1}{d(A, z)},$$

Now (3.3) implies the required result.

If A is self-adjoint, then by Lemma 6

$$||(A - zI)^{-1}|| \le \frac{1}{d(A, z)}$$

 $(z \not\in \sigma(A)).$ 

Furthermore, since  $\frac{1}{a_k-z}$  are the eigenvalues of  $(A-zI)^{-1}$ , making use of the Weyl inequalities, we have

$$N_2^2((S-zI)^{-1}) = \sum_{k=1}^{\infty} \frac{1}{|a_k - z|^2} \le N_2^2((A-zI)^{-1}) \quad (z \notin \sigma(A)). \tag{3.4}$$

Let Re  $a_k = b_k$  and Im  $a_k = c_k$ . Then for any  $t \ge 0$ ,  $d(A, -t) \ge \beta(A) + t$  and

$$|a_k + t|^2 = (b_k + t)^2 + c_k^2 = \ge (b_k^2 + 2b_k t + t^2 + c_k^2 = |a_k|^2 + 2b_k t + t^2.$$

Hence,  $N_2((S+t)^{-1}) = \tau(A,t)$ , where

$$\tau(A,t) := \left(\sum_{k=1}^{\infty} \frac{1}{|a_k|^2 + 2b_k t + t^2}\right)^{1/2} \le \tau(A,0) = \tau(A). \tag{3.5}$$

Thus Lemma 6 yields

Corollary 7. Let conditions (1.1), (1.2) hold. Then

$$\|(A+tI)^{-1}\| \le \frac{1}{(t+\beta(A))}\phi(2\|A_I\|\tau(A,t))$$

$$\leq \frac{1}{(t+\beta(A))}\phi(2||A_I||\tau(A)) \ (t\geq 0).$$

and therefore,

$$||h_{\mu}(A)|| \leq \int_{0}^{\infty} \frac{\phi(2||A_{I}||\tau(A,t))}{t + \beta(A)} d\mu(t).$$

*Proof of Theorem 1.* The assertion of the theorem follows from Corollary 7 and (3.5).

## 4. Perturbations of the Hirsch type functions

Let  $\tilde{A}$  and A be linear operators on  $\mathcal{H}$  with the same dense domain:

$$D(\tilde{A}) = D(A)$$
 and  $q := ||A - \tilde{A}|| < \infty$ . (4.1)

**Lemma 8.** Let the conditions (4.1),

$$\|(A+t)^{-1}\| \le \frac{m_1}{1+t} \quad (m_1 = const, t \ge 0),$$
 (4.2)

and

$$qm_1 < 1 \tag{4.3}$$

hold. Then

$$||h_{\mu}(\tilde{A})|| \leq \frac{m_1}{1 - qm_1} \int_0^{\infty} \frac{d\mu(t)}{1 + t}$$

and

$$||h_{\mu}(\tilde{A}) - h_{\mu}(A)|| \le \frac{qm_1^2}{1 - qm_1} \int_0^\infty \frac{d\mu(t)}{(1+t)^2}.$$

*Proof.* Since  $||(A+It)^{-1}|| \le m_1$   $(t \ge 0)$ , by the Hilbert identity

$$(\tilde{A} + It)^{-1} - (\tilde{A} + It)^{-1} = (\tilde{A} + It)^{-1}(\tilde{A} + It)^{-1}(A - \tilde{A})(\tilde{A} + It)^{-1}$$
(4.4)

and (4.3) we get

$$\|(\tilde{A}+It)^{-1}\| \le \frac{\|(A+It)^{-1}\|}{1-q\|(A+It)^{-1}\|} \le \frac{\|(A+It)^{-1}\|}{1-qm_1} \ (t \ge 0).$$

Hence, condition (4.2) implies

$$\|(\tilde{A}+It)^{-1}\| \le \frac{m_1}{(1-qm_1)(1+t)}.$$

Moreover, by (4.4)

$$\|(\tilde{A} + It)^{-1} - (A + It)^{-1}\| \le \frac{qm_1^2}{(1 - qm_1)(1 + t)^2}.$$

Consequently,

$$||h_{\mu}(\tilde{A})|| \le \int_{0}^{\infty} ||(\tilde{A} + it)^{-1}||d\mu(t)| \le m_{1} \int_{0}^{\infty} \frac{d\mu(t)}{(1 - qm_{1})(1 + t)}$$

and

$$||h_{\mu}(\tilde{A}) - h_{\mu}(A)|| \le \int_{0}^{\infty} ||(A+it)^{-1} - (\tilde{A}+it)^{-1}||d\mu(t)||$$

$$\leq \frac{qm_1^2}{1 - qm_1} \int_0^\infty \frac{d\mu(t)}{(1+t)^2},$$

as claimed.

