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# EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A DISCRETE FRACTIONAL BOUNDARY VALUE PROBLEM

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**Abstract:** This present work discusses existence and uniqueness of solutions for the following discrete fractional antiperiodic boundary value problem of the form

$$ {}_0^C\Delta_k^\alpha x(k) = f\left(k+\alpha-1, x(k+\alpha-1)\right), $$

for $k \in [0, \ell+2]_{N_0} = \{0, 1, ..., \ell+2\}$, with boundary conditions $x(\alpha-3) = -x(\alpha+\ell)$, $\Delta x(\alpha-3) = -\Delta x(\alpha+\ell)$, $\Delta^2 x(\alpha-3) = -\Delta^2 x(\alpha+\ell)$, where $f : [\alpha-2, \alpha+\ell]_{N_{\alpha-2}} \times R \to R$ is continuous and ${}_0^C\Delta_k^\alpha$ is the Caputo fractional difference operator with order $2 < \alpha \leq 3$. Finally, the main results are illustrated by suitable examples.

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**Key Words:** existence, uniqueness, discrete, fractional order, boundary value problem

## 1. Introduction

The theory of fractional calculus is one of the branches of study which involves integrals and derivatives of an arbitrary order. In the past few decades, fractional calculus is one of the most novel types of tools applied to problems in engineering, science and technology (see [16], [17]). It also played a key role for

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the improvement of the specialized research in mathematical modeling of several phenomena such as enzyme kinetics, nonlinear oscillation of earthquake, blood flow problems, aerodynamics, cancer modeling being among the popular ones, control theory, electrical circuits, fluid-dynamic traffic model, regular variation in thermodynamics, polymer rheology, arterial and heart disease modeling, etc (see [13], [14]). In recent years, the theory of discrete fractional calculus has been developed by a very few researchers (see [8], [9], [11], [15]) and they obtained many important properties and interesting fruitful results.

Anti-periodic boundary value problems (classical and fractional), recently have received great attention as they occur in the mathematical modeling of a diversity of physical processes (see [2], [3], [4], [7]). As a matter of fact, many numerical problems converge faster when anti-periodic boundary conditions are used instead of periodic boundary conditions. However, the concept of non local anti-periodic boundary conditions has not been addressed yet. In the last two decades, many researchers have focused their attention on the study of stability of solutions, existence and uniqueness, positive and multiplicity of solutions of boundary value problems (BVP's) for nonlinear fractional differential and difference equations by using fixed point techniques such as Krasnoselskii fixed point theorem, Brower fixed point theorem and contraction mapping principle (see [1], [3], [5], [6], [11], [12], [15]).

Although the discrete boundary value problems with fractional order $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$ have been studied by researchers, very little is known in the literature about discrete fractional equation with order $2 < \alpha \leq 3$. Motivated by the aforesaid work, in this present work, we investigate the existence and uniqueness of solutions for the following discrete fractional order antiperiodic boundary value problem of the form

$$ {}_{0}^{C}\Delta_{k}^{\alpha} x(k) = f\left(k+\alpha-1, x(k+\alpha-1)\right), \tag{1} $$

for $k \in [0, \ell+2]_{N_0} = \{0, 1, ..., \ell+2\}$, subject to the conditions

$$ \begin{aligned} &\text{(i) } x(\alpha-3) = -x(\alpha+\ell), \text{ (ii) } \Delta x(\alpha-3) = -\Delta x(\alpha+\ell), \text{ and} \\ &\text{(iii) } \Delta^2 x(\alpha-3) = -\Delta^2 x(\alpha+\ell), \end{aligned} \tag{2} $$

where $f : [\alpha-2, \alpha+\ell]_{N_{\alpha-2}} \times R \to R$ is a continuous, ${}_{0}^{C}\Delta_{k}^{\alpha}$ is the Caputo fractional difference operator with order $2 < \alpha \leq 3$.

The plan of this paper is as follows. Some definitions and theorems needed to prove the main results are provided in Section 2. In Section 3, a form of solutions of boundary value problem (1)-(2) is obtained. In Section 4, existence and uniqueness of a solution to equation (1)-(2) are established by using the

contraction mapping principle and Brouwer fixed point theorem. In Section 5, suitable examples to illustrate the main results are presented and the paper ends with a conclusion.

