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ON THE PRODUCT AND RATIO OF PARETO AND HYPEREXPONENTIAL RANDOM VARIABLES

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Abstract: The distributions of products and ratios of random variables are of interest in many areas of the sciences. In this paper, we find analytically the probability distributions of the product XY and the ratio X/Y, when X and Y are two independent random variables following Pareto and Hyperexponential distributions, respectively. To the best of our knowledge, this is the first study on the combination of these two distributions.

AMS Subject Classification: 62Exx

Key Words: product distribution; ratio distribution; Pareto distribution; Hyperexponential distribution; probability density function; moment of order r; survival function; hazard function

1. Introduction

Applied sciences like Engineering, Physics, Economics, Order statistics, Classification, Ranking, Selection, Number theory, Genetics, Biology, Medicine, Hydrology, Psychology, these all employ the distribution of product and ratio of random variables [1],[2]. As an example of the use of the product of random variables in physics, Sornette [27] mentions: To mimic system size limitation, Takayasu, Sato, and Takayasu introduced a threshold x_c and found a stretched exponential truncating the power-law pdf beyond x_c . Frisch and Sornette recently developed a theory of extreme deviations generalizing the central limit theorem which, when applied to the multiplication of random variables, pre-

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dicts the generic presence of stretched exponential pdfs. The problem thus boils down to determining the tail of the pdf for a product of random variables. Several authors have studied the product distributions for independent random variables that come from the same family or different families, see [21] for t and Rayleigh families, [4] for Pareto and Kumaraswamy families, [6] for the t and Bessel families, and [22] for the independent generalized Gamma-ratio family, [28] for Pareto and Rayleigh families. As an example of the use of the ratio of random variables include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, and inventory ratios in economics. Several authors have studied the ratio distributions for independent random variables come from the same family or different families. the historical review, see [9], [10] for the Normal family, [11] for Students t family, [12] for the Weibull family, [13] for the noncentral Chisquared family, [14] for the Gamma family, [15] for the Beta family, [16] for the Logistic family, [17] for the Frechet family, [3] for the inverted Gamma family, [18] for Laplace family, [7] for the generalized-F family, [19] for the Hypoexponential family, [2] for the Gamma and Rayleigh families, and [20] for Gamma and Exponential families, [28] for Pareto and Rayleigh families. In this paper, the analytical probability distributions are derived of XY and X/Y, when X and Y are two independent Pareto and Hyperexponential distributions respectively, with probability density functions (p.d.f.s)

$$f_X(x) = \frac{ca^c}{x^{c+1}},\tag{1}$$

$$f_Y(y) = \sum_{j=1}^k p_j \lambda_j e^{-\lambda_j y},$$
(2)

respectively, for $a \le x < \infty$, a > 0, c > 0, y > 0, $\lambda_j > 0$, $0 \le p_j \le 1$, and $\sum_{j=1}^k p_j = 1.$

Notations and Preliminaries

Recall some special mathematical functions, these will be used repeatedly throughout this article.

• The upper incomplete Gamma function is defined by:

$$\Gamma(a,x) = \int_{x}^{\infty} \exp(-t)t^{a-1}dt;$$
 (3)

• The lower incomplete Gamma function is defined by:

$$\gamma(a,x) = \int_0^x \exp(-t)t^{a-1}dt, \tag{4}$$

• The Generalized hypergeometric function is denoted by:

$$_{p}F_{q}(a_{1}, a_{2}, ..., a_{p}; b_{1}, b_{2}, ..., b_{q}; z) = \sum_{k=0}^{\infty} \frac{(a_{1})_{k}(a_{2})_{k}...(a_{p})_{k}}{(b_{1})_{k}(b_{2})_{k}...(b_{p})_{k}} \frac{z^{k}}{k!},$$
 (5)

where $(a)_k$, $(b)_k$ represent Pochhammers symbols given by:

$$(a)_k = a(a+1)...(a+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)};$$
 (6)

• The Exponential integral is generalized, for n = 0, 1, 2, ..., x > 0, to

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt,\tag{7}$$

where n is the order of the integral.

The calculations of this paper involve several lemmas.

