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BOUNDARY VALUE PROBLEM FOR A TWO-TIME-SCALE NONLINEAR DISCRETE SYSTEM

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Abstract: In this work, an algorithmic procedure is given to implement the solution of a two-point boundary value problem for a nonlinear two-time-scale discrete-time system.

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Key Words: discrete system, time-scale, iterative methods, boundary value problem, asymptotic expansions

1. Introduction

Singularly perturbed discrete systems (SPDSs) occur in engineering and applied mathematics. Contrary to the continuous-time counterpart, they could be characterized by numerous models (see [3], [12], [13], [14], [15], [16]). The details on the recent progress of the theory and applications of SPDSs can be found from the survey paper [3] and the references therein. In this article we study a class of nonlinear SPDS introduced in [6] and [9],

$$\begin{cases} x(t+1) = \varepsilon f\left(x(t), y(t), \varepsilon, t\right), \\ y(t+1) = g\left(x(t), y(t), \varepsilon, t\right), \end{cases} \quad t \in I_{N-1},$$
 (1)

where ε is a small real parameter, and $I_N = \{0, 1, \dots, N\}$, N a positive integer. Let $F(I_N, X)$ and $G(I_N, Y)$ denote the space of all mappings of I_N into the Banach spaces $(X, \|.\|)$ and $(Y, \|.\|)$, respectively, $\mathcal{U} := F(I_N, X) \times G(I_N, Y) \times G(I_N, Y)$

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 $(-1,1) \times I_N$; the mappings $f: \mathcal{U} \to X$ and $g: \mathcal{U} \to Y$ are supposed to be n-differentiable in their arguments. In the next section, we associate to (1) two known boundary conditions

$$x(t=0) = \alpha(\varepsilon), \qquad y(t=N) = \beta(\varepsilon).$$
 (2)

For sufficiently small values of the perturbation parameter ε , we study the existence and uniqueness of a solution $(x(t,\varepsilon),y(t,\varepsilon)), t \in I_N$, and we seek its asymptotic expansion

$$x(t,\varepsilon) = x(t)^{(0)} + \varepsilon x(t)^{(1)} + \dots + \varepsilon^n x(t)^{(n)} + \mathcal{O}(\varepsilon^{n+1}),$$

$$y(t,\varepsilon) = y(t)^{(0)} + \varepsilon y(t)^{(1)} + \dots + \varepsilon^n y(t)^{(n)} + \mathcal{O}(\varepsilon^{n+1}),$$
(3)

by using an algorithmic technique we developed firstly for perturbed difference equations (see [10], [11], [16], [17], [18], [19]). Recently, many researchers have studied discrete versions of boundary value problems (BVPs) (see [1], [4], [5], [8]), and applications of two-point BVP algorithms arise in pollution control problems, nuclear reactor heat transfer and vibration. For an abbreviated writing, we denote the partial derivative $\frac{\partial^{k_1+k_2+\cdots+k_p}f(x_1,x_2,\cdots,x_p)}{\partial x_1^{k_1}\partial x_2^{k_2}\cdots\partial x_p^{k_p}}$ by $D_1^{k_1}D_2^{k_2}\cdots D_n^{k_p}f$.

2. Main Result

This section contains the main results of the article, we describe the algorithmic procedure providing the asymptotic solutions. For $|\varepsilon| < \delta \le 1$, we assume that the boundary values $\alpha(\varepsilon)$ and $\beta(\varepsilon)$, have the asymptotic representations

$$\alpha(\varepsilon) = \alpha^{(0)} + \varepsilon \alpha^{(1)} + \dots + \varepsilon^n \alpha^{(n)} + \mathcal{O}(\varepsilon^{n+1}),$$

$$\beta(\varepsilon) = \beta^{(0)} + \varepsilon \beta^{(1)} + \dots + \varepsilon^n \beta^{(n)} + \mathcal{O}(\varepsilon^{n+1}).$$
(4)

2.1. Reduced Problem

By setting the small parameter to zero in (5)–(6), results the lower order subsystem, said reduced problem

$$\begin{cases} x^{(0)}(t+1) = 0, \\ y^{(0)}(t+1) = g\left(x^{(0)}(t), y^{(0)}(t), 0, t\right), \end{cases} \quad t \in I_{N-1},$$
 (5)

$$x^{(0)}(0) = \alpha(0), \qquad y^{(0)}(N) = \beta(0).$$
 (6)

