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THE THREE-PARAMETER BLACK AND SCHOLES PROCESS

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Abstract: In this paper, we propose a new study of the Black and Scholes Process (BSP). The main objective is to add a threshold parameter to the Black and Scholes Process. Using Kolmogorov equations, we obtain the probability density function and the moments of the process. Estimators of the parameters are studied by considering discrete sampling of the sample trajectories of the model and then using the maximum likelihood method and the Wicksell method.

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Key Words: Black and Scholes model; three-parameter lognormal process; maximum likelihood; diffusion process

1. Introduction

Nowadays, financial sciences play a major role in increasing the wealth of a country and in its competitiveness. The increasingly efficient tools to equip them allow it to manage the risks. Their role and importance are extending to all aspects of the life in World.

However, the existence of numerous financial crises contributes to some questioning of its social importance. Therefore it appears important for finances to be based on solid models to assess risks and prices.

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For these purposes, the Black and Scholes Model has been established as a reference since 1973, in the calculation of the option. This model is very successful because of having many advantages: its simplicity of application and its important use by market operators in increasing the wealth of a country and in its competitiveness. The increasingly efficient tools that are equipped allow to manage the risks.

2. The basic model

Let $(\Omega, (\mathscr{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions and $W = \{W(t), t \geq 0\}$ a standard Brownian movement suitable for the filtration $(\mathscr{F}_t) = (\sigma(w(s), s \leq t))$.

We define the process $S=\{S(t), t\geq 0\}$ by the solution of the following stochastic differential equation:

$$\begin{cases} dS(t) = \mu S(t).dt + \sigma S(t).dW(t) \\ S(t_0) = S_0 \end{cases}.$$

The process $S = \{S(t), t \geq 0\}$ is a stochastic process with values in \mathbb{R}^+ representing the price of a financial asset.

We put
$$Y(t) = \log(S(t))$$
 and $\eta = \mu - \frac{\sigma}{2}$.

By applying Ito's lemma, we obtain the following stochastic differential equation:

$$\begin{cases} dY(t) = \eta.dt + \sigma.dW(t) \\ Y(0) = y_0 = \log(S_0) \end{cases}.$$

It follows that $Y = \{Y(t), t \geq 0\}$ is a generalized Wienner process of drift η and variance σ^2 , such that

$$Y(t) - Y(0) = \log \frac{S(t)}{S(0)} \sim N(\eta t, \sigma^2 t),$$

therefore S(t) is a lognormal diffusion one-dimensional process characterized by the following transition density function.

For $s \leq t$:

$$S(s) = y$$

$$f(x,t/y,s) = \frac{1}{\sigma x \sqrt{2\pi(t-s)}} \exp\left\{-\frac{1}{2\sigma^2(t-s)} [\log x - \log y\right\}$$

$$-(\mu - \frac{\sigma^2}{2})(t-s)]^2$$
.

We can conclude that:

$$S(t) \sim \Lambda[\log y + \eta(t-s); \sigma^2(t-s)]$$
 with $\eta = \mu - \frac{\sigma^2}{2}$.

3. Black and Scholes process with three parameters

3.1. Definition of the model:

Let $(\Omega, (\mathscr{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions and $W = \{W(t), t \geq 0\}$ a standard Brownian movement suitable for the filtration $(\mathscr{F}_t) = (\sigma(w(s), s \leq t))$.

We define the process $S = \{S(t), t_0 \le t \le T\}, (S \in \mathbb{R}^+)$ by the solution of the following stochastic differential equation:

$$\begin{cases} dS(t) = \mu[S(t) - \gamma].dt + \sigma[S(t) - \gamma].dW(t) \\ S(t_0) = S_0 \end{cases}$$

Let $S = \{S(t), t_0 \le t \le T\}$ be a Markov process with values in $]\gamma, \infty[$ with almost certain continuous trajectories and transition probability given by:

$$P(x, t/y, s) = P[S(t) = x/S(s) = y],$$

where $x > \gamma$, $y > \gamma$ and $\gamma \in \mathbb{R}$.

