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ON THE IRREDUCIBLITY OF THE REPRESENTATION OF THE PURE BRAID GROUP ON THREE STRANDS

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Abstract: Consider a representation $\rho : B_3 \to GL_6(\mathbb{C})$ constructed by M. Al-Tahan and M. Abdulrahim. We construct a representation ϕ equivalent to the restriction of ρ on P_3 and show that ϕ is a direct sum of irreducible subrepresentations, which are not equivalent to the reduced Burau representation restricted to P_3. Also, we show that the subrepresentations of ϕ are unitary relative to unique invertible hermitian matrices.

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Key Words: braid group, pure braid group, irreducible, unitary

1. Introduction

Let B_n be the braid group on n strands. There exists an obvious surjective group homomorphism $\pi : B_n \to S_n$. The kernel of π is referred to as the pure braid group P_n with $\frac{n(n-1)}{2}$ generators. Burau constructed a representations of B_n of degrees n and $n-1$, known as Burau and reduced Burau representations respectively [4]. The reduced Burau representation of B_n was proved to be irreducible [5].

Also, researchers gave a great value for representations of the pure braid group P_n. M. Abdulrahim gave a necessary and sufficient condition for the irreducibility of the complex specialization of the reduced Gassner representation of

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P_n, [1]. Moreover, M. Al-Tahan and M. Abdulrahim constructed an irreducible representation of B_3 of degree 6, namely, $\rho : B_3 \to GL_6(\mathbb{C})$, [2]. In our work, we define a representation ϕ equivalent to the restriction of the representation ρ of B_3 on P_3 and show that ϕ is a direct sum of three irreducible subrepresentations of degree two each, namely ϕ_1, ϕ_2, and ϕ_3 (Theorem 2). Moreover, we show that each of ϕ_1, ϕ_2, and ϕ_3 is not equivalent to the complex specialization of the reduced Burau representation restricted to P_3 of dimension 2 (Theorem 3). Finally we show that each of the irreducible subrepresentations of ϕ is unitary relative to a unique invertible hermitian matrix (Theorem 4).

2. Definitions and preliminaries

Definition 1. Let $\mathbb{C}^r = r \times 1$ (or column) vectors, $\overline{\mathbb{C}}^r = 1 \times r$ (or row) vectors. A matrix $X \in M_r(\mathbb{C})$ is a pseudoreflection if $X - I$ has rank 1. If X is a pseudoreflection, then $X = I - AB$, where $A \in \mathbb{C}^r$ and $B \in \overline{\mathbb{C}}^r$.

Notation 1. Let $(*) : M_n\left(\mathbb{C}\left[t^{\pm 1}\right]\right)$ be an involution defined as follows:

$$(h_{ij}(t))^* = h_{ji}(t^{-1}),\ h_{ij}(t) \in \mathbb{C}\left[t^{\pm 1}\right]$$

Definition 2. Let N and U be elements of $GL_n(\mathbb{C})$, U is called unitary relative to N if $UNU^\star = N$.

Definition 3. ([3]) The braid group on n strings, B_n, is the abstract group with presentation $B_n = \{\sigma_1, ..., \sigma_{n-1}; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, for $i = 1, 2, ..., n-2, \sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| < 1\}$.

The generators $\sigma_1, ..., \sigma_{n-1}$ are called the standard generators of B_n.

Definition 4. ([3]) The pure braid group, P_n, is defined as the kernel of the homomorphism $B_n \longrightarrow S_n$, defined by $\sigma_i \longrightarrow (i, i+1), 1 \le i \le n-1$. It has the following generators: $A_{ij} = \sigma_{j-1}\sigma_{j-2}...\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}...\sigma_{j-2}^{-1}\sigma_{j-1}^{-1},\ 1 \le i, j \le n$.

