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HYERS-ULAM STABILITY OF A PERTURBED GENERALISED LIENARD EQUATION

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Abstract: In this paper, we consider the Hyers-Ulam stability of a perturbed generalized Lienard equation, using a nonlinear extension of Gronwall-Bellman integral inequality called the Bihari integral inequality.

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1. Introduction

Generalised Lienard equation has been considered by many researchers. These include: Kroopnick (see [10], [11]) who studied properties of solutions to a generalized Lienard equations with forcing term and also studied bounded L^p-solutions of generalized Lienard equation, Nkashama [13] considered periodically perturbed non conservative system of Lienard type. In 2014, Ogundare and Afuwape [15] studied conditions which guarantee boundedness and stability properties of solutions of generalized Lienard equations. However, none of these researchers have studied the Hyers-Ulam stability of the perturbed generalized

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Lienard equations of the form

$$u'' + c(t)f(u(t))u'(t) + a(t)g(u(t)) = P(t, u(t)), \tag{1}$$

where $f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $g \in C(\mathbb{R}_+, \mathbb{R}_+)$, $c, a \in C(\mathbb{I}, \mathbb{R}_+)$, for $\mathbb{R}_+ = [t_0, \infty)$, $\mathbb{I} = (t_0, b)(b \leq \infty)$, $P \in C(\mathbb{I} \times \mathbb{R}_+, \mathbb{R}_+)$. In this paper, we shall consider Hyers-Ulam stability of (1) and also the case where $P(t, u(t)) = 0$.

The stability problem of functional equation started with the question concerning stability of group homomorphism proposed by Ulam [18] in 1940 during a talk before a Mathematical Colloquium at the University of Wincosin, Madison. In 1941, Hyers [7] gave a solution of Ulam's problem for the case of approximate additive mappings in the context of Banach spaces. The result obtained by Hyers opened up research in Hyers-Ulam stability. Rassias [16] in 1978 generalized the theorem of Hyers by considering the stability problem of the unbounded Cauchy differences

$$||f(x + y) - f(x) - f(y)|| \leq \epsilon(||x||^p + ||y||^p).\ \ t > 0\ \ p \in [0, 1). \tag{2}$$

This phenomenon of the stability that was introduced by Rassias leads to Hyers-Ulam-Rassias stability (or the generalized Hyers-Ulam stability), see [8].

Thereafter, the result reported by Rassias was improved, see [14], [2], [5], [1], [17], [6], [19], [9].

2. Preliminaries

We present the following definitions, lemmas and theorems for subsequent use in this work.

Definition 1. Equation (1) is Hyers-Ulam stable, if there exists a constant $K > 0$ and $\epsilon > 0$ such that for $u(t) \in C^2(\mathbb{I}, \mathbb{R}_+)$, satisfying

$$|u'' + c(t)f(u(t))u'(t) + a(t)g(u(t)) - P(t, u(t))| \leq \epsilon, \tag{3}$$

there exists a solution $u_0(t) \in C^2(\mathbb{I}, \mathbb{R}_+)$ of the equation (1), such that $|u(t) - u_0(t)| \leq K\epsilon$, where K is called Hyers-Ulam constant with initial condition

$$u(t) = u'(t) = 0. \tag{4}$$

Theorem 2. (Generalized First Mean Value Theorem, [12]) *If $f(t)$ and $g(t)$ are continuous in $[t_0, t] \subseteq \mathbb{I}$ and $f(t)$ does not change sign in the interval, then there is a point $\xi \in [t_0, t]$ such that* $\int_{t_0}^{t} g(s)f(s)ds = g(\xi)\int_{t_0}^{t} f(s)ds.$

Definition 3. A function $\omega : [0,\infty) \to [0,\infty)$ is said to belong to a class S if:

i $\omega(u)$ is nondecreasing and continuous for $u \geq 0$.

ii $(\frac{1}{v})\omega(u) \leq \omega(\frac{u}{v})$ for all u and $v \geq 1$.

iii there exists a function ϕ, continuous on $[0,\infty)$ with $\omega(\alpha u) \leq \phi(\alpha)\omega(u)$ for $\alpha \geq 0$.