We assume the following conditions:

$$\bullet \lim_{h\to 0} \frac{1}{h} P(x, t+h/y, t) = 0,$$

•
$$\lim_{h\to 0} \frac{1}{h} \int_{|x-y|<\epsilon} (x-y)P(x,t+h/y,t) = A_1(y,t;\gamma) = \mu(y-\gamma),$$

•
$$\lim_{h\to 0} \frac{1}{h} \int_{|x-y|<\epsilon} (x-y)^2 P(x,t+h/y,t) = A_2(y,t;\gamma) = \sigma^2(y-\gamma)^2$$
,

• The higher-order infinitesimal moments are null.

Remark 1. Where $\mu > 0$, $\sigma > 0$ such that $\mu, \sigma \in \mathbb{R}$, the infinitesimal moments of the process are:

$$A_1(y,t;\gamma) = \mu(y-\gamma),$$

$$A_2(y,t;\gamma) = \sigma^2(y-\gamma)^2.$$

Using the fact that process $S'(t) = S(t) - \gamma$ is a lognormal diffusion process with two parameters, we obtain the backward and forward Kolmogorov equations:

$$\frac{\partial P}{\partial s} + \frac{\sigma^2}{2} (y - \gamma)^2 \frac{\partial^2 P}{\partial y^2} + \mu (y - \gamma) \frac{\partial P}{\partial y} = 0, \tag{1}$$

$$-\frac{\partial P}{\partial t} + \frac{\sigma^2}{2} \frac{\partial (x - \gamma)^2 P}{\partial x^2} - \mu \frac{\partial (x - \mu) P}{\partial x} = 0.$$
 (2)

Using Ricciardi's theorem (see [13]), the common solution to these equations, with the initial condition $P(x, t/y, s) = \delta(x - y)$ is:

$$P(x, t/y, s) = \frac{1}{(x - \gamma)\sigma} \frac{1}{\sqrt{2\pi(t - s)}}$$

$$\times \exp\{-\frac{1}{2\sigma^2(t - s)} (\log(x - \gamma) - \log(y - \gamma) - \eta(t - s))^2\}, \tag{3}$$

with
$$\eta = \mu - \frac{\sigma^2}{2}$$
.

The distibution of random variables S(t)/S(s) = y is one-dimensional three-parameter lognormal distibution:

$$S(t) \sim \Lambda[\gamma; \log(y - \gamma) + \eta(t - s); \sigma^2(t - s)].$$

3.2. Moments of the process:

The moments of the lognormal diffusion process with three parameters are obtained from the moments of the lognormal diffusion process with two parameters $S'(t) = S(t) - \gamma$.

We have then.

$$E[(S(t) - \gamma)^k / S(s) = y_s] = \exp\{k \log(y_s - \gamma) + k\eta(t - s) + \frac{k^2 \sigma^2}{2}(t - s)\}$$

$$= (y_s - \gamma)^k \exp\{(k\eta + \frac{k^2\sigma^2}{2})(t - s)\}.$$

Taking into account that:

$$S^{k}(t) = (S(t) - \gamma + \gamma)^{k},$$

applying Newton's binomial we have:

$$S^{k}(t) = \sum_{i=0}^{k} {k \choose i} (S(t) - \gamma)^{i} \gamma^{k-i}.$$

Thus,

$$E[S^{k}(t)/S(s) = y_{s}] = \sum_{i=0}^{k} {k \choose i} (y_{s} - \gamma)^{i} \gamma^{k-i} \exp\{(i\eta + \frac{i^{2}\sigma^{2}}{2}(t - s))\}.$$
 (4)

3.2.1. The mean function:

We take the case k = 1 in equation (4), so we have:

$$E[S(t)/S(t) = y_s] = (y_s - \gamma) \exp\{(\eta + \frac{\sigma^2}{2}(t - s))\} + \gamma$$

= $(y_s - \gamma) \exp\{\mu(t - s)\} + \gamma$.

Thus, the mean function of the process will be:

$$E[S(t)] = E[E[S(t)/S(t_0)]]$$

$$= E[(S(t_0) - \gamma) \exp{\{\mu(t - t_0)\}} + \gamma]$$

$$= E[(S(t_0) - \gamma)] \exp{\{\mu(t - t_0)\}} + \gamma.$$

By using the initial condition $P[S(t_0) = y_0]$ we obtain:

$$E[(S(t))] = (s_0 - \gamma) \exp{\{\mu(t - t_0)\}} + \gamma.$$
(5)

3.2.2. The variance function:

To calculate the variance of the process, we need to calculate previously the second order moment: $E[S^2(t)]$.