Lemma 1. For $\alpha \geq 0$, $r \in \mathbb{R}$, and $b \in \mathbb{R}_+^*$

$$I(\alpha, r, b) = \int_{\alpha}^{\infty} x^r e^{-bx} dx = \frac{1}{b^{r+1}} \Gamma(r+1, b\alpha).$$
 (8)

Proof. Let u = bx, then

$$I(\alpha, r, b) = \int_{b\alpha}^{+\infty} \frac{u^r}{b^{r+1}} e^{-u} du = \frac{1}{b^{r+1}} \Gamma(r+1, b\alpha). \tag{9}$$

Lemma 2. For $t \in \mathbb{R}$,

$$\frac{d}{dx}\Gamma(t,v(x)) = -v(x)^{t-1}e^{-v(x)}\frac{d}{dx}v(x). \tag{10}$$

Proof.

$$\frac{d}{dx}\Gamma(t,v) = \frac{d}{dv}\Gamma(t,v)\frac{dv}{dx},\tag{11}$$

$$\frac{d}{dv}\Gamma(t,v) = -v^{t-1}e^{-v}. (12)$$

Lemma 3. For $\alpha \geq 0$, $r \in \mathbb{R}$, and $b \in \mathbb{R}_+^*$

$$\int_0^\alpha x^r e^{-bx} dx = \frac{1}{b^{r+1}} \gamma(r+1, b\alpha). \tag{13}$$

Proof. For u = bx,

$$\int_0^{b\alpha} \frac{u^r}{b^{r+1}} e^{-u} du = \frac{1}{b^{r+1}} \gamma(r+1, b\alpha).$$

Lemma 4. The Exponential integral (7)

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt,$$

is closely related to the incomplete gamma function as follows:

$$E_n(x) = x^{n-1}\Gamma(1-n, x), \text{ for } n = 0, 1, 2, ..., x > 0.$$
 (14)

Proof. For u = xt,

$$E_n(x) = x^{n-1} \int_x^\infty e^{-u} u^{-n} du = x^{n-1} \Gamma(-n+1).$$

2. Distribution of the Product XY

Theorem 5. Suppose X and Y are independent and distributed according to (1) and (2), respectively. Then for z > 0, the cumulative distribution function c.d.f. of Z = XY can be expressed as:

$$F_Z(z) =$$

$$\begin{cases}
0, & \text{if } z < 0, \\
1 - \sum_{j=1}^{k} p_{j} e^{-\lambda_{j} z / a} - \frac{a^{c}}{z^{c}} \sum_{j=1}^{k} \frac{p_{j}}{\lambda_{j}^{c}} \left[\Gamma(c+1) - \Gamma(c+1, \frac{\lambda_{j} z}{a}) \right], & \text{if } z > 0.
\end{cases}$$
(15)

Proof. The c.d.f. corresponding to (1) is $F_X(x) = 1 - (\frac{a}{x})^c$, Thus, one can write the c.d.f. of XY as:

$$Pr(XY \le z) = \int_0^\infty F_X(\frac{z}{y}) f_Y(y) dy$$

$$= \int_0^{z/a} (1 - (\frac{ay}{z})^c) f_Y(y) dy$$

$$= \int_0^{z/a} f_Y(y) dy - \frac{a^c}{z^c} \int_0^{z/a} y^c f_Y(y) dy.$$
(16)

Let

$$I_1 = \int_0^{z/a} f_Y(y) dy, \tag{17}$$

and

$$I_2 = \frac{a^c}{z^c} \int_0^{z/a} y^c f_Y(y) dy.$$
 (18)

Then

$$F_Z(z) = I_1 - I_2.$$

Calculus of I_1

$$I_{1} = \int_{0}^{z/a} \sum_{j=1}^{k} p_{j} \lambda_{j} e^{-\lambda_{j} y} dy$$

$$= 1 - \sum_{j=1}^{k} p_{j} e^{-\lambda_{j} z/a}.$$

$$(19)$$

Calculus of I_2

$$I_2 = \frac{a^c}{z^c} \int_0^{z/a} y^c f_Y(y) dy$$

$$= \frac{a^c}{z^c} \int_0^{z/a} y^c \sum_{i=1}^k p_j \lambda_j e^{-\lambda_j y} dy.$$
(20)

Using Lemma 3 in the integral above, then we get

$$I_2 = \frac{a^c}{z^c} \sum_{j=1}^k \frac{p_j}{\lambda_j^c} \left[\Gamma(c+1) - \Gamma(c+1, \frac{\lambda_j z}{a}) \right], \tag{21}$$

then

$$F_Z(z) = I_1 - I_2$$

$$= 1 - \sum_{j=1}^k p_j e^{-\lambda_j z/a} - \frac{a^c}{z^c} \sum_{j=1}^k \frac{p_j}{\lambda_j^c} \Big[\Gamma(c+1) - \Gamma(c+1, \frac{\lambda_j z}{a}) \Big].$$
(22)

Corollary 6. Let X and Y be independent and distributed according to (1) and (2), respectively. Then for z > 0, the probability density function p.d.f. of Z = XY can be expressed as:

$$f_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ \sum_{j=1}^k \frac{p_j c a^c}{\lambda_j^c z^{c+1}} \left[\Gamma(c+1) - \Gamma(c+1, \lambda_j z/a) \right], & \text{if } z > 0. \end{cases}$$
 (23)

Proof. The probability density function $f_Z(z)$ in (23) follows by differentiation using Lemma 2.

