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THE SIZE MULTIPARTITE RAMSEY NUMBERS $m_j(P_n, K_{j \times b})$

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Abstract: For given two graphs G_1 and G_2 , the size Ramsey multipartite numbers $m_j(G_1, G_2)$ is the smallest integer t such that every factorization of graph $K_{j \times t} := F_1 \oplus F_2$ satisfies the following condition: either F_1 contains G_1 as a subgraph or F_2 contains G_2 as a subgraph. In this paper, we determine for the size multipartite Ramsey numbers $m_j(P_n, K_{j \times b})$ with integers $j, n \geq 3$ and $b \geq 2$.

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1. Introduction

Let G = (V, E) be a graph with the vertex-set V(G) and edge-set E(G). All graphs in this paper are finite and simple. For any set $S \subset V(G)$, the *induced* subgraph G[S] of G by S is the maximal subgraph of G with the vertex-set

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S. If $e = uv \in E$ then u is called *adjacent* to v. A graph G is said to be factorable into factors G_1, \dots, G_n if these factors are pairwise edge-disjoint and $\bigcup_{i=1}^n E(G_i) = E(G)$. If G is factored into G_1, \dots, G_n , then $G := G_1 \oplus \dots \oplus G_n$, which is called a factorization of G.

For two given graphs G_1 and G_2 , the Ramsey number $r(G_1, G_2)$ is the smallest positive integer n such that for any graph G of order n, the following statement is always true: either G contains G_1 or \overline{G} contains G_2 , where \overline{G} is the complement of G. Burger and Vuuren [1] studied one of the generalizations of this Ramsey number as follows. Let j, l, n, s and t be natural numbers with $n, s \geq 2$. Let $K_{n \times l}$ denote a complete, balanced, multipartite graph consisting of n partite sets and l vertices in each partite set. The size multipartite Ramsey number $m_j(K_{n \times l}, K_{s \times t})$ is the smallest natural number ζ such that any red-blue coloring of the edges of $K_{j \times \zeta}$ forces either a red $K_{n \times l}$ or a blue $K_{s \times t}$ as a subgraph.

In this paper, we consider a generalization of this concept by releasing completeness requirement in the forbidden graphs as follows. Let $j \geq 2$ be a natural number. For graphs G_1, \dots, G_k , the (generalized) size multipartite Ramsey number $m_j(G_1, \dots, G_k)$ is the smallest natural number m so that any k-coloring of the edges of $K_{j\times m}$ will yield a copy of G_i in the ith color for some i. Hattingh and Henning [3] determined the exact values of bipartite Ramsey numbers $b(P_m, K_{1,n})$. For the case of k = 2, with G_1 and G_2 are complete balanced multipartite graphs, the numbers can be derived from result Burger and van Vuuren [1]. Gyárfás et al. [2] studied the Ramsey-type problem in directed and bipartite graphs. Furthermore, Syafrizal et al. [4] determined Path-path size multipartite Ramsey numbers. For the combination path versus complete, balanced, multipartite graphs, Syafrizal [5] showed lower bound of $m_j(P_n, K_{j\times b})$ for integers $j, n \geq 3$ and $b \geq 2$.

Theorem 1. For integers $j, n \geq 3$ and $b \geq 2$,

$$m_{j}(P_{n}, K_{j \times b}) \ge \begin{cases} (n-1)b, & \text{if } 3 \le n \le j, \\ jb, & \text{if } j < n < jb, \\ (j-1)\lfloor \frac{n-2}{2} \rfloor + b, & \text{if } n \ge jb. \end{cases}$$

2. Main results

For any natural number j, clearly that $m_j(P_2, K_{j \times 1}) \geq 1$. Consider the factorization of $K_{j \times 1} = F_1 \oplus F_2$ such that F_1 contains no P_2 . Thus, F_1 consist of independent vertices. This implies $F_2 \cong K_{j \times 1}$.

The aim of this paper is to derive some upper bounds for the size multipartite Ramsey numbers $m_j(P_n, K_{j \times b})$. In this note, we prove the following theorem.

Theorem 2. For integers $j, n \geq 3$ and $b \geq 2$,

$$m_{j}(P_{n}, K_{j \times b}) = \begin{cases} (n-1)b, & \text{if } 3 \leq n \leq j, \\ jb, & \text{if } j < n < jb, \\ (j-1)\lfloor \frac{n-2}{2} \rfloor + b, & \text{if } n \geq jb. \end{cases}$$

Proof. We consider three cases.