It is seen that the states $x^{(0)}(t), 1 \le t \le N$, are fixed

$$x^{(0)}(0) = \alpha(0), \quad x^{(0)}(1) = x^{(0)}(2) = \dots = x^{(0)}(N) = 0,$$
 (7)

whereas the states $y^{(0)}(t), 0 \le t \le N$, satisfy the recurrence

$$\begin{cases} y^{(0)}(1) = g\left(\alpha(0), y^{(0)}(t), 0, 0\right), \\ y^{(0)}(t+1) = g\left(0, y^{(0)}(t), 0, t\right), \ 1 \le t \le N-1, \ y^{(0)}(N) = \beta(0), \end{cases}$$
(8)

which needs only the final value to be solved backwards, thus the boundary layer occurs at the initial value $x^{(0)}(0) = \alpha(0)$. To resolve BVP (5)–(6), we assume the following hypothesis.

H1
$$D_2g(x(t), y(t), 0, t) \neq 0, \forall (x, y) \in X \times Y, t = 0, ..., N-1.$$

Proposition 1. If H1 holds, then BVP(5)-(6) has a unique solution.

2.2. Preliminaries

We use the notation

$$\mathcal{X} = (x(0), y(0), x(1), y(1), \dots, x(N), y(N)),$$

thus system (1)-(2) can be written in the form $\mathcal{F}(\varepsilon,\mathcal{X})=0$, where

$$\mathcal{F}: (-1,1) \times X^{2N+2} \longrightarrow X^{2N+2}, \tag{9}$$

$$\mathcal{F}\left(\varepsilon,\mathcal{X}\right) = \left(f_0\left(\varepsilon,\mathcal{X}\right), g_0\left(\varepsilon,\mathcal{X}\right), \dots, f_N\left(\varepsilon,\mathcal{X}\right), g_N\left(\varepsilon,\mathcal{X}\right)\right),$$

$$\begin{cases}
f_0(\varepsilon, \mathcal{X}) = x(0) - \alpha(\varepsilon), \\
f_t(\varepsilon, \mathcal{X}) = x(t+1) - \varepsilon f(x(t), y(t), \varepsilon, t), \\
g_t(\varepsilon, \mathcal{X}) = y(t+1) - g(x(t), y(t), \varepsilon, t), \\
g_N(\varepsilon, \mathcal{X}) = y(N) - \beta(\varepsilon),
\end{cases} t = 0, \dots, N-1.$$

For a sufficiently small parameter ε , with Hypothesis H1 and other appropriate assumptions, the classical Implicit Function Theorem [2] assures the existence of a function

$$\Gamma(\varepsilon) = (\phi_0(\varepsilon), \psi_0(\varepsilon), \phi_1(\varepsilon), \psi_1(\varepsilon), \dots, \phi_N(\varepsilon), \psi_N(\varepsilon)),$$

of class C^n , such that $\mathcal{F}(\varepsilon, \Gamma(\varepsilon)) = 0$. Therefore, we have

$$\begin{cases}
\phi_{t+1}(\varepsilon) = \varepsilon f \left(\phi_t(\varepsilon), \psi_t(\varepsilon), \varepsilon, t \right), \\
\psi_{t+1}(\varepsilon) = g \left(\phi_t(\varepsilon), \psi_t(\varepsilon), \varepsilon, t \right),
\end{cases} \quad t \in I_{N-1},$$
(10)

$$\phi_0(\varepsilon) = \alpha(\varepsilon), \quad \psi_N(\varepsilon), = \beta(\varepsilon).$$
 (11)