Corollary 7. Let X and Y be independent and distributed according to (1) and (2), respectively. Then for c > r, the Moment of order r of Z = XY can be expressed as:

$$E[Z^r] = \sum_{j=1}^k \left[\frac{p_j c a^c \left[\Gamma(c+1) - \Gamma(c+1, \lambda_j \alpha/a) \right]}{\alpha^{c-r} \lambda_j^c (c-r)} + \frac{p_j c a^r \Gamma(r+1, \lambda_j \alpha/a)}{\lambda_j^r (c-r)} \right].$$
(24)

Proof.

$$E[Z^r] = \int_{-\infty}^{+\infty} z^r f_Z(z) dz$$

$$= \lim_{\alpha \to 0^+} \left[\int_{\alpha}^{+\infty} \sum_{j=1}^k \frac{p_j c a^c}{\lambda_j^c z^{c+1-r}} \left[\Gamma(c+1) - \Gamma(c+1, \lambda_j z/a) dz \right] \right], \tag{25}$$

let

$$I_1 = \int_{\alpha}^{\infty} \sum_{i=1}^{k} \frac{p_j c a^c}{\lambda_j^c z^{c-r+1}} \Gamma(c+1) dz, \qquad (26)$$

and

$$I_2 = \int_{\alpha}^{\infty} \sum_{j=1}^{k} \frac{p_j c a^c}{\lambda_j^c z^{c+1-r}} \Gamma(c+1, \lambda_j z/a) dz.$$
 (27)

Then

$$E[Z^r] = I_1 - I_2.$$

Calculus of I_1 :

$$I_1 = \int_{\alpha}^{\infty} \sum_{j=1}^{k} \frac{p_j ca^c}{\lambda_j^c z^{c-r+1}} \Gamma(c+1) dz = \sum_{j=1}^{k} \frac{p_j ca^c \Gamma(c+1)}{\lambda_j^c \alpha^{c-r}(c-r)}.$$
 (28)

Calculus of I_2 :

$$I_2 = \sum_{j=1}^k \frac{p_j c a^c}{\lambda_j^c} \int_{\alpha}^{\infty} z^{r-1-c} \Gamma(c+1, \lambda_j z/a) dz.$$
 (29)

Integration by parts of the integral above implies:

$$-\frac{\alpha^{r-c}}{r-c}\Gamma(c+1,\lambda_j\alpha/a) + \frac{\lambda_j^{c+1}}{a^{c+1}(r-c)} \int_{\alpha}^{\infty} z^r e^{-\lambda_j z/a} dz, \tag{30}$$

using Lemma 1, eq. (8), in the integral above then

$$I_2 = \sum_{j=1}^k \frac{p_j c a^c \Gamma(c+1, \lambda_j \alpha/a)}{\lambda_j^c(c-r) \alpha^{c-r}} - \sum_{j=1}^k \frac{p_j c a^r \Gamma(r+1, \lambda_j \alpha/a)}{\lambda_j^r(c-r)},$$
 (31)

and finally,

$$E[Z^r] = \sum_{j=1}^k \left[\frac{p_j c a^c \Big[\Gamma(c+1) - \Gamma(c+1, \lambda_j \alpha/a) \Big]}{\alpha^{c-r} \lambda_j^c(c-r)} + \frac{p_j c a^r \Gamma(r+1, \lambda_j \alpha/a)}{\lambda_j^r(c-r)} \right].$$
(32)

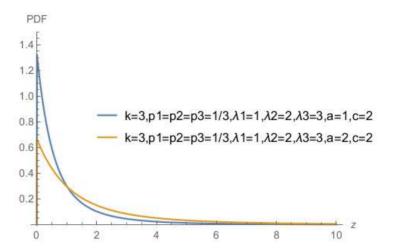


Figure 1: Plots of the pdf (23) for k = 3, a = 1, c = 2, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $p_1 = 1/3$, $p_2 = 1/3$, $p_3 = 1/3$.