## 2. Preliminaries

This section presents some required definitions and theorems which are needed throughout this paper.

**Definition 1.** (see [8], [10]) Let $\alpha > 0$. The $\alpha^{th}$ fractional sum of $f : N_a \to R$ is defined by

$$\Delta^{-\alpha} f(k) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{k-\alpha} (k-s-1)^{\underline{\alpha-1}} f(s), \tag{3}$$

for all $k \in \{a+\alpha, a+\alpha+1, \ldots\} =: N_{a+\alpha}$ and $k^{\underline{\alpha}} := \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}$.

**Definition 2.** (see [8]) Let $\alpha > 0$ and set $\mu = n - \alpha$. Then, the $\alpha^{th}$ fractional Caputo difference operator is defined as

$${}_0^C\Delta_k^{\alpha} f(k) = \Delta^{-\mu}\left(\Delta^n f(k)\right) = \frac{1}{\Gamma(\mu)} \sum_{s=a}^{k-\mu} (k-s-1)^{\underline{\mu-1}}\, \Delta^n f(s), \tag{4}$$

for all $k \in \{a+\mu, a+\mu+1, \ldots\} =: N_{a+\mu}$ and $n-1 < \alpha < n$, where $n = \lceil \alpha \rceil$, $\lceil . \rceil$ is the ceiling function.

**Theorem 3.** (see [8]) *Assume that $\alpha > 0$ and $f$ is defined on $N_a$. Then*

$$\begin{aligned} {}_0^C\Delta_k^{\alpha}\Delta^{-\alpha} f(k) &= f(k) - \sum_{j=0}^{n-1} \frac{(k-a)^{\underline{j}}}{j!} \Delta^j f(a) \\ &= f(k) + c_0 + c_1 k + \ldots + c_{n-1} k^{\underline{n-1}}, \end{aligned} \tag{5}$$

*for some $c_i \in R$, where $i = 1, 2, \ldots, n-1$.*

**Theorem 4.** (see [8]) *Let $f : N_{a+\alpha} \times N_a \to R$ be given. Then*

$$\Delta\left(\sum_{s=a}^{k-\alpha} f(k,s)\right) = \sum_{s=a}^{k-\alpha} \Delta_k f(k,s) + f(k+1, k+1-\alpha), \tag{6}$$

*for all $k \in \{a+\alpha, a+\alpha+1, \ldots\} =: N_{a+\alpha}$.*

## 3. Solution for antiperiodic BVP

In order to discuss the main results, now we state and prove important theorems which will aid to obtain a form of the solution of (1)-(2), provided the solution exists.

**Theorem 5.** *Let $2 < \alpha \leq 3$ and $f : [\alpha-2, \cdots, \alpha+\ell]_{N_{\alpha-2}} \to R$ be given. A function $x(k)$ is a solution of a discrete fractional antiperiodic boundary value problem*

$$\begin{aligned} {}_0^C\Delta_k^\alpha x(k) &= f(k+\alpha-1), \\ x(\alpha-3) &= -x(\alpha+\ell),\ \Delta x(\alpha-3) = -\Delta x(\alpha+\ell), \\ \Delta^2 x(\alpha-3) &= -\Delta^2 x(\alpha+\ell), \end{aligned} \tag{7}$$

*where $k \in [0, \ell+2]_{N_0}$, if and only if $x(k)$, for $k \in [\alpha-3, \alpha+\ell]_{N_{\alpha-3}}$ has the form*

$$\begin{aligned} x(k) &= \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{k-\alpha} (k-s-1)^{\underline{\alpha-1}} f(s+\alpha-1) \\ &- \frac{1}{2\Gamma(\alpha)} \sum_{s=0}^{\ell} (\alpha+\ell-s-1)^{\underline{\alpha-1}} f(s+\alpha-1) \\ &+ \frac{A(k)}{4\Gamma(\alpha-1)} \sum_{s=0}^{\ell+1} (\alpha+\ell-s-1)^{\underline{\alpha-2}} f(s+\alpha-1) \\ &+ \frac{B(k)}{8\Gamma(\alpha-2)} \sum_{s=0}^{\ell+2} (\alpha+\ell-s-1)^{\underline{\alpha-3}} f(s+\alpha-1), \end{aligned} \tag{8}$$