Lemma 4. (see [3], [4]) *Let $u(t)$, $f(t)$ be positive continuous functions defined on $a \leq t \leq b, (\leq \infty)$ and $K > 0$, $M \geq 0$, further let $\omega(u)$ be a nonnegative nondecreasing continuous function for $u \geq 0$, then the inequality*

$$u(t) \leq K + M\int_a^t f(s)\omega(u(s))ds, \quad a \leq t < b. \tag{5}$$

implies the inequality

$$u(t) \leq \Omega^{-1}\left(\Omega(k) + M\int_a^t f(s)ds\right), \quad a \leq t \leq b' \leq b. \tag{6}$$

where

$$\Omega(u) = \int_{u_0}^{u} \frac{dt}{\omega(t)}, \quad 0 < u_0 < u. \tag{7}$$

In the case $\omega(0) > 0$ or $\Omega(0+)$ is finite, one may take $u_0 = 0$ and Ω^{-1} is the inverse function of Ω and t must be in the subinterval $[a, b']$ of $[a, b]$ such that

$$\Omega(k) + M\int_a^t f(s)ds \in Dom(\Omega^{-1}). \tag{8}$$

3. Main Result

The main results of this work are given in the following theorems.

Theorem 5. *Let the functions a, f, c, g and P be as defined earlier such that $a(t) \geq \delta$, $a'(t) \leq 0$ on $\mathbb{I}$ with $f \in S$. Suppose that*

$$\lim_{t\to\infty}\int_{t_0}^{t} c(s)ds = M < \infty \tag{9}$$

and

$$G(u(t)) = \int_{t_0}^{t} g(u(s))ds < \infty, \tag{10}$$

then equation (1) is Hyers-Ulam stable with the Hyers-Ulam constant K given by

$$K = \frac{1}{\delta}\left(L + LA|u(\xi)|\right)\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}M\right), \tag{11}$$

where Ω is as defined in (7).

Proof. It follows from inequality (3) that

$$-\epsilon \leq u''(t) + c(t)f(u(t))u'(t) + a(t)g(u(t))u'(t) - P(t, u(t) \leq \epsilon. \tag{12}$$

Multiplying (12) by $u'(t)$, gives

$$\begin{aligned} &-\epsilon u'(t) \leq \\ &u''(t)u'(t) + c(t)f(u(t))(u'(t))^2 + a(t)g(u(t)u'(t) - P(t, u(t))u'(t) \leq \epsilon u'(t). \end{aligned} \tag{13}$$

Since $G(u(t))$ in (10) is nondecreasing, monotonic and belongs to class S, we have from (13) that

$$\begin{aligned} &-\epsilon u'(t) \leq \\ &u''(t)u'(t) + c(t)f(u(t))(u'(t))^2 + a(t)\frac{d}{dt}G(u(t)) - P(t, u(t))u'(t) \leq \epsilon u'(t). \end{aligned} \tag{14}$$

Integrating (14) from t_0 to t, we have

$$\begin{aligned} -\epsilon \int_{t_0}^{t} u'(s)ds \leq \frac{1}{2}(u'(s))^2 + \int_{t_0}^{t} c(s)f(u(s))(u'(s))^2 ds \\ + \int_{t_0}^{t} a(s)\frac{d}{ds}G(u(s))ds - \int_{t_0}^{t} P(s, u(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} u'(s)ds. \end{aligned} \tag{15}$$

It follows that

$$\begin{aligned} &\int_{t_0}^{t} c(s)f(u(s))(u'(s))^2 ds \\ &\quad + \int_{t_0}^{t} a(s)\frac{d}{ds}G(u(s))ds - \int_{t_0}^{t} P(s, u(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} u'(s)ds. \end{aligned} \tag{16}$$

Integrating (14) by parts, we have

$$\int_{t_0}^{t} c(s)f(u(s))(u'(s))^2 ds + a(t)G(u(t)) - \int_{t_0}^{t} a'(s)G(u(s))ds - \int_{t_0}^{t} P(s,u(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} u'(s)ds, \quad (17)$$

that is

$$a(t)G(u(t)) \leq \epsilon \int_{t_0}^{t} u'(s)ds - \int_{t_0}^{t} c(s)f(u(s))(u'(s))^2 ds + \int_{t_0}^{t} a'(s)G(u(s))ds + \int_{t_0}^{t} P(s,u(s))u'(s)ds. \quad (18)$$

Since $a'(t) \leq 0$ and $a(t) \geq \delta$, we have

$$\delta G(u(t)) \leq \epsilon \int_{t_0}^{t} u'(s)ds - \int_{t_0}^{t} c(s)f(u(s))(u'(s))^2 ds + \int_{t_0}^{t} P(s,u(s))u'(s)ds. \quad (19)$$