Corollary 8. Let X and Y be independent and distributed according to (1) and (2), respectively. Then for c > 1, the Expected value of Z = XY can be expressed as: for r = 1,

$$E[Z] = \sum_{j=1}^{k} \left[\frac{p_j ca^c \left[\Gamma(c+1) - \Gamma(c+1, \lambda_j \alpha/a) \right]}{\alpha^{c-1} \lambda_j^c(c-1)} + \frac{p_j ca \Gamma(2, \lambda_j \alpha/a)}{\lambda_j(c-1)} \right].$$
(33)

Corollary 9. Let X and Y be independent and distributed according to (1) and (2), respectively. Then for c > 2, the Variance of Z = XY can be expressed as:

$$\sigma^{2} = \sum_{j=1}^{k} \left[\frac{p_{j} c a^{c} \left[\Gamma(c+1) - \Gamma(c+1, \lambda_{j} \alpha/a) \right]}{\alpha^{c-2} \lambda_{j}^{c}(c-2)} + \frac{p_{j} c a^{2} \Gamma(3, \lambda_{j} \alpha/a)}{\lambda_{j}^{2}(c-2)} \right]$$

$$\left[\sum_{j=1}^{k} \left[\frac{p_{j} c a^{c} \left[\Gamma(c+1) - \Gamma(c+1, \lambda_{j} \alpha/a) \right]}{\alpha^{c-1} \lambda_{j}^{c}(c-1)} + \frac{p_{j} c a \Gamma(2, \lambda_{j} \alpha/a)}{\lambda_{j}(c-1)} \right] \right]^{2}.$$
(34)

Proof. The variance of Z = XY easy follows from:

$$\sigma^2 = E[Z^2] - E[Z]^2, \tag{35}$$

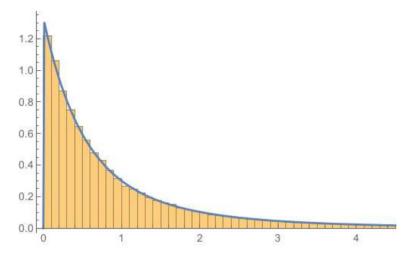


Figure 2: Monte Carlo simulation of the product of Pareto and Hyperexponential random variables for $k=3, a=1, c=2, \lambda_1=1, \lambda_2=2, \lambda_3=3, p_1=1/3, p_2=1/3, p_3=1/3.$

for
$$c > 2$$
.

Corollary 10. Let X and Y be independent and distributed according to (1) and (2), respectively. Then for z > 0, the Survival function of Z = XY can be expressed as:

$$S_{Z}(z) = \begin{cases} 1, & \text{if } z < 0, \\ \sum_{j=1}^{k} p_{j} e^{-\lambda_{j} z / a} + \frac{a^{c}}{z^{c}} \sum_{j=1}^{k} \frac{p_{j}}{\lambda_{j}^{c}} \left[\Gamma(c+1) - \Gamma(c+1, \frac{\lambda_{j} z}{a}) \right], & \text{if } z > 0. \end{cases}$$
(36)

Proof. By definition of the Survival function $S_Z(z) = 1 - F_Z(z)$.

Corollary 11. Let X and Y be independent and distributed according to (1) and (2), respectively. Then for z > 0, the Hazard function of Z = XY can

be expressed as:

$$h_{Z}(z) = \begin{cases} 0, & \text{if } z < 0, \\ \sum_{j=1}^{k} \frac{p_{j}ca^{c}}{\lambda_{j}^{c}z^{c+1}} \left[\Gamma(c+1) - \Gamma(c+1, \lambda_{j}z/a) \right], & \text{if } z > 0. \end{cases}$$
(37)

Here,

$$A = \sum_{j=1}^{k} p_j \frac{a^c}{z^c(\lambda_j)^c} \left[\Gamma(c+1) - \Gamma(c+1, \frac{\lambda_j z}{a}) \right] + \sum_{j=1}^{k} p_j e^{-\frac{\lambda_j z}{a}}.$$

Proof. By definition of the Hazard function $h_Z(z) = \frac{f_Z(z)}{S_Z(z)}$.