*where $A(k) = [(2\alpha-3)+(\ell-2k)]$ and $B(k) = [2(\ell k - k^2) + 4k(\alpha-1) - 2\alpha(\alpha+\ell-2) + (5\ell+3)]$.*

*Proof.* Suppose that $x(k)$ defined on $[\alpha-3, \alpha+\ell]_{N_{\alpha-3}}$ is a solution of (7). From Theorem 3, we obtain a general solution for (7) as

$$\begin{aligned} x(k) &= \Delta^{-\alpha} f(k+\alpha-1) - c_0 - c_1 k - c_2 k^{\underline{2}} \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{k-\alpha} (k-s-1)^{\underline{\alpha-1}} f(s+\alpha-1) - c_0 - c_1 k - c_2 k^{\underline{2}}, \end{aligned} \tag{9}$$

for some $c_0$, $c_1$, $c_2 \in R$. Using Theorem 4, we obtain

$$\Delta x(k) = \frac{1}{\Gamma(\alpha-1)} \sum_{s=0}^{k-\alpha+1} (k-s-1)^{\underline{\alpha-2}} f(s+\alpha-1) - c_1 - 2c_2 k,$$

$$\Delta^2 x(k) = \frac{1}{\Gamma(\alpha-2)} \sum_{s=0}^{k-\alpha+2} (k-s-1)^{\underline{\alpha-3}} f(s+\alpha-1) - 2c_2.$$

In view of $x(\alpha-3) = -x(\alpha+\ell)$, $\Delta x(\alpha-3) = -\Delta x(\alpha+\ell)$ and $\Delta^2 x(\alpha-3) = -\Delta^2 x(\alpha+\ell)$, the value of $c_0$, $c_1$ and $c_2$ as follows:

$$\begin{aligned}
c_0 &= \frac{1}{2\Gamma(\alpha)} \sum_{s=0}^{\ell} (\alpha+\ell-s-1)^{\underline{\alpha-1}} f(s+\alpha-1) \\
&\quad - \frac{(2\alpha+\ell-3)}{4\Gamma(\alpha-1)} \sum_{s=0}^{\ell+1} (\alpha+\ell-s-1)^{\underline{\alpha-2}} f(s+\alpha-1) \\
&\quad + \frac{[2\alpha(\alpha+\ell-2)-(5\ell+3)]}{8\Gamma(\alpha-2)} \sum_{s=0}^{\ell+2} (\alpha+\ell-s-1)^{\underline{\alpha-3}} f(s+\alpha-1), \\
c_1 &= \frac{1}{2\Gamma(\alpha-1)} \sum_{s=0}^{\ell+1} (\alpha+\ell-s-1)^{\underline{\alpha-2}} f(s+\alpha-1) \\
&\quad - \frac{(2\alpha+\ell-3)}{4\Gamma(\alpha-2)} \sum_{s=0}^{\ell+2} (\alpha+\ell-s-1)^{\underline{\alpha-3}} f(s+\alpha-1), \text{ and} \\
c_2 &= \frac{1}{4\Gamma(\alpha-2)} \sum_{s=0}^{\ell+2} (\alpha+\ell-s-1)^{\underline{\alpha-3}} f(s+\alpha-1).
\end{aligned}$$

Substituting the values of $c_0$, $c_1$ and $c_2$ into (9), we obtain (8). This shows that if (7) is a solution, then it can be represented by (8).