Case 1. For $3 \le n \le j$.

By Theorem 1, we have $m_j(P_n, K_{j \times b}) \ge (n-1)b$. To show that $m_j(P_n, K_{j \times b}) \le (n-1)b$, consider $F \cong K_{j \times t}$. Let $F_1 \oplus F_2$ be any factorization of F such that F_1 contains no P_n . To show that F_2 contains $K_{j \times b}$, we can do the following process.

Let $V_1 = \{v_{1,1}, \ldots, v_{1,(n-1)b}\}, \cdots, V_j = \{v_{j,1}, \ldots, v_{j,(n-1)b}\}$ be the partite sets of F. Since F_1 contains no P_n then the longest path in F_1 is at most (n-1) vertices. Since $|F_1| = j(n-1)b$, then there are at most jb longest path in F_1 . Without loss of generality, suppose

$$F_1 \cong \bigcup_{l=1}^j F[V_l^*]$$
 where $V_l^* \cong \bigcup_{i=1}^b V_l(K_{n-1}^i)$

where $l \in \{1,2,\ldots,j\}$ and $V_l(K_{n-1}^i) = \{u_{1,i},u_{2,i},\ldots,u_{(n-1),i}\}$ is the set of vertices of complete graph K_{n-1}^i on (n-1) vertices with $i \in \{1,2,\ldots,b\}$. Let P_l^i be the longest path in $V_l(K_{n-1}^i)$ and let $u_{li}^* \in V_l(K_{n-1}^i)$ be the end vertex of P_l^i in F_1 for every $l=1,2,\ldots,j$ and $i=1,2,\ldots,b$. Since $u_{li}^* \in V_l(K_{n-1}^i)$ is a end vertex of P_l^i then u_{li}^* is not adjacent to every other vertex in F_2 . Since the number of complete graph K_{n-1}^i is b, then we have the new partite sets $V_1^* = \{u_{1,1}^*, u_{1,2}^*, \ldots, u_{1,b}^*\}, \ V_2^* = \{u_{2,1}^*, u_{2,2}^*, \ldots, u_{2,b}^*\}, \ \cdots, \ V_j^* = \{u_{j,1}^*, u_{j,2}^*, \ldots, u_{j,b}^*\}$ will be form $K_{j\times b}$ in F_2 . Therefore, $m_j(P_n, K_{j\times b}) \leq (n-1)b$. For example of Case 1, we can see as follows.

Example 1. $m_5(P_5, K_{5\times 2}) = 8$.

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Proof. By Theorem 1, we have $m_5(P_5, K_{5\times 2}) \geq 8$. To show that $m_5(P_5, K_{5\times 2}) \leq 8$, consider $F \cong K_{5\times 8}$. Let $F_1 \oplus F_2$ be any factorization of F such that F_1 contains no P_5 . We will show that F_2 contains $K_{5\times 2}$ as follows. Let $V_1 = \{v_{1,1}, v_{1,2}, \dots, v_{1,8}\}, \dots, V_5 = \{v_{5,1}, v_{5,2}, \dots, v_{5,8}\}$, be the partite sets of F. Since F_1 contains no P_5 , without loss of generality, suppose $F_1 \cong 8K_3 \cup 2K_3$. Now, we construct the new partite sets of F_2 as follows

$$V_l^* = V_l(K_4^1) \cup V_l(K_4^2),$$

where $l \in \{1, 2, 3, 4, 5\}$ and $V_l(K_4^i) = \{u_{1,i}, u_{2,i}, \dots, u_{4,i}\}$ is the set of vertices of complete graph K_4^i on 4 vertices with $i \in \{1, 2\}$. Next, we take one vertex $u_{li}^* \in V_l(K_4^i)$ for every i = 1, 2 and $l = 1, \dots, 5$. Since $3 \le n \le j$, then we have the new partite sets $V_1^* = \{u_{1,1}^*, u_{1,2}^*\}, \dots, V_5^* = \{u_{5,1}^*, u_{5,2}^*\}$ will be form $K_{5\times 2}$ in F_2 . Therefore, $m_5(P_5, K_{5\times 2}) \le 8$.

Case 2. For j < n < jb.

By Theorem 1, we have $m_j(P_n, K_{j \times b}) \geq jb$. We will show that $m_j(P_n, K_{j \times b}) \leq jb$, consider $F \cong K_{j \times jb}$. Let $F_1 \oplus F_2$ be any factorization of F such that F_1 contains no P_n . We will to show that F_2 contains $K_{j \times b}$.