3. Distribution of the Ratio X/Y

Theorem 12. Suppose X and Y be independent and distributed according to (1) and (2), respectively. Then for z > 0, the cumulative distribution function c.d.f. of Z = X/Y can be expressed as:

$$F_{Z}(z) = \begin{cases} 0, & \text{if } z \leq 0, \\ \sum_{j=1}^{k} p_{j}e^{-\lambda_{j}a/z} & \text{if } z > 0, c < 1, \\ -\frac{a^{c}}{z^{c}} \sum_{j=1}^{k} p_{j}\lambda_{j}^{c}\Gamma(-c+1,\lambda_{j}a/z), & \\ \sum_{j=1}^{k} p_{j}e^{-\lambda_{j}a/z} & \text{if } z > 0, \\ -\frac{a}{z} \sum_{j=1}^{k} p_{j}\lambda_{j}E_{c}(\lambda_{j}a/z), & \\ \sum_{j=1}^{k} p_{j}e^{-\lambda_{j}a/z} & \\ \sum_{j=1}^{k} p_{j}e^{-\lambda_{j}a/z} - & \\ \left[\int_{0}^{\infty} t^{-c} \left[e^{-t} - \sum_{i=0}^{n-1} \frac{(-t)^{i}}{i!} \right] dt & \text{if } z > 0, \\ -n < 1 - c < -n + 1. & \\ -\int_{0}^{\lambda a/z} t^{-c} \left[e^{-t} - \sum_{i=0}^{n-1} \frac{(-t)^{i}}{i!} \right] dt \\ -\sum_{i=0}^{n-1} (-1)^{i} \frac{(\lambda a/z)^{1-c+i}}{(1-c+i)i!} \right]. \end{cases}$$

Proof. The c.d.f. corresponding to (1) is $F_X(x) = 1 - (\frac{a}{x})^c$,

$$F_{Z}(z) = pr(X/Y \le z)$$

$$= \int_{0}^{\infty} F_{X}(zy) f_{Y}(y) dy$$

$$= \int_{a/z}^{\infty} (1 - (\frac{a}{yz})^{c}) f_{Y}(y) dy$$

$$= \int_{a/z}^{\infty} f_{Y}(y) dy - \frac{a^{c}}{z^{c}} \int_{a/z}^{\infty} \frac{f_{Y}(y)}{y^{c}} dy.$$

$$(39)$$

Let

$$I_1 = \int_{a/z}^{\infty} \sum_{j=1}^{k} p_j \lambda_j e^{-\lambda_j y} dy = \sum_{j=1}^{k} p_j e^{-\lambda_j a/z},$$

and

$$I_2 = \frac{a^c}{z^c} \int_{a/z}^{\infty} \frac{\sum_{j=1}^k p_j \lambda_j e^{-\lambda_j y}}{y^c} dy.$$

Then we can write

$$F_Z(z) = I_1 - I_2.$$

Calculus of I_2 :

$$I_2 = \frac{a^c}{z^c} \int_{a/z}^{\infty} \frac{\sum_{j=1}^k p_j \lambda_j e^{-\lambda_j y}}{y^c} dy$$
$$= \frac{a^c}{z^c} \sum_{j=1}^k p_j \lambda_j \int_{a/z}^{\infty} y^{-c} e^{-\lambda_j y} dy.$$

Using Lemma 1 in the integral above, then we get

$$I_2 = \frac{a^c}{z^c} \sum_{j=1}^k p_j \lambda_j^c \Gamma(-c+1, \lambda_j a/z). \tag{40}$$

Finally we get

$$F_Z(z) = \sum_{i=1}^k p_j e^{-\lambda_j a/z} - \frac{a^c}{z^c} \sum_{i=1}^k p_j \lambda_j^c \Gamma(-c+1, \lambda_j a/z), \tag{41}$$

for c < 1.

For c = 1, 2, 3, ..., using Lemma 4 we have

$$\Gamma(1 - c, \lambda_j a/z) = (\frac{\lambda_j a}{z})^{1-c} E_c(\lambda_j a/z),$$

and

$$F_Z(z) = \sum_{j=1}^{k} p_j e^{-\lambda_j a/z} - \frac{a}{z} \sum_{j=1}^{k} p_j \lambda_j E_c(\lambda_j a/z), \tag{42}$$

where c is the order of the integral.