Conversely, every function of the form (8) is a solution of (7). The proof is completed. □

**Theorem 6.** *If $\alpha$ and $k$ are any numbers then*

$$\sum_{s=0}^{k-\alpha} (k-s-1)^{\underline{\alpha-1}} = \frac{\Gamma(k+1)}{\alpha\Gamma(k-\alpha+1)}.$$

*Proof.* For $u > m$, $u, m \in R$, $m > -1$, $u > -1$, we have

$$\frac{\Gamma(u+1)}{\Gamma(m+1)\Gamma(u-m+1)} = \frac{\Gamma(u+2)}{\Gamma(m+2)\Gamma(u-m+1)} - \frac{\Gamma(u+1)}{\Gamma(m+2)\Gamma(u-m)}.$$

That is $\dfrac{\Gamma(u+1)}{\Gamma(u-m+1)} = \dfrac{1}{m+1}\left[\dfrac{\Gamma(u+2)}{\Gamma(u-m+1)} - \dfrac{\Gamma(u+1)}{\Gamma(u-m)}\right]$.
Then,

$$\sum_{s=0}^{k-\alpha}(k-s-1)^{\underline{\alpha-1}} = \sum_{s=0}^{k-\alpha-1}(k-s-1)^{\underline{\alpha-1}} + \Gamma(\alpha). \tag{10}$$

Now we find

$$\sum_{s=0}^{k-\alpha-1}(k-s-1)^{\underline{\alpha-1}} = \frac{1}{\alpha}\left[\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} - \frac{\Gamma(\alpha+1)}{\Gamma(1)}\right]. \tag{11}$$

Substituting equation (11) in (10), we get

$$\sum_{s=0}^{k-\alpha}(k-s-1)^{\underline{\alpha-1}} = \frac{1}{\alpha}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}.$$

□

**Corollary 7.** *The following results are immediate consequences of Theorem 6:*

1. $\sum\limits_{s=0}^{\ell}(\alpha+\ell-s-1)^{\underline{\alpha-1}} = \dfrac{\Gamma(\alpha+\ell+1)}{\alpha\Gamma(\ell+1)}$.

2. $\sum\limits_{s=0}^{\ell+1}(\alpha+\ell-s-1)^{\underline{\alpha-2}} = \dfrac{\Gamma(\alpha+\ell+1)}{(\alpha-1)\Gamma(\ell+2)}$.

3. $\sum\limits_{s=0}^{\ell+2}(\alpha+\ell-s-1)^{\underline{\alpha-3}} = \dfrac{\Gamma(\alpha+\ell+1)}{(\alpha-2)\Gamma(\ell+3)}$.

## 4. Existence and uniqueness of solution

In this section, we prove that under certain conditions, discrete fractional antiperiodic boundary value problem (1)-(2) has at least one solution. We observe that boundary value problem (1)-(2) may be recast as an equivalent summation equation. It follows from Theorem 5 that $x(k)$ is a solution of (1)-(2) if and

only if $x(k)$ is a fixed point of the operator $T : R^{\ell+4} \to R$, where

$$\begin{aligned}(Tx)(k) &= \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{k-\alpha}(k-s-1)^{\underline{\alpha-1}}f(s+\alpha-1,x(s+\alpha-1))\\ &-\frac{1}{2\Gamma(\alpha)}\sum_{s=0}^{\ell}(\alpha+\ell-s-1)^{\underline{\alpha-1}}f(s+\alpha-1,x(s+\alpha-1))\\ &+\frac{A(k)}{4\Gamma(\alpha-1)}\sum_{s=0}^{\ell+1}(\alpha+\ell-s-1)^{\underline{\alpha-2}}f(s+\alpha-1,x(s+\alpha-1))\\ &+\frac{B(k)}{8\Gamma(\alpha-2)}\sum_{s=0}^{\ell+2}(\alpha+\ell-s-1)^{\underline{\alpha-3}}f(s+\alpha-1,x(s+\alpha-1)),\end{aligned} \tag{12}$$

for $k \in [\alpha-3, \alpha+\ell]_{N_{\alpha-3}}$, where $A(k) = [(2\alpha-3)+(\ell-2k)]$ and $B(k) = [2(\ell k - k^2) + 4k(\alpha-1) - 2\alpha(\alpha+\ell-2) + (5\ell+3)]$.