3. Representations of B_3 and P_3

3.1. Reduced Burau representation of B_3 of dimension 2

Definition 5. ([6]) The reduced Burau representation of the braid group on three strands $\beta_3 : B_3 \longrightarrow GL_2(Z[t^{\pm 1}])$, where $Z[t^{\pm 1}]$ is a Laurent polynomial ring, is the matrix representation defined on the generators σ_1, σ_2 of B_3 by

$$\beta_3(\sigma_1) = \begin{pmatrix} -t & 0 \\ -1 & 1 \end{pmatrix}, \ \beta_3(\sigma_2) = \begin{pmatrix} 1 & -t \\ 0 & -t \end{pmatrix}.$$

Proposition 1. ([5]) *For $z \in \mathbb{C}^*$, the complex specialization of the reduced Burau representation of B_n, namely $\beta_n(z) : B_n \longrightarrow GL_{n-1}(\mathbb{C})$, is irreducible if and only if z is not a root of $f_n(t) = t^{n-1} + t^{n-2} + ... + t + 1$.*

Definition 6. For any $z \in \mathbb{C}^*, \beta_3(z) : B_3 \longrightarrow GL_2(\mathbb{C})$ is the representation obtained from β_3 by the specialization $t \rightarrow z$.

Definition 7. Let $\beta_3(z) : P_3 \longrightarrow GL_2(\mathbb{C})$ be the complex specialization of the reduced Burau representation restricted to the pure braid group P_3 defined as follows:

$$\beta_3(z)(A_{12}) \rightarrow \begin{pmatrix} z^2 & 0 \\ z-1 & 1 \end{pmatrix}, \ \beta_3(z)(A_{23}) \rightarrow \begin{pmatrix} 1 & z(z-1) \\ 0 & z^2 \end{pmatrix},$$

and

$$\beta_3(z)(A_{13}) \rightarrow \begin{pmatrix} z & 1-z \\ -z(z-1) & z^2-z+1 \end{pmatrix}.$$

Theorem 1. ([5]) *Let $X_1 = I - A_1B_1, ..., X_r = I - A_rB_r$ be r invertible pseudoreflections in $M_r(\mathbb{C})$, where $r \geq 2$. Let Γ be the directed graph whose vertices are 1,2,...,r, and which has a directed edge from i to j ($i \neq j$) precisely when $B_iA_j \neq 0$. Let G be the subgroup of $GL_r(\mathbb{C})$ generated by $X_1, ..., X_r$. Then the following are equivalent:*
1) G is an irreducible subgroup of $GL_r(\mathbb{C})$.
2) For each $i \neq j$, with $1 \leq i, j \leq r$, the graph Γ contains a directed path from i to j, and $(B_iA_j) \in M_r(\mathbb{C})$ is invertible.

Proposition 2. ([7]) (Shur's Lemma) *Suppose that F is $n \times n$ matrix such that $F\alpha(g) = \alpha(g)F$ for every $g \in G$, where α is an irreducible representation* *of the group G. Then $F = \lambda I$ for some $\lambda \in \mathbb{C}$, where I is the $n \times n$ identity matrix.*

Proposition 3. *The complex specialization of the reduced Burau representation restricted to P_3, namely $\beta_3(z)$, is irreducible if and only if $z^3 \neq 1$.*

Proof. Consider U_3 the free normal subgroup of P_3 with generatores A_{13} and A_{23}. It is easy to see that $\beta_3(z)(A_{13})$ and $\beta_3(z)(A_{23})$ are pseudoreflections.

We can see that $\beta_3(z)(A_{13}) = I_2 - A_1B_1$ and $\beta_3(z)(A_{23}) = I_2 - A_2B_2$, where

$$A_1 = \begin{pmatrix} 1 \\ -z \end{pmatrix}, A_2 = \begin{pmatrix} z \\ z+1 \end{pmatrix}, B_1 = \begin{pmatrix} 1-z & z-1 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 1-z \end{pmatrix}.$$

Let F be the inner product $< B_iA_j >$. The determinant of F equals to $z^3 - 1$. Thus $\beta_3(z)$ restricted to U_3 is irreducible if and only if $z^3 - 1 \neq 0$ (see Theorem 1). It follows that $\beta_3(z)$ restricted to P_3 is irreducible if $z^3 - 1 \neq 0$. Now, let us show if $z^2 + z + 1 = 0$ or $z - 1 = 0$ then $\beta_3(z)$ restricted to P_3 is reducible. If $z - 1 = 0$ then $A_{12} = A_{13} = A_{23} = I_2$. Thus it is reducible.