In the same way, and by using the initial condition $P[S(t_0) = y_0]$, the variance function will be:

$$Var[S(t)] = E[S^{2}(t)] - [E[S(t)]^{2}$$

= $(s_{0} - \gamma)^{2} \exp\{2\mu(t - t_{0})\} [\exp\{\sigma^{2}(t - t_{0})\} - 1].$

3.2.3. The covariance function:

The covariance function is given by:

$$Cov[S(t)S(s)] = E[S(t)S(s)] - E[S(t)E[S(s)]].$$

So, the covariance function has the following form:

$$Cov[S(t)S(s)] = (y_0 - \gamma)^2 \exp{\{\mu(t - t_0) + (s - t_0)\}}$$
$$\times [\exp{\{\sigma^2(\min(t, s) - t_0)\}} - 1].$$

4. Estimation of the parameters

4.1. Maximum likelihood method (M.L.M):

We consider a discrete sampling of the process $S \in \mathbb{R}^+$ for the instants t_1, \dots, t_n :

$$\{S(t_1) = s_1, S(t_2) = s_2, \cdots, S(t_n) = s_n\},\$$

with the initial condition $P[S(t_1) = s_1] = 1$.

The maximum likelihood function associated is thus

$$\mathbb{L}(s_1, \dots, s_n; \eta, \sigma^2, \gamma) = \prod_{i=2}^{n} P(s_i, t_i / s_{i-1}, t_{i-1}).$$

This function tends to infinity when γ tends to $s_{(1)} = \inf_{0 \le j \le n} (s_j)$

We introduce the Log in equation (3), we obtain then the log-likelihood function:

$$\log[\mathbb{L}(s_1,\dots,s_n;\eta,\sigma^2,\gamma)] = -\frac{n-1}{2}\log(2\pi\sigma^2)\sum_{i=2}^n\log(s_i-\gamma)$$

$$-\frac{1}{2}\sum_{i=2}^{n}(t_i-t_{i-1})-\frac{1}{2\sigma^2}\sum_{i=2}^{n}\frac{1}{(t_i-t_{i-1})}$$

$$\times [\log(s_i - \gamma) - \log(s_{i-1} - \gamma) - \eta((t_i - t_{i-1}))]^2.$$
 (6)

We obtain then the likelihood equations for η , σ^2 and γ from (6), by differentiating with respect of parameters and setting the result equal to zero:

$$\hat{\eta} = \frac{\log(s_n - \hat{\gamma}) - \log(s_1 - \hat{\gamma})}{t_n - t_1},\tag{7}$$

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=2}^n \frac{\left[\log(s_n - \hat{\gamma}) - \log(s_{i-1} - \hat{\gamma}) - \hat{\eta}(t_i - t_{i-1}) \right]^2}{t_i - t_{i-1}},\tag{8}$$

$$\lambda(\hat{\gamma}) = \hat{\sigma^2} \sum_{i=2}^n \frac{1}{(s_i - \hat{\gamma})} - \sum_{i=2}^n \frac{(s_i - s_{i-1})[\log(s_i - \hat{\gamma}) - \log(s_{i-1} - \hat{\gamma})]}{(s_i - \hat{\gamma})(s_{i-1} - \hat{\gamma})(t_i - t_{i-1})} + \hat{\eta} \frac{s_n - s_1}{(s_n - \hat{\gamma})(s_1 - \hat{\gamma})}, \tag{9}$$

where $\lambda(\hat{\gamma}) = 0$ and $\gamma < s_{(1)}$.

4.2. Reparametrization of the process, according to Wingo:

The estimation of the third parameter introduced in the process is not obvious. The main theoretical difficulty is that the likelihood function reaches global maxima at points where the parameters take inadmissible values and where the global maxima value of the likelihood function is $+\infty$.