**Theorem 8.** *Define* $\|x\| = \max_{k\in[\alpha-3,\alpha+\ell]_{N_{\alpha-3}}} |x(k)|$. *Assume that* $f(k,x)$ *is Lipschitz in* $x$. *That is, there exists* $L > 0$ *such that* $|f(k,x) - f(k,y)| \le L|x-y|$ *whenever* $x, y \in R$. *Then it follows that discrete fractional antiperiodic boundary value problem (1)-(2) has a unique solution provided that the condition*

$$\Lambda = \beta L \frac{\Gamma(\alpha+\ell+1)}{2\Gamma(\alpha-1)\Gamma(\ell+1)} < 1, \tag{13}$$

*where* $\beta = \left(\frac{3}{\alpha(\alpha-1)} + \frac{(\ell+3)}{2(\alpha-1)(\ell+1)} + \frac{(\ell+3)}{4(\ell+1)(\ell+2)}\right)$ *holds.*

*Proof.* We show that $T$ is a contraction mapping. To achieve this, we notice that for given $x$ and $y$,

$$\begin{aligned}\|Tx - Ty\| &\le L\|x-y\| \max_{k\in[\alpha-3,\alpha+\ell]_{N_{\alpha-3}}} \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{k-\alpha}(k-s-1)^{\underline{\alpha-1}}\\ &+ L\|x-y\| \max_{k\in[\alpha-3,\alpha+\ell]_{N_{\alpha-3}}} \frac{1}{2\Gamma(\alpha)}\sum_{s=0}^{\ell}(\alpha+\ell-s-1)^{\underline{\alpha-1}}\end{aligned}$$

$$+ L\|x-y\| \max_{k\in[\alpha-3,\alpha+\ell]_{N_{\alpha-3}}} \frac{|A(k)|}{4\Gamma(\alpha-1)} \sum_{s=0}^{\ell+1} (\alpha+\ell-s-1)^{\underline{\alpha-2}}$$

$$+ L\|x-y\| \max_{k\in[\alpha-3,\alpha+\ell]_{N_{\alpha-3}}} \frac{|B(k)|}{8\Gamma(\alpha-2)} \sum_{s=0}^{\ell+2} (\alpha+\ell-s-1)^{\underline{\alpha-3}}.$$

Since $|A(k)| = |(2\alpha-3)+(\ell-2k)| = (\ell+3)$ and $|B(k)| = \left|2(\ell k-k^2)+4k(\alpha-1)-2\alpha(\alpha+\ell-2)+(5\ell+3)\right| = (\ell+3)$. By an application of Theorem 6 and Corollary 7, we obtain

$$\begin{aligned}\|Tx-Ty\| \le{}& \frac{3L\|x-y\|}{2}\left[\frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+1)\Gamma(\ell+1)}\right] \\ &+ \frac{L\|x-y\|(\ell+3)}{4}\left[\frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha)\Gamma(\ell+2)}\right] \\ &+ \frac{L\|x-y\|(\ell+3)}{8}\left[\frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha-1)\Gamma(\ell+3)}\right].\end{aligned}$$

Above inequality leads to the conclusion

$$\|Tx-Ty\| \le \left[\beta L \frac{\Gamma(\alpha+\ell+1)}{2\Gamma(\alpha-1)\Gamma(\ell+1)}\right]\|x-y\|.$$

When the condition (13) holds, boundary value problem (1) - (2) has a unique solution, which completes the proof of the theorem. □

**Theorem 9.** *Assume that there exists a constant $M>0$ such that $f(k,x)$ satisfies the inequality*

$$\max_{(k,x)\in[\alpha-3,\alpha+\ell]_{N_{\alpha-3}}\times[-M,M]} |f(k,x)| \le \frac{M}{\beta\left[\dfrac{\Gamma(\alpha+\ell+1)}{2\Gamma(\alpha-1)\Gamma(\ell+1)}\right]}, \tag{14}$$

*where* $\beta = \left(\dfrac{3}{\alpha(\alpha-1)} + \dfrac{(\ell+3)}{2(\alpha-1)(\ell+1)} + \dfrac{(\ell+3)}{4(\ell+1)(\ell+2)}\right)$. *Then it follows that discrete fractional antiperiodic boundary value problem (1) - (2) has at least one solution $x_0$ satisfying $|x_0(k)| \le M$ for all $k \in [\alpha-3,\alpha+\ell]_{N_{\alpha-3}}$.*