Otherwise, if $z^2 + z + 1 = 0$ then the reducibility on P_3 follows from reducibility on B_3 (see Proposition 1). □

3.2. Representation of B_3 of Dimension 6

A new six dimensional representation of B_3 was constructed by M. Al-Tahan and M. Abdulrahim.

Definition 8. ([2]) Let z be a non zero complex number with $z^2 \neq 1$. We consider the representation ρ of B_3 given by

$$\rho(\sigma_1) = \begin{pmatrix} 1-z & z & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^{-1} & 0 & 0 \\ 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1-z^{-1} & 1 & 0 \end{pmatrix}$$

and

$$\rho(\sigma_2) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 & 0 & z^{-1} \\ 1 & z-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & -z & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proposition 4. ([2]) *The representation of B_3, $\rho : B_3 \to GL_6(\mathbb{C})$ is irreducible.*

4. Reducibility of the representation ρ restricted to P_3

We construct a representation ϕ equivalent to the restriction of ρ on P_3. More precisely, we conjugate by the matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to see that ϕ is a direct sum of three representations, namely ϕ_1, ϕ_2, and ϕ_3. Then we prove that ϕ_1, ϕ_2, and ϕ_3 are irreducible. Moreover, we show that each of ϕ_1, ϕ_2, and ϕ_3 is not equivalent to the complex specialization of the reduced Burau representation restricted to P_3 of dimension 2. Finaly we show that ϕ_1, ϕ_2, and ϕ_3 are unitary relative to unique invertible hermitian matrices.

Definition 9. let z be a non-zero complex number, with $z^2 \neq 1$. Consider $\phi : P_3 \to GL_6(\mathbb{C})$, defined as follows:

$$\phi(A_{12}) = \begin{pmatrix} z^2-z+1 & -z(z-1) & 0 & 0 & 0 & 0 \\ -z+1 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & z-1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{z-1}{z} & 1 \end{pmatrix},$$

$$\phi(A_{23}) = \begin{pmatrix} 1 & z-1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2z-1}{z} & \frac{z-1}{z} & 0 & 0 \\ 0 & 0 & \frac{1-z}{z} & z^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -z(z-1) \\ 0 & 0 & 0 & 0 & 0 & z^2 \end{pmatrix},$$

and

$$\phi(A_{13}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{z-1}{z} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1-z & 0 & 0 \\ 0 & 0 & 0 & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & z & z-1 \\ 0 & 0 & 0 & 0 & 1-z & 2-z \end{pmatrix}.$$

Theorem 2. *The representation ϕ is a direct sum of irreducible subrepresentations.*

Proof. The representation ϕ is reducible because there is an invariant subspace spanned by e_1 and e_2. Moreover, we write $\phi = \phi_1 \oplus \phi_2 \oplus \phi_3$, where ϕ_1, ϕ_2, and ϕ_3 are given by

$$\phi_1(A_{13}) = \begin{pmatrix} 1 & 0 \\ \frac{z-1}{z} & 1 \end{pmatrix}, \ \phi_1(A_{12}) = \begin{pmatrix} z^2-z+1 & -z(z-1) \\ -z+1 & z \end{pmatrix},$$

$$\phi_1(A_{23}) = \begin{pmatrix} 1 & z-1 \\ 0 & 1 \end{pmatrix},$$

$$\phi_2(A_{12}) = \begin{pmatrix} 1 & 0 \\ z-1 & 1 \end{pmatrix}, \ \phi_2(A_{23}) = \begin{pmatrix} \frac{2z-1}{z} & \frac{z-1}{z} \\ \frac{1-z}{z} & z^{-1} \end{pmatrix},$$

$$\phi_2(A_{13}) = \begin{pmatrix} 1 & 1-z \\ 0 & z^2 \end{pmatrix},$$

$$\phi_3(A_{12}) = \begin{pmatrix} 1 & 0 \\ \frac{z-1}{z} & 1 \end{pmatrix}, \ \phi_3(A_{23}) = \begin{pmatrix} 1 & -z(z-1) \\ 0 & z^2 \end{pmatrix},$$

$$\phi_3(A_{13}) = \begin{pmatrix} z & z-1 \\ 1-z & 2-z \end{pmatrix}.$$

Now, we use Theorem 1 to show that ϕ_1, ϕ_2, and ϕ_3 are irreducible. Consider U_3 the free normal subgroup of P_3 with generatores A_{13} and A_{23}. It is easy to see that $\phi_k(A_{13})$ and $\phi_k(A_{23})$ are pseudoreflections, where $k \in \{1, 2, 3\}$. We see that