For the computational part, we have convergence problems due to the iterative numerical method used for ML estimation. In other words, when we try to find a solution for equation (4) by means of numerical approximation method, and since the initial estimate of γ is not close enough to the solution, this method will converge towards the degenerate solution $\gamma = -\infty$.

For the case of three-parameter lognormal distribution (see [15]), Wingo proposed a computational algorithm based on the reparametrization of the likelihood function by using a parametric transform, to reduce the interval of the real slope on which many finite local maxima of the log-likelihood function are located.

We consider the following transform:

$$\gamma(\theta) = s_1 - \exp(-\theta), \quad]-\infty, +\infty[, \tag{10}$$

where s_1 is the minimum of the values in the sample.

It is clear that $\gamma \longrightarrow s_1$ when $\theta \longrightarrow +\infty$, and $\gamma \longrightarrow -\infty$ when $\theta \longrightarrow -\infty$.

This transformation is substituted into the log-likelihood function (6), we obtain then $\log[\mathbb{L}(s_1, \dots, s_n; \eta, \sigma^2, \gamma(\theta))]$, denoted by $\mathbb{L}^*(\eta, \sigma^2, \theta)$.

The computational algorithm is simple:

- We obtain a global maximum $\hat{\theta}$ by maximazing $\mathbb{L}^*(\eta, \sigma^2, \theta)$ in a compact interval on the real slope.
- We calculate the local maximum likelihood estimator $\hat{\gamma}$ by means of $\hat{\gamma}(\theta) = s_1 \exp(-\hat{\theta})$.
- We calculate the remaining maximum likelihood estimators by substituting $\hat{\gamma}$ in the equations (7) and (8).

By applying the Wingo transformation to our process, the log-likelihood function would be:

$$\mathbb{L}^*(\eta, \sigma^2, \theta) = -\frac{n-1}{2} \log(2\pi\sigma^2) - \sum_{i=2}^n \log(s_i - [s_1 - \exp(-\theta)])$$
$$-\frac{1}{2} \sum_{i=2}^n \log(t_i - t_{i-1}) - \sum_{i=2}^n \frac{1}{2\sigma^2(t_i - t_{i-1})} [\log(s_i - [s_1 - \exp(-\theta)])$$
$$-\log(s_{i-1} - [s_1 - \exp(-\theta)]) - \eta(t_i - t_{i-1})]^2.$$

By differentiating the log-likelihood function compared to θ , and setting the result equal to zero we obtain an estimator for θ .

We substitute then in other likelihood equations, we obtain the estimators for remaining parameters, as described above:

$$\hat{\eta} \left(\frac{1}{s_n - \hat{\gamma}(\theta)} - \frac{1}{s_1 - \hat{\gamma}(\theta)} \right) = \sum_{i=2}^n \frac{\hat{\sigma}^2 \exp(-\theta)}{s_i - \hat{\gamma}(\theta)}$$
$$- \sum_{i=2}^n \frac{(s_i - s_{i-1})[\log(s_i - \hat{\gamma}(\theta))\log(s_{i-1} - \hat{\gamma}(\theta))]}{(s_i - \hat{\gamma}(\theta))(s_{i-1} - \hat{\gamma}(\theta))(t_i - t_{i-1})}.$$

4.3. Wicksell line method:

We have previously seen that S_t is a lognormal process with three parameters so we can apply our alternative estimation method: the wicksell line method that we have already discussed in our last article (see [4]).

We pose $S_{t_i} = s_i \in \mathbb{R}^m \ (m \in \mathbb{N}^*)$ a multidimensional three-parametric lognormal process at an instant i.

Let
$$Z_{t_i}(s_i) = \frac{\log(s_i - \gamma_i) - \mu_i}{\sigma_i}$$
 for $1 \le i \le n$ and n fixed.

We have already seen that $S_{t_i} = s_i \sim \Lambda(\gamma_i; \mu_i, \sigma_i)$, which implies that $s_i = \gamma_i + \exp(\sigma_i Z_{t_i} + \mu_i)$.