*Proof.* Consider the Banach space $B := \left\{x \in R^{\ell+4} : \|x\| \le M\right\}$. Let $T$ be the operator defined in (12). It is clear that $T$ is a continuous operator. Thus, the main purpose is to prove that $T : B \to B$. That is, whenever $\|x\| \le M$, it follows that $\|Tx\| \le M$. Once this is proved, we use the Brouwer fixed point

theorem to obtain the conclusion. Suppose that inequality (14) holds for given $f$. For convenience, we assume

$$\Phi := \frac{M}{\beta\left[\dfrac{\Gamma(\alpha+\ell+1)}{2\Gamma(\alpha-1)\Gamma(\ell+1)}\right]}, \tag{15}$$

which is a strictly positive constant. Then we obtain

$$\begin{aligned}\|Tx\| \;\leq\; & \frac{\Phi}{\Gamma(\alpha)}\sum_{s=0}^{k-\alpha}(k-s-1)^{\underline{\alpha-1}} + \frac{\Phi}{2\Gamma(\alpha)}\sum_{s=0}^{\ell}(\alpha+\ell-s-1)^{\underline{\alpha-1}} \\ & + \frac{\Phi\,|A(k)|}{4\Gamma(\alpha-1)}\sum_{s=0}^{\ell+1}(\alpha+\ell-s-1)^{\underline{\alpha-2}} \\ & + \frac{\Phi\,|B(k)|}{8\Gamma(\alpha-2)}\sum_{s=0}^{\ell+2}(\alpha+\ell-s-1)^{\underline{\alpha-3}}.\end{aligned}$$

Since $|A(k)| = |(2\alpha-3)+(\ell-2k)| = (\ell+3)$ and $|B(k)| = \left|2(\ell k-k^2)+4k(\alpha-1)-2\alpha(\alpha+\ell-2)+(5\ell+3)\right| = (\ell+3)$. By an application of Theorem 6 and Corollary 7, we get

$$\begin{aligned}\|Tx\| \leq\; & \frac{3\Phi}{2}\left[\frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+1)\Gamma(\ell+1)}\right] + \frac{\Phi(\ell+3)}{4}\left[\frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha)\Gamma(\ell+2)}\right] \\ & + \frac{\Phi(\ell+3)}{8}\left[\frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha-1)\Gamma(\ell+3)}\right].\end{aligned}$$

From the above inequality, we have

$$\|Tx\| \leq \beta\left[\frac{\Gamma(\alpha+\ell+1)}{2\Gamma(\alpha-1)\Gamma(\ell+1)}\right]\Phi. \tag{16}$$

Finally, by the definition of $\Phi$ given in (15), we get that (16) implies that

$$\|Tx\| \leq M. \tag{17}$$

Thus, from (17) we deduce that $T : B \to B$. Consequently, it follows from Brouwer fixed point theorem that there exists a fixed point of the map $T$ , say $Tx_0 = x_0$ with $x_0 \in B$. So this $x_0$ is a solution of (1) - (2). Moreover, $x_0$ satisfies the bound $|x_0(k)| \leq M$, for each $k \in [\alpha-3, \alpha+\ell]_{N_{\alpha-3}}$. Proof of the theorem is completed. □

## 5. Examples

This section provides two examples to illustrate Theorems 8 and 9. We start with an example illustrating Theorem 8, subsequently by an example illustrating Theorem 9.

**Example 10.** Suppose that $\alpha = \frac{5}{2}$ and $\ell = 5$. Let $f(k, x) = \frac{\cos x(k)}{1000+k^2}$ and $L = \frac{1}{1000}$. Then discrete fractional antiperiodic boundary value problem (1)-(2) becomes

$$ {}_0^C\Delta_k^{\frac{5}{2}} x(k) = \frac{\cos\left(x\left(k+\frac{3}{2}\right)\right)}{1000+\left(k+\frac{3}{2}\right)^2},\ k \in [0,7], \tag{18} $$

subject to the conditions

$$ \begin{aligned} &x\left(-\frac{1}{2}\right) = -x\left(\frac{15}{2}\right),\ \Delta x\left(-\frac{1}{2}\right) = -\Delta x\left(\frac{15}{2}\right), \\ &\Delta^2 x\left(-\frac{1}{2}\right) = -\Delta^2 x\left(\frac{15}{2}\right). \end{aligned} \tag{19} $$