Suppose t = jb. Let $V_1 = \{v_{1,1}, v_{1,2}, \dots, v_{1,t}\}, \dots, V_j = \{v_{j,1}, v_{j,2}, \dots, v_{j,t}\}$ be the partite sets of F. Since F_1 contains no P_n , without loss of generality, suppose $F_1 \cong (n-1)bK_{n-1} \cup (b(j-1)-n+1)K_{n-1}$. Now, we construct the new partite sets of F_2 as follows

$$V_l^* = \bigcup_{i=1}^b V_l(K_{n-1}^i),$$

where $l \in \{1,2,\ldots,j\}$ and $V_l(K_{n-1}^i) = \{u_{1,i},u_{2,i},\ldots,u_{(n-1),i}\}$ is the set of vertices of complete graph K_{n-1}^i on (n-1) vertices with $i \in \{1,2,\ldots,b\}$. Next, we take one vertex $u_{li}^* \in V_l(K_{n-1}^i)$ for every $i=1,2,\ldots,b$ and $l=1,2,\ldots,j$. Since $j \leq n \leq jb$, then we have the new partite sets $V_1^* = \{u_{1,1}^*,u_{1,2}^*,\ldots,u_{1,b}^*\},\cdots,V_j^* = \{u_{j,1}^*,u_{j,2}^*,\ldots,u_{j,b}^*\}$ will be form $K_{j\times b}$ in F_2 . Therefore, $m_j(P_n,K_{j\times b}) \leq jb$. For example of Case 2, we can see as follows.

Example 2. $m_5(P_7, K_{5\times 2}) = 10.$

Proof. By Theorem 1, we have $m_5(P_7, K_{5\times 2}) \geq 10$. To show that $m_5(P_7, K_{5\times 2}) \leq 10$, consider $F \cong K_{5\times 10}$. Let $F_1 \oplus F_2$ be any factorization of F such that F_1 contains no P_7 . We will show that F_2 contains $K_{5\times 2}$ as follows.

Let $V_1 = \{v_{1,1}, v_{1,2}, \dots, v_{1,10}\}, \dots, V_5 = \{v_{5,1}, v_{5,2}, \dots, v_{5,10}\}$ be the partite sets of F. Since F_1 contains no P_7 , without loss of generality, suppose $F_1 \cong 8K_3 \cup 2K_3$. Now, we construct the new partite sets of F_2 thats is

$$V_l^* = V_l(K_4^1) \cup V_l(K_4^2),$$

where $l \in \{1, 2, 3, 4, 5\}$ and $V_l(K_4^i) = \{u_{1,i}, u_{2,i}, \dots, u_{4,i}\}$ is the set of vertices of complete graph K_4^i on 4 vertices with $i \in \{1, 2\}$. Next, we take one vertex $u_{li}^* \in V_l(K_4^i)$ for every i = 1, 2 and $l = 1, \dots, 5$. Thus, we have the new partite sets $V_1^* = \{u_{1,1}^*, u_{1,2}^*\}, \ V_2^* = \{u_{2,1}^*, u_{2,2}^*\}, \ \dots, \ V_5^* = \{u_{5,1}^*, u_{5,2}^*\}$ will be form $K_{5\times 2}$ in F_2 . Therefore, $m_5(P_5, K_{5\times 2}) \leq 10$.

Case 3. For $n \geq jb$.

Let $t = (j-1)\lfloor \frac{n-2}{2} \rfloor + b$, by Theorem 1, we have $m_j(P_n, K_{j \times b}) \geq t$. To show that $m_j(P_n, K_{j \times b}) \leq t$, consider $F \cong K_{j \times t}$. Let $F_1 \oplus F_2$ be any factorization of F such that F_1 contains no P_n . To show that F_2 contains $K_{j \times b}$, we can do the following process.