Let
$$N_i(Z_{t_i}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Z_{t_i}} \exp(-\frac{t_i^2}{2}) dt_i$$
 the distribution function of $N(0,1)$, and $F_i(S_{t_i}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Z_{t_i}(S_{t_i})} \exp(-\frac{t_i^2}{2}) dt_i$ the distribution function of $\Lambda(\gamma_i; \mu_i, \sigma_i)$.

$$F(S_{t_i}) = N(Z_i(S_{t_i})). \tag{11}$$

If A_i is the mean of S_{t_i} , then:

$$A_i = \gamma_i + \exp\left(\mu_i + \frac{{\sigma_i}^2}{2}\right),$$

$$Z_i(A_i) = \frac{\sigma_i}{2},$$

$$F_i(A_i) = N_i\left(\frac{\sigma_i}{2}\right).$$

Finally,

$$\sigma_i = 2N_i^{-1}[F_i(A_i)].$$
 (12)

Let $\{(S_{t_{ij}}, F_{ij}(S_{t_{ij}})) : j = 1, \dots, n\}$ a given random sample.

We denote by S_{ij} the j^{th} observation of the process at the instant i and t_{ij} the instant of the j^{th} observation of the process at the instant i.

We calculate first the mean A_i , then by the interpolation method we calculate $F_i(A_i)$.

From (11) and (12) we determinate σ_i and $Z(S_{t_{ij}})$.

Finally for $j = 1, \dots, n$, we determinate u_j such that

$$u_j = \exp[\sigma_i Z_i(S_{t_{ij}})].$$

We make then a linear approximation to the data by

$$\{(S_{t_{ij}}, u_j) : j = \{1, \dots, n\} \text{ with } u = cx + d\},\$$

to determine the parameters μ_i and γ_i .

If we make an approximation of u = cx + d with quadratic mean, then we can get c and d minimizing:

$$f(c,d) = \sum_{j=1}^{n} (u - u_j)^2 = \sum_{j=1}^{n} (cS_{ij} + d - u_j)^2.$$

The partial derivatives of f(c, d) are:

$$\frac{\partial f(c,d)}{\partial d} = 2\sum_{j=1}^{n} (cS_{ij} + d - u_j),$$

$$\frac{\partial f(c,d)}{\partial c} = 2\sum_{j=1}^{n} S_{ij}(cS_{ij} + d - u_j).$$

If these partial derivatives are equal to zero we obtain the following system:

$$\begin{cases} \sum_{j=1}^{n} (cS_{ij} + d - u_j) = 0 \\ \sum_{j=1}^{n} x_{ij} (cS_{ij} + d - u_j) = 0 \end{cases},$$

which is equivalent to:

$$\begin{cases} \left(\sum_{j=1}^{n} S_{ij}\right) c + nd = \sum_{j=1}^{n} u_j \\ \left(\sum_{j=1}^{n} S_{ij}^2\right) c + \left(\sum_{j=1}^{n} S_{ij}\right) d = \sum_{j=1}^{n} S_{ij} u_j \end{cases},$$

thus

$$c = \frac{n \sum_{j=1}^{n} S_{ij} u_j - \sum_{j=1}^{n} S_{ij} \sum_{j=1}^{n} u_j}{n \sum_{j=1}^{n} S_{ij}^2 - \left(\sum_{j=1}^{n} S_{ij}\right)^2},$$

and

$$d = \frac{1}{n} \sum_{j=1}^{n} u_j - \frac{1}{n} \sum_{j=1}^{n} S_{ij}c.$$

Finally, we have

$$\begin{cases} \mu_i = -\log c, \\ \gamma_i = -\frac{d}{c}, \\ \sigma_i = \frac{1}{n} \sum_{j=1}^n \left(\log(S_j - \gamma_i) - \mu_i \right) \left(\log(S_j - \gamma_i) - \mu_i \right)' \end{cases}.$$

5. Conclusion

The basic goal of this study is to define a new stochastic process: the three-parameter Black and Scholes process. To do this, we determine the inferential and probabilistic results: we estimate the parameters of the process based on discrete sampling of the process using two methods: the maximum likelihood method and Wicksell line method. The problems we encountered when applying the ML method are solved via the method of reparametrization according to Wingo to reduce the interval of the threshold parameter.

Our next paper will be devoted to a numerical study of estimation methods. We will apply the model to real data, and therefore determine which of these methods is the most exact.

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