Taking the absolute value of both sides, we get

$$\delta|G(u(t))| \leq \epsilon \int_{t_0}^{t} |u'(s)|ds + \int_{t_0}^{t} c(s)f(|u(s)|)(|u'(s)|)^2 ds + \int_{t_0}^{t} |P(s,u(s))||u'(s)|ds. \quad (20)$$

Suppose $|G(u(t)| \geq |u(t)|$, $|P(t,u(t))| \leq A|u(t)|$ and $\int_{t_0}^{t} |u'(s)|ds \leq L$ for $L > 0$. It follows that

$$|u(t)| \leq \frac{1}{\delta}\epsilon L + \frac{1}{\delta}\int_{t_0}^{t} c(s)f(|u(s)|)(|u'(s)|)^2 ds + \frac{1}{\delta}A\int_{t_0}^{t} |u(s))||u'(s)|ds. \quad (21)$$

By Theorem (2), for $t_0 < \xi < t$, we have

$$|u(t)| \leq \frac{1}{\delta}\epsilon L + \frac{1}{\delta}\int_{t_0}^{t} c(s)f(|u(s)|)(|u'(s)|)^2 ds + \frac{1}{\delta}Au(\xi)\int_{t_0}^{t} |u'(s)|ds. \quad (22)$$

This gives

$$|u(t)| \leq \frac{1}{\delta}\epsilon L + \frac{1}{\delta}LA|u(\xi)| + \frac{1}{\delta}\int_{t_0}^{t} c(s)f(|u(s)|)|u'(t)|^2 ds. \tag{23}$$

It follows that

$$|u(t)| \leq \frac{1}{\delta}\epsilon L + \frac{1}{\delta}LA|u(\xi)| + \frac{(|u'(t)|)^2}{\delta}\int_{t_0}^{t} c(s)f(|u(s)|)ds. \tag{24}$$

Let $|u'(t)| \leq \lambda$, for $\lambda > 0$ this gives

$$|u(t)| \leq \frac{1}{\delta}\epsilon L + \frac{1}{\delta}LA|u(\xi)| + \frac{\lambda^2}{\delta}\int_{t_0}^{t} c(s)f(|u(s)|)ds. \tag{25}$$

Let us set

$$R = \frac{1}{\delta}\epsilon\left(L + LA|u(\xi)|\right) \quad \text{and} \quad \epsilon \geq 1. \tag{26}$$

Using (26) and the fact $f \in S$, (25) becomes

$$\frac{|u(t)|}{R} \leq 1 + \frac{\lambda^2}{\delta}\int_{t_0}^{t} c(s)f(\frac{|u(s)|}{R})ds. \tag{27}$$

Setting $\frac{|u(t)|}{R} = z(t)$, then (27) becomes

$$z(t) \leq 1 + \frac{\lambda^2}{\delta}\int_{t_0}^{t} c(s)f(z(s))ds \tag{28}$$

Let $\omega(z(t)) = f(z(t))$, By (7), we obtain

$$z(t) \leq \Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}\int_{t_0}^{t} c(s)ds\right).$$

Substituting for $z(t)$, we have

$$|u(t)| \leq R\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}\int_{t_0}^{t} c(s)ds\right).$$

Replacing R by (26), we obtain

$$|u(t)| \leq \epsilon\frac{1}{\delta}\left(L + LA|u(\xi)|\right)\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}\int_{t_0}^{t} c(s)ds\right).$$

By (9), we have

$$|u(t)| \leq \epsilon\frac{1}{\delta}\left(L + LA|u(\xi)|\right)\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}M\right).$$

Hence,

$$K = \frac{1}{\delta}\left(L + LA|u(\xi)|\right)\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}M\right).$$

Since,

$$|u(t) - u_0(t)| \leq |u(t)| \leq K\epsilon.$$

Therefore,

$$|u(t) - u_0(t)| \leq K\epsilon.$$

□

Example 6. *Consider the equation*

$$u''(t) + (t+1)^{-2}u^2u' + t^4u^4 = 2u^2(t).$$

The equation is Hyers-Ulam stable by the conditions of Theorem 5.

Next we consider the case $P(t, u(t)) = 0$.