In this case, inequality (13) takes the form

$$ \Lambda = \beta L \frac{\Gamma(\alpha+\ell+1)}{2\Gamma(\alpha-1)\Gamma(\ell+1)} \leq 0.0853 < 1. $$

Therefore, from Theorem 8 we conclude that boundary value problem (18) - (19) has a unique solution. For the values of $\alpha \in (2,3]$ (with an increment of 0.1), the corresponding values of $\Lambda$ are computed and tabulated in Table 1. Taking the values of $\alpha$ on the horizontal axis, the computed values of $\Lambda$ are plotted in Figure 1. The resulting curve is nonlinear and shows monotonically increasing trend in $\alpha \in (2,3]$ (with an increment of 0.025).

| $\alpha$ | 2.1 | 2.2 | 2.3 | 2.4 | 2.5 |
|---|---|---|---|---|---|
| $\Lambda = \beta L \frac{\Gamma(\alpha+\ell+1)}{2\Gamma(\alpha-1)\Gamma(\ell+1)}$ | 0.0528 | 0.0597 | 0.0674 | 0.0759 | 0.0853 |
| $\alpha$ | 2.6 | 2.7 | 2.8 | 2.9 | 3 |
| $\Lambda = \beta L \frac{\Gamma(\alpha+\ell+1)}{2\Gamma(\alpha-1)\Gamma(\ell+1)}$ | 0.0956 | 0.1069 | 0.1194 | 0.1331 | 0.1480 |

Table 1: Illustration of $2 < \alpha \leq 3$ and $\Lambda$.

When $\alpha \in (2,3]$, condition provided by Theorem 8 for the existence of unique solution for the boundary value problem (18) - (19) is satisfied, see Table 1 and Figure 1.

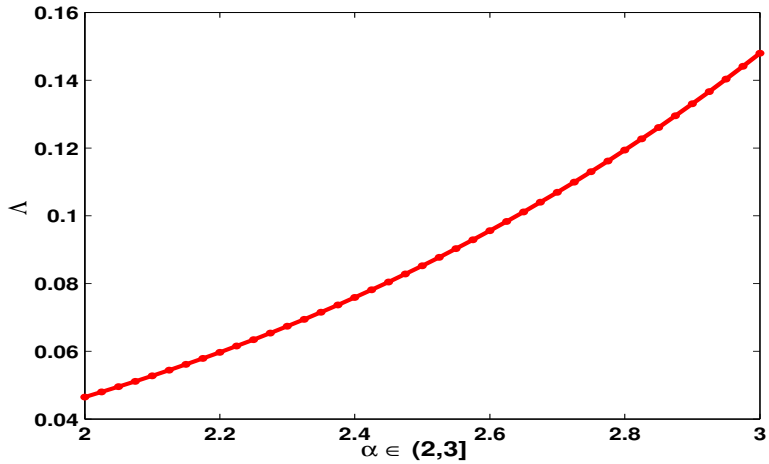


Figure 1: $\alpha$ versus $\Lambda$.

**Example 11.** Suppose that $\alpha = \frac{14}{5}$, $\ell = 7$ and $M = 1000$ with $f(k,x) = \frac{1}{21}ke^{-\frac{1}{10}x^2(k)}$. Then discrete fractional antiperiodic boundary value problem (1)-(2) takes the form

$$ {}_0^C\Delta_k^{\frac{14}{5}} x(k) = \frac{1}{21}\left(k+\frac{9}{5}\right)e^{-\frac{1}{10}x^2\left(k+\frac{9}{5}\right)},\ k \in [0,9], \tag{20} $$

subject to the conditions

$$ \begin{aligned} &x\left(-\frac{1}{5}\right) = -x\left(\frac{49}{5}\right),\ \Delta x\left(-\frac{1}{5}\right) = -\Delta x\left(\frac{49}{5}\right), \\ &\Delta^2 x\left(-\frac{1}{5}\right) = -\Delta^2 x\left(\frac{49}{5}\right). \end{aligned} \tag{21} $$

The Banach space is $B := \left\{x \in R^{11} : \|x\| \leq 1000\right\}$. We note that

$$ \frac{M}{\beta\left[\dfrac{\Gamma(\alpha+\ell+1)}{2\Gamma(\alpha-1)\Gamma(\ell+1)}\right]} = \frac{1000}{236.4345} \approx 4.2295. $$

It is clear that $|f(k,x)| \leq \frac{49}{105} < 4.2295$, whenever $x \in [-1000, 1000]$. Therefore by Theorem 9, we conclude that the boundary value problem (20) - (21) has at least one solution.