Let

$$V_1 = \{v_{1,1}, v_{1,2}, \dots, v_{1,t}\}, \ V_2 = \{v_{2,1}, v_{2,2}, \dots, v_{2,t}\}, \cdots, V_j = \{v_{j,1}, v_{j,2}, \dots, v_{j,t}\}$$

be the partite sets of F. Since F_1 contains no P_n then the longest path in F_1 is at most (n-1) vertices. Since $|F_1| = jt$, then there are at most jt/(n-1) longest path in F_1 . Without loss of generality, suppose

$$F_1 \cong \bigcup_{l=1}^{j} F[V_l^*]$$
 where $V_l^* \cong \bigcup_{i=1}^{b} V_l(K_{n-1}^i)$,

where $l \in \{1,2,\ldots,j\}$ and $V_l(K_{n-1}^i) = \{u_{1,i},u_{2,i},\ldots,u_{(n-1),i}\}$ is the set of vertices of complete graph K_{n-1}^i on (n-1) vertices with $i \in \{1,2,\ldots,b\}$. Let P_l^i be the longest path in $V_l(K_{n-1}^i)$ and let $u_{li}^* \in V_l(K_{n-1}^i)$ be the end vertex of P_l^i for every $i=1,2,\ldots,b$ and $l=1,2,\ldots,j$. Since $u_{li}^* \in V_l(K_{n-1}^i)$ is a end vertex of P_l^i then u_{li}^* is not adjacent to every other vertex in F_2 . Since the number of complete graph K_{n-1}^i is b, then we have the new partite sets $V_1^* = \{u_{1,1}^*, u_{1,2}^*, \ldots, u_{1,b}^*\}, \cdots, V_j^* = \{u_{j,1}^*, u_{j,2}^*, \ldots, u_{j,b}^*\}$ will be form $K_{j \times b}$ in F_2 . Therefore, $m_j(P_n, K_{j \times b}) \leq t$. For example of Case 3, we can see as follows.

Example 3. $m_5(P_{10}, K_{5\times 2}) = 18.$

Proof. By Theorem 1, we have $m_5(P_{10}, K_{5\times 2}) \ge 18$. To show that $m_5(P_{10}, K_{5\times 2}) \le 18$, consider $F \cong K_{5\times 18}$. Let $F_1 \oplus F_2$ be any factorization of F

such that F_1 contains no P_{10} . We will to show that F_2 contains $K_{5\times 2}$. Let $V_1=\{v_{1,1},v_{1,2},\ldots,v_{1,18}\},\ V_2=\{v_{2,1},v_{2,2},\ldots,v_{2,18}\},\ \cdots,\ V_5=\{v_{5,1},v_{5,2},\ldots,v_{5,18}\}$ be the partite sets of F. Since F_1 contains no P_{18} then the longest path in F_1 is at most 17 vertices on every path. Since $|F_1|=5(18)$, then there are at most 10 longest path in F_1 . Without loss of generality, suppose $(u_{1,1}^*,u_{1,2})^*,(u_{2,1}^*,u_{2,2}^*),(u_{3,1}^*,u_{3,2}^*),(u_{4,1}^*,u_{4,2}^*),$ and $(u_{5,1}^*,u_{5,2}^*)$ are end vertices of longest path in V_1^*,V_2^*,V_3^*,V_4^* , and V_5^* respectively. All these vertices $(u_{1,1}^*,u_{1,2})^*,(u_{2,1}^*,u_{2,2}^*),(u_{3,1}^*,u_{3,2}^*),(u_{4,1}^*,u_{4,2}^*),$ and $(u_{5,1}^*,u_{5,2}^*)$ will form $K_{5\times 2}$ in F_2 . Therefore $m_5(P_{10},K_{5\times 2})\leq 18$.

This completes the proof of Theorem 2.

As a consequence of this theorem, we have a corollary as follows.

Corollary 3. For integers $n \geq 3$ and $j, b \geq 2$,

$$m_j(C_n, K_{j \times b}) = \begin{cases} (n-1)b, & \text{if } 3 \le n \le j, \\ jb, & \text{if } j < n < jb, \\ (j-1)\lfloor \frac{n-2}{2} \rfloor + b, & \text{if } n \ge jb. \end{cases}$$

Proof. Let $F \cong K_{j \times r}$ with r are (n-1)b (for $3 \le n \le j$), jb (for j < n < jb), and $(j-1)\lfloor \frac{n-2}{2} \rfloor + b$ (for $n \ge jb$). Let $F \cong F_1 \oplus F_2$ such that F_1 contains no $K_{j \times b}$. We will show that F_1 contains path P_n . For j=2, since F is bipartite graph then F contains no odd cycle. Hence, $m_2(C_n, K_{j \times b}) = \infty$ for any odd $n \ge 5$.

Next, for j=2 and n is even, and for $j\geq 3$. Let ${}_uP_v$ be the final path obtained in the proof of Theorem 1. This path consists of at least n vertices. Since uv is a edge F_1 then by joining the two vertices u and v, we have a cycle C_n in F_1 with at least n vertices.

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