For $-n < 1 - c < -n + 1, 1 - c \neq -1, -2, ...$, we have

$$\Gamma(1-c,\lambda_{j}a/z) = \int_{0}^{\infty} t^{-c} \left[e^{-t} - \sum_{i=0}^{n-1} \frac{(-t)^{i}}{i!}\right] dt$$

$$- \int_{0}^{\lambda_{j}a/z} t^{-c} \left[e^{-t} - \sum_{i=0}^{n-1} \frac{(-t)^{i}}{i!}\right] dt$$

$$- \sum_{i=0}^{n-1} (-1)^{i} \frac{(\lambda_{j}a/z)^{1-c+i}}{(1-c+i)i!},$$
(43)

and

$$F_{Z}(z) = \sum_{j=1}^{k} p_{j} e^{-\lambda_{j} a/z}$$

$$- \frac{a^{c}}{z^{c}} \sum_{j=1}^{k} p_{j} \lambda_{j}^{c} \left[\int_{0}^{\infty} t^{-c} \left[e^{-t} - \sum_{i=0}^{n-1} \frac{(-t)^{i}}{i!} \right] dt \right.$$

$$- \int_{0}^{\lambda a/z} t^{-c} \left[e^{-t} - \sum_{i=0}^{n-1} \frac{(-t)^{i}}{i!} \right] dt - \sum_{i=0}^{n-1} (-1)^{i} \frac{(\lambda a/z)^{1-c+i}}{(1-c+i)i!} \right].$$

Corollary 13. Let X and Y be independent and distributed according to (1) and (2), respectively. Then for z > 0, the probability density function p.d.f. of Z = X/Y can be expressed as:

$$f_{Z}(z) = \begin{cases} 0, & \text{if } z \leq 0, \\ \sum_{j=1}^{k} \frac{p_{j}c(a\lambda_{j})^{c}}{z^{c+1}} \Gamma(1 - c, a\lambda_{j}/z), & \text{if } z > 0, \ c < 1, \end{cases}$$

$$f_{Z}(z) = \begin{cases} \sum_{j=1}^{k} \frac{p_{j}\lambda_{j}a}{z^{2}} e^{-\lambda_{j}a/z} \\ + \sum_{j=1}^{k} \frac{p_{j}(a\lambda_{j})^{2}}{z^{3}} E_{c-1}(\lambda_{j}a/z) & \text{if } z > 0, \ c = 1, 2, 3, \dots \end{cases}$$

$$- \sum_{j=1}^{k} \frac{p_{j}a\lambda_{j}}{z^{2}} E_{c}(\lambda_{j}a/z), \qquad (44)$$

Proof. The probability density function $f_Z(z)$ in (44) follows by differentiation using Lemma 2, eq. (10), and

$$\frac{d}{dz}E_n(z) = -E_{n-1}(z),\tag{45}$$

for n = 1, 2, 3, ...

Corollary 14. Let X and Y be independent and distributed according to (1) and (2), respectively. Then for z > 0, the Survival function of Z = X/Y can be expressed as:

ressed as:
$$\begin{cases} 1, & \text{if } z \leq 0, \\ 1 - \sum_{j=1}^{k} p_{j} e^{-\lambda_{j} a/z} \\ & \text{if } z > 0, \ c < 1, \\ + \frac{a^{c}}{z^{c}} \sum_{j=1}^{k} p_{j} \lambda_{j}^{c} \Gamma(-c+1, \lambda_{j} a/z), \\ 1 - \sum_{j=1}^{k} p_{j} e^{-\lambda_{j} a/z} \\ & \text{if } z > 0, \\ c = 1, 2, 3, \dots \end{cases}$$

$$S_{Z}(z) = \begin{cases} 1 - \sum_{j=1}^{k} p_{j} \lambda_{j}^{c} E_{c}(\lambda_{j} a/z), \\ 1 - \sum_{j=1}^{k} p_{j} \lambda_{j}^{c} E_{c}(\lambda_{j} a/z), \end{cases}$$

$$1 - \sum_{j=1}^{k} p_{j} e^{-\lambda_{j} a/z}$$

$$1 - \sum_{j=1}^{k} p_{j} \lambda_{j}^{c}$$

$$1 - \sum_{j=1}^{k} p_{j} \lambda_{j}^{c$$

Proof. By definition of the Survival function,

$$S_Z(z) = 1 - F_Z(z).$$

Corollary 15. Let X and Y be independent and distributed according to (1) and (2), respectively. Then for z > 0, the Hazard function of Z = X/Y can be expressed as:

$$h_{Z}(z) = \begin{cases} 0, & \text{if } z \leq 0, \\ \frac{\sum_{j=1}^{k} p_{j} c(a\lambda_{j})^{c} \Gamma(1-c, a\lambda_{j}/z)}{B}, & \text{if } z > 0, \ c < 1, \\ \frac{z}{\sum_{j=1}^{k} p_{j} \lambda_{j} a e^{-\lambda_{j} a/z}}{C} \\ + \frac{\sum_{j=1}^{k} p_{j} (a\lambda_{j})^{2} E_{c-1}(\lambda_{j} a/z)}{C}, & \text{if } z > 0, \ c = 1, 2, \dots \\ - \frac{z}{\sum_{j=1}^{k} p_{j} a \lambda_{j} E_{c}(\lambda_{j} a/z)}{C} \\ - \frac{z}{C} & \text{if } z > 0, \ c = 1, 2, \dots \end{cases}$$