## 6. Conclusions

Arguments in this paper are based on a contraction mapping theorem and Brouwer fixed point theorem for Banach spaces. Conditions for the existence and uniqueness of solutions for the fractional order difference equations in the presence of antiperiodic boundary conditions are obtained. Results obtained in the main results are illustrated with suitable examples supported with numerical calculations.

## References

[1] M.S. Abdo, A.M. Saeed, S.K. Panchal, Caputo fractional integro-differential equation with nonlocal conditions in banach space, *International Journal of Applied Mathematics*, **32**, No 2 (2019), 279-288; DOI: 10.12732/ijam.v32i2.9.

[2] R.P. Agarwal, B. Ahmad, Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions, *Computers & Mathematics with Applications*, **62** (2011), 1200-1214.

[3] B. Ahmad, Existence of solutions for fractional differential equations of order $q \in (2, 3]$ with anti-periodic boundary conditions, *J. Appl. Math. Comput.*, **34** (2010), 385-391.

[4] B. Ahmad, J.J. Nieto, Anti-periodic fractional boundary value problems, *Computers & Mathematics with Applications*, **62** (2011), 1150-1156.

[5] M.B. Akorede, P.O. Arawomo, Existence of positive solutions for a coupled system of nonlinear boundary value problems of fractional order with integral boundary conditions, *International Journal of Applied Mathematics*, **30**, No 4 (2017), 273-288; DOI: 10.12732/ijam.v32i2.9.

[6] F.M. Atici, P.W. Eloe, Two-point boundary value problems for finite fractional difference equations, *J. Differ. Equ. Appl.*, **17** (2011), 445-456.

[7] J. Cao, Q. Yang, Z. Huang, Existence of anti-periodic mild solutions for a class of semilinear fractional differential equations, *Communications in Nonlinear Science and Numerical Simulation*, **17** (2012), 277-283.

[8] H. Chen, Z. Jin, S. Kang, Existence of positive solution for caputo fractional difference equation, *Advances in Difference Equations*, **2015** (2015), 1-12.

[9] A. George Maria Selvam, R. Dhineshbabu, Uniqueness of solutions of a discrete fractional order boundary value problem, *AIP Conference Proceeding*, **2095** (2019), 1-7.

[10] A. George Maria Selvam, R. Janagaraj, Oscillation criteria of a class of fractional order damped difference equations, *International Journal of Applied Mathematics*, **32**, No 3 (2019), 433-441; DOI: 10.12732/ijam.v32i3.5.

[11] C.S. Goodrich, Existence and uniqueness of solutions to a fractional difference equation with non local conditions, *Computers & Mathematics with Applications*, **61** (2011), 191-202.

[12] A.A. Hamoud, K.P. Ghadle, M.Sh.B. Issa, G. Giniswamy, Existence of uniqueness theorems for fractional volterra-fredholm integro-differential equations, *International Journal of Applied Mathematics*, **31**, No 3 (2018), 333-348; DOI: 10.12732/ijam.v32i3.5.

[13] S. He, K. Sun, Y. Peng, L. Wang, Modeling of discrete fracmemristor and its application, *AIP Advances*, **10** (2020), 1-6.

[14] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies 204, Elsevier, Amsterdam (2006).

[15] Y. Pan, Z. Han, S. Sun, C. Hou, The existence of solutions to a class of boundary value problems with fractional difference equations, *Advances in Difference Equations*, **2013** (2013), 1-20.

[16] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York (1999).

[17] J. Tenreiro Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus, *Communications in Nonlinear Science and Numerical Simulation*, **16** (2011), 1140-1153.