Let conditions (1.1), (1.2) hold. Then from Corollary 7 it follows that

$$\|(A+It)^{-1}\| \le \phi(2\|A_I\|\tau(A))\frac{1}{\beta(A)+t} \le \frac{m_0(A)}{1+t},$$

where

$$m_0(A) = \phi(2||A_I||\tau(A)) \sup_{t>0} \frac{1+t}{\beta(A)+t}.$$

Simple calculations show that

$$m_0(A) = \phi_{\mu}(2||A_I||\tau(A)) \begin{cases} 1 & \text{if } \beta(A) > 1, \\ \frac{1}{\beta(A)} & \text{if } \beta(A) \le 1. \end{cases}$$

Now Lemma 8 implies

Corollary 9. Let the conditions (1.1), (1.2), (4.1) and  $qm_0(A) < 1$  hold. Then,

$$||h_{\mu}(\tilde{A})|| \le \frac{m_0(A)}{1 - qm_0(A)} \int_0^\infty \frac{d\mu(t)}{1 + t}$$

and

$$||h_{\mu}(\tilde{A}) - h_{\mu}(A)|| \le \frac{qm_0^2(A)}{1 - qm_0(A)} \int_0^{\infty} \frac{d\mu(t)}{(1+t)^2}.$$

#### 5. Fractional powers

Recall that a closed densely defined operator A is called a non-negative one, if  $(-\infty,0)$  is included in the resolvent set and  $\sup_{t>0} ||t(A+t)^{-1}|| < \infty$ , cf. [9, p. 1]. For a closed non-negative operator A, the negative fractional power can be a defined by the formula

$$A^{-\nu} = \frac{\sin(\pi\nu)}{\pi} \int_0^\infty t^{-\nu} (A + It)^{-1} dt \ (0 < \nu < 1), \tag{5.1}$$

cf. [9]. Under condition (4.2) A is non-negative.

If conditions (1.1), (1.2) hold, then due to Corollary 3.2, A is non-negative and by Theorem 1

$$||A^{-\nu}|| \le \frac{\sin(\pi\nu)}{\pi} \phi(2||A_I||\tau(A)) \int_0^\infty \frac{t^{-\nu}dt}{t + \beta(A)}$$

$$\le m_0(A) \frac{\sin(\pi\nu)}{\pi} \int_0^\infty \frac{t^{-\nu}dt}{1 + t}.$$
(5.2)

If, in addition, the conditions (4.1) and  $qm_0(A) < 1$  are fulfilled, then by Corollary 4.2,

$$\|\tilde{A}^{-\nu}\| \le \frac{m_0(A)}{1 - qm_0(A)} \int_0^\infty \frac{t^{-\nu}dt}{1 + t}$$
 (5.3)

and

$$\|\tilde{A}^{-\nu} - A^{-\nu}\| \le \frac{qm_0^2(A)}{1 - qm_0(A)} \int_0^\infty \frac{t^{-\nu}dt}{(1+t)^2}.$$
 (5.4)

#### 6. Logarithm

By [9, formula (10.3), p. 245] for a non-negative operator with a dense range R(A), we can define the logarithm by

$$\log(A)y = \int_0^\infty (t+A)^{-1} (Ay - y) \frac{dt}{t+1} \quad (y \in D(A) \cap R(A))$$
 (6.1)

(see also [9, p. 246, line -7]).

Let conditions (1.1), (1.2) hold. Then by Theorem 1

$$\|\int_0^\infty (t+A)^{-1} \frac{dt}{t+1}\| \le \phi(2\|A_I\|\tau(A))J_l,$$

where

$$J_l(A) := \int_0^\infty \frac{dt}{(t+\beta(A))(t+1)} \le m_0(A) \int_0^\infty \frac{dt}{(t+1)^2} = m_0(A).$$

Since A is boundedly invertible we have  $R(A) = \mathcal{H}$ . Now (6.1) implies

$$\|\log(A)y\| \le \phi(2\|A_I\|\tau(A))J_l\|Ay - y\| \le m_0(A)\|Ay - y\|$$

$$(y \in D(A)), \tag{6.2}$$

provided conditions (1.1), (1.2) hold. Furthermore, according to (6.1),

$$\log(\tilde{A})y - \log(A)y = \int_0^\infty [(t+\tilde{A})^{-1}(\tilde{A}y - y) - (t+A)^{-1}(\tilde{A}y - y)] \frac{dt}{t+1}$$
$$= \int_0^\infty [(t+\tilde{A})^{-1} - (t+A)^{-1}] \frac{dt}{t+1} (\tilde{A}y - y) + \int_0^\infty (t+A)^{-1} \frac{dt}{t+1} (\tilde{A} - A)y.$$

$$\|\log(\tilde{A})y - \log(A)y\| \le \|\int_0^\infty [(t+\tilde{A})^{-1} - (t+A)^{-1}] \frac{dt}{t+1} \|\|\tilde{A}y - y\|\|$$
$$+ \|\int_0^\infty (t+A)^{-1} \frac{dt}{t+1} \|\|(A-\tilde{A})y\|.$$

Making use of Corollary 4.2, we can assert that

So

$$\|\log(\tilde{A})y - \log(A)y\| \le \frac{qm_0^2(A)}{(1 - qm_0(A))} \int_0^\infty \frac{dt}{(t+1)^3} \|\tilde{A}y - y\|$$

$$+qm_0(A)\|y\| = \frac{qm_0^2(A)}{2(1 - qm_0(A))} \|\tilde{A}y - y\| + qm_0(A)\|y\|,$$
(6.3)

provided the conditions (1.1), (1.2), (4.1) and  $qm_0(A) < 1$  hold.

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