Theorem 7. *Let all the conditions of Theorem 5 remain valid with*

$$P(t, u(t)) = 0.$$

Equation (1) is Hyers-Ulam stable with Hyers-Ulam constant defined as

$$K = \frac{1}{\delta}(L)\left(\Omega(1) + \frac{\lambda^2}{\delta}M\right).$$

Proof. From inequality (3), we have

$$-\epsilon \leq u''(t) + c(t)f(u(t))u'(t) + a(t)g(u(t))u'(t) \leq \epsilon. \tag{29}$$

Since

$$P(t, u(t)) = 0,$$

using equation (10), we have

$$-\epsilon \leq u''(t) + c(t)f(u(t))u'(t) + a(t)\frac{d}{dt}G(u(t)) \leq \epsilon. \tag{30}$$

Multiplying (30) by $u'(t)$, we obtain

$$-\epsilon u'(t) \leq u''(t)u'(t) + c(t)f(u(t))(u'(t))^2 + a(t)\frac{d}{dt}G(u(t))u'(t) \leq \epsilon. \tag{31}$$

Integrating (31) from t_0 and t, we get

$$-\epsilon\int_{t_0}^{t}u'(s)ds\leq\frac{1}{2}u'^2(t)$$
$$+\int_{t_0}^{t}c(s)f(u(s))(u'(s))^2+\int_{t_0}^{t}a(s)\frac{d}{ds}(G(u(s)))ds\leq\epsilon\int_{t_0}^{t}u'(s)ds.$$

It follows that

$$\int_{t_0}^{t}c(s)f(u(s))(u'(s))^2ds$$
$$+\int_{t_0}^{t}a(s)\frac{d}{ds}G(u(s))ds\leq\epsilon\int_{t_0}^{t}u'(s)ds.$$

Integrating by part, we get

$$\int_{t_0}^{t}c(s)f(u(s))(u'(s))^2ds$$
$$+a(t)G(u(t))-\int_{t_0}^{t}a'(s)G(u(s))ds\leq\epsilon\int_{t_0}^{t}u'(s)ds.$$

Since $a'(t)\leq 0$ and $a(t)\geq\delta>0$, we obtain

$$\delta G(u(t))\leq\epsilon\int_{t_0}^{t}u'(s)ds-\int_{t_0}^{t}c(s)f(u(s))(u'(s))^2ds. \tag{32}$$

Taking the absolute value (32), we have

$$\delta|G(u(t))|\leq\epsilon\int_{t_0}^{t}|u'(s)|ds+\int_{t_0}^{t}c(s)f(|u(s)|)(|u'(s)|)^2ds. \tag{33}$$

Setting $\int_{t_0}^{t}|u'(s)|ds\leq L$, for $L>0$, we obtain

$$|G(u(t))|\leq\frac{1}{\delta}\epsilon L+\frac{1}{\delta}\int_{t_0}^{t}c(s)f(|u(s)|)(|u'(s)|)^2ds. \tag{34}$$

Suppose $|G(u(t))|\geq|u(t)|$, then (34) becomes

$$\frac{|u(t)|}{P}\leq 1+\frac{1}{\delta}\int_{t_0}^{t}c(s)f(\frac{|u(s)|}{P})(|u'(s)|)^2ds \tag{35}$$

for

$$P = \frac{\epsilon}{\delta} L, \tag{36}$$

and it follows that

$$\frac{|u(t)|}{P} \leq 1 + \frac{(|u'(t)|)^2}{\delta} \int_{t_0}^{t} c(s) f(\frac{|u(s)|}{P}) ds. \tag{37}$$

Let $|u'(t)| \leq \lambda$, using this in (3.31), we get

$$\frac{|u(t)|}{P} \leq 1 + \frac{\lambda^2}{\delta} \int_{t_0}^{t} c(s) f(\frac{|u(s)|}{P}) ds. \tag{38}$$

Setting $\frac{|u(t)|}{P} = z(t)$, (37) becomes

$$z(t) \leq 1 + \frac{\lambda^2}{\delta} \int_{t_0}^{t} c(s) f(z(s) ds. \tag{39}$$

Using Lemma 4, for $\omega(z(t)) = f(z(t))$ with Ω defined as in (7), we obtain

$$z(t) \leq \Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\delta} \int_{t_0}^{t} c(s) ds \right).$$

By (9), we have

$$z(t) \leq \Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\delta} M \right).$$

Substituting for $z(t)$, we have

$$|u(t)| \leq P\Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\delta} M \right).$$

Replacing P, with (36), we have

$$|u(t)| \leq \frac{\epsilon}{\delta} (L) \Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\delta} M \right),$$

where

$$K = \frac{1}{\delta} (L) \Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\delta} M \right).$$

Therefore,

$$|u(t) - u_0(t)| \leq |u(t)| \leq K\epsilon$$

with condition (4). $\square$

Example 8. *Consider the equation*

$$u'' + t^{-2}u^2u' + t^{-4}u^2 = 0, \quad \textit{for } \ t > 0,$$

This equation is Hyers-Ulam stable by all the properties of Theorem 7.

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