Here

$$B = \sum_{j=1}^{k} z(a\lambda_j)^c p_j \Gamma(1 - c, a\lambda_j/z) + z^{c+1} (1 - \sum_{j=1}^{k} p_j e^{-\lambda_j a/z})$$

and

$$C = z^{3} - z^{3} \sum_{j=1}^{k} p_{j} e^{-\lambda_{j} a/z} + z^{2} a \sum_{j=1}^{k} p_{j} \lambda_{j} E_{c}(\lambda_{j} a/z)$$

Proof. By definition of the Hazard function $h_Z(z) = \frac{f_Z(z)}{S_Z(z)}$.

4. Applications

4.1. Electric circuit

In several applications, it is necessary to know the properties of the product of random variables: this occurs in particular when the random variables involved have dimensions of a ratio like fuel consumption per mile, cost of a structure per 1 lb. of payload, amplification ratio, tolerances expressed in percentages

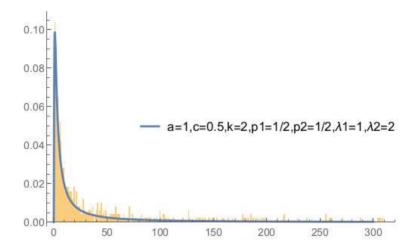


Figure 3: Monte Carlo simulation of the ratio of Pareto and Hyper-exponential for $k=2,\ a=1,\ c=0.5,\ \lambda_1=1,\ \lambda_2=2,\ p_1=\frac{1}{2},\ p_2=\frac{1}{2}.$

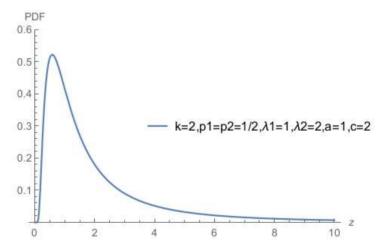


Figure 4: Plots of the pdf (44) for $k=2,\ a=1,\ c=2,\ \lambda_1=1,\ \lambda_2=2,\ p_1=\frac{1}{2},\ p_2=\frac{1}{2}.$

of the desired value, etc. Thus, for instance, if the number of accidents in a period can be regarded as a random variable and if the same applies to the number of days spent in hospital by an accident victim and to the total cost per one day-patient then the total cost is a product of these three random variables. Another application occurs, for instance, in the case when amplifiers

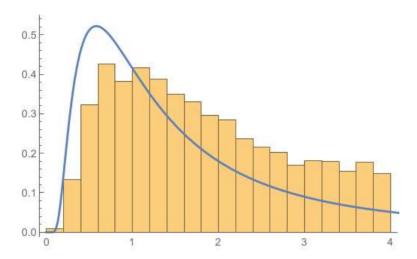


Figure 5: Monte Carlo simulation of the ratio of Pareto and Hyper-exponential (44) for $k=2,\ a=1,\ c=2,\ \lambda_1=1,\ \lambda_2=2,\ p_1=\frac{1}{2},$ $p_2=\frac{1}{2}.$

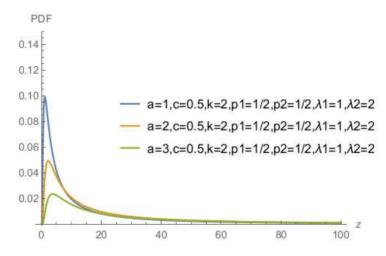


Figure 6: Plots of the pdf (44) for $k=2, a=1,2,3, c=0.5, \lambda_1=1, \lambda_2=2, p_1=\frac{1}{2}, p_2=\frac{1}{2}.$

are connected in series. If x, is the random variable describing the amplification of the *ith* amplifier then the total amplification $x = x_1x_2...x_n$ is also a random variable and it is important to know the distribution of this product. Example, suppose an electric circuit with two amplifiers in series, X_1 is a random variable