$$\phi_k(A_{13}) = I_2 - A_1^{(k)} B_1^{(k)},\ \phi_k(A_{23}) = I_2 - A_2^{(k)} B_2^{(k)},$$

where

$$A_1^{(1)} = \begin{pmatrix} 0 \\ \frac{1-z}{z} \end{pmatrix},\ A_2^{(1)} = \begin{pmatrix} 1-z \\ 0 \end{pmatrix},\ B_1^{(1)} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

$$B_2^{(1)} = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

$$A_1^{(2)} = \begin{pmatrix} -1 \\ 1+z \end{pmatrix},\ A_2^{(2)} = \begin{pmatrix} 1-z \\ z-1 \end{pmatrix},\ B_1^{(2)} = \begin{pmatrix} 0 & 1-z \end{pmatrix},$$

$$B_2^{(2)} = \begin{pmatrix} 1/z & 1/z \end{pmatrix}.$$

$$A_1^{(3)} = \begin{pmatrix} 1-z \\ z-1 \end{pmatrix},\ A_2^{(3)} = \begin{pmatrix} -z \\ 1+z \end{pmatrix},\ B_1^{(3)} = \begin{pmatrix} 1 & 1 \end{pmatrix},$$

$$B_2^{(3)} = \begin{pmatrix} 0 & 1-z \end{pmatrix}.$$

Let F_k be the inner product $< B_i^{(k)} A_j^{(k)} >$. It is easy to see that $\det(F_1) = \frac{-(z-1)^2}{z} \neq 0$ and $\det(F_2) = \det(F_3) = (z-1)^2 \neq 0$. Thus ϕ_k's, restricted to U_3, are irreducible (Theorem 1). It follows that ϕ_1, ϕ_2, and ϕ_3 are irreducible. □

Theorem 3. *The representations ϕ_1, ϕ_2, and ϕ_3 are not equivalent to the complex specializations of the irreducible reduced Burau representation restricted to the pure braid group P_3.*

Proof. For any non-zero complex number z with $z^2 \neq 1$, it is easy to see that the images of the generators of P_3 under the complex specialization of the irreducible reduced Burau representation restricted to P_3 have two distinct eigenvalues 1 and z^2 (see Definition 7). But each of the images of A_{23} under ϕ_1, and A_{12} under ϕ_2 and ϕ_3 has only one eigenvalue equals to 1. □

It was shown that the reduced Burau representation of the braid group is unitary relative to a hermitian matrix (see [8]). We show that ϕ_1, ϕ_2, and ϕ_3, which are irreducible, are unitary relative to hermitian matrices. Thus, we get the following theorem.

Theorem 4. *The representations ϕ_1, ϕ_2, and ϕ_3 are unitary relative to unique invertible hermitian matrices N_1, N_2, and N_3 respectively.*

Proof. We define the matrices N_1, N_2, and N_3 as follows:
$N_1 = \begin{pmatrix} 0 & z+1 \\ z^{-1}+1 & 0 \end{pmatrix}$, $N_2 = \begin{pmatrix} 0 & z^{-1}+1 \\ z+1 & \frac{-(z+1)^2}{z} \end{pmatrix}$,

$$N_3 = \begin{pmatrix} 0 & z+1 \\ z^{-1}+1 & \frac{-(z+1)^2}{z} \end{pmatrix}.$$

For $i \in \{1,2,3\}$, $\det(N_i) = \frac{-(z+1)^2}{z} \neq 0$. Also, it is easy to see that $N_i^* = N_i$. Thus N_i's are invertible and hermitian. We also have for $i \in \{1,2,3\}$

$$\phi_i(A_{12})N_i(\phi_i(A_{12}))^* = N_i, \phi_i(A_{23})N_i(\phi_i(A_{23}))^* = N_i,$$

$$\phi_i(A_{13})N_i(\phi_i(A_{13}))^* = N_i.$$

Moreover, ϕ_i's are irreducible (Theorem 2), and the uniqueness of N_i's, up to scalar multiplication, follows from Shur's lemma. □

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