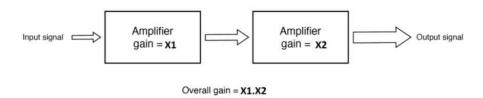


Figure 7: An electric circuit with two amplifiers in series.

follows Pareto distribution with parameter c=3 and a=1, and X_2 is a random variable follows Hyperexponential distribution with parameter $p_1=\frac{1}{2},\ p_2=\frac{1}{2},\ \lambda_1=1,\ \lambda_2=1$, then the total amplification gain is $Z=X_1.X_2$, and by using our result their pdf is

$$f_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ \frac{3[\Gamma(4) - \Gamma(4, z)]}{z^4}, & \text{if } z > 0. \end{cases}$$
 (48)

4.2. Selection of a space workhorse booster

Many experts agree that all long-range space explorations to be attempted in the intermediate future will best be initiated from an earth or parking orbit. In this respect, the success of such space missions will then greatly depend upon the successful completion of a very important logistics operation. This operation will involve the transportation of all necessary equipment and supplies into the earth's orbit and the assembly of it there. The selection of a suitable "workhorse" booster for this phase of the project represents only one of the myriad of complex decisions facing the project planners. But, since this decision must be made years ahead of most others and will involve great expenditures of money, manpower, and time, it perhaps represents the most important presentday space industry problem. Let us assume that, at present, the choice of a workhorse booster is restricted to typical systems such as the Nova, Saturn 0-5, and the Titan III systems. The concepts of these boosters differ considerably, and any operational boosters forthcoming from thee projects will most assuredly differ in respect to payload capacity, costs, velocities, reliability, date of assembly of the first operational booster, etc. Thus the problem of an "Optimal" selection of one booster system for the transportation job must be based on a very realistic evaluation model. One meaningful index for an evaluation model of this type is the "dollar" cost per pound of equipment placed in orbit.

This index would be obtained, of course, by dividing the total system cost, C_{system} , by "the payload weight placed in orbit, $W_{Payload}$, or

$$I = \frac{C_{system}}{W_{Pauload}},\tag{49}$$

a closer investigation of C_{system} and $W_{payload}$ reveals that these components are functions of variable components. For example, total system costs include both research and development costs and operational costs estimates of which are highly uncertain. Uncertainties in R & D cost estimates intervene in the form of unanticipated differences in estimated costs of parts and increased expenditures caused by modifications in the design to meet new or revised performance specifications. Operational costs are highly susceptible to the failure rates of the parts and, more generally, to the level of "Sophistication" in the supporting logistics system. The weight and space of payload placed in orbit are functionally dependent upon various characteristics of the missile such as Specific Impulse, Isp, the weight-to-stage weight ratio, λ_1 and other aerodynamic characteristics, λ_i . It becomes apparent from these considerations that equation (49) represents a complex random variable and would be better represented by

$$I = \frac{cost_{R\&D} + C_{Operational}}{f(Isp, \lambda_1, \lambda_2, ..., v)},$$
(50)

additionally, each component of C_{system} and $W_{payload}$ is random in respect to measurement or estimate error. For instance, experimental data collected from operational booster systems to date show that the Isp of an "average" missile of the system varies from the parameter value the designer had intended. Without the knowledge of how to combine the functional forms of the components of I, only a point estimate of I will be possible. On this basis, the choice of a booster system would be greatly influenced by the relative differences in the point estimates of I for these systems. On the other hand, the c.d.f.'s of I, F(I), for each booster may be obtained with knowledge of random variable techniques. For instance, if the total system cost C_{system} is a random variable follows Pareto distribution with parameter c=0.5 and a=3, and the payload weight placed in orbit $W_{payload}$ is a random variable follows Hyperexponential distribution with parameter $p_1=\frac{1}{2},\ p_2=\frac{1}{2},\ \lambda_1=1,\ \lambda_2=1$. then by using our result their pdf is

$$f_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ \frac{3^{0.5}\Gamma(1/2, 3/z)}{2z^{1.5}}, & \text{if } z > 0. \end{cases}$$
 (51)

5. Conclusion

This paper has derived the analytical expressions of the PDF, CDF, the rth moment function, the variance, the survival function, and the hazard function, for the distributions of XY and X/Y when X and Y are Pareto and Hyper-exponential random variables distributed independently of each other. we illustrate our results with some graphics of the pdf for the distributions of product and ratio. Finally, we have discussed two examples of engineering applications for the distribution of the product and ratio, and we assured our results using Monte Carlo simulation.

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