We aim to determine the coefficients of the polynomial expansions (Taylor/Maclaurin)

$$\phi_t(\varepsilon) = \phi_t(0) + \frac{\varepsilon}{1!} \frac{d\phi_t}{d\varepsilon}(0) + \frac{\varepsilon^2}{2!} \frac{d^2\phi_t}{d\varepsilon^2}(0) + \dots + \frac{\varepsilon^n}{n!} \frac{d^n\phi_t}{d\varepsilon^n}(0) + \mathcal{O}(\varepsilon^{n+1}),$$

$$\psi_t(\varepsilon) = \psi_t(0) + \frac{\varepsilon}{1!} \frac{d\phi_t}{d\varepsilon}(0) + \frac{\varepsilon^2}{2!} \frac{d^2\psi_t}{d\varepsilon^2}(0) + \dots + \frac{\varepsilon^n}{n!} \frac{d^n\psi_t}{d\varepsilon^n}(0) + \mathcal{O}(\varepsilon^{n+1}),$$
(12)

therefore, we have to find explicitly the sequential differentiation of (10). It is achieved in the following Lemma by using the formula of Faa di Bruno [7].

Lemma 2. Suppose that ϕ and ψ satisfy (10), and that all necessary derivatives are defined. Then we have for $n \geq 2$,

$$\frac{d^n \phi_{t+1}}{d\varepsilon^n}(0) = \sum_0 \sum_1 \cdots \sum_{n-1} \frac{n! D_1^{p_1} D_2^{p_2} D_3^{p_3} f_t \prod_{i=1}^{n-1} (\frac{d^i \phi_t(0)}{d\varepsilon^i})^{q_1} (\frac{d^i \psi_t(0)}{d\varepsilon^i})^{q_{i2}} \delta_i}{\prod_{i=1}^{n-1} (i!)^{k_i} \prod_{i=1}^{n-1} \prod_{i=1}^{3} q_{ij}!}$$
(13)

$$\frac{d^n \psi_{t+1}}{d \varepsilon^n}(0) = \sum_0 \sum_1 \cdots \sum_n \frac{n! D_1^{p_1} D_2^{p_2} D_3^{p_3} g_t \times \prod_{i=1}^n (\frac{d^i \phi_t(0)}{d \varepsilon^i})^{q_{i1}} (\frac{d^i \psi_t(0)}{d \varepsilon^i})^{q_{i2}} \delta_i}{\prod_{i=1}^n (i!)^{k_i} \prod_{i=1}^n \prod_{j=1}^3 q_{ij}!}, \quad (14)$$

where the coefficients k_i , q_{ij} and p_j , $i = 0, \dots, n$, j = 1, 2, 3, are all nonnegative integer solutions of the Diophantine equations

$$\sum_{0} d_{1} + 2k_{2} + \dots + nk_{n} = n,$$

$$\sum_{i} d_{1} + q_{i2} + q_{i3} = k_{i}, \quad i = 1, 2, \dots, n,$$

$$p_{j} = q_{1j} + q_{2j} + \dots + q_{nj}, \quad j = 1, 2, 3,$$

$$k = p_{1} + p_{2} + p_{3} = k_{1} + k_{2} + \dots + k_{n},$$

$$(15)$$

and

$$f_t := f(\phi_t(0), \psi_t(0), 0, t), \quad g_t := g(\phi_t(0), \psi_t(0), 0, t),$$
 (16)

$$\delta_i := \left(\varepsilon^{(i)}\right)^{q_{i3}} = \begin{cases} 1, & i = 1 \quad \lor \quad q_{i3} = 0, \\ 0, & i \ge 2 \quad \land \quad q_{i3} \ne 0. \end{cases}$$
 (17)

Proof. We expand Faa Di Bruno's Formula into (10), and we verify by induction

$$\frac{d^n}{d\varepsilon^n} \left(\varepsilon f \left(\phi_t(\varepsilon), \psi_t(\varepsilon), \varepsilon, t \right) \right) |_{\varepsilon = 0} = n \frac{d^{n-1}}{d\varepsilon^{n-1}} \left(f \left(\phi_t(\varepsilon), \psi_t(\varepsilon), \varepsilon, t \right) \right) |_{\varepsilon = 0}. \tag{18}$$

2.3. Description of the Method

We find an algorithmic process giving the coefficients of (3) when we substitute, into (13) and (14), for $0 \le t \le N$, l = 0, 1, by

$$x^{(i)}(t+l) := \frac{1}{i!} \frac{d^i \phi_{t+l}}{d\varepsilon^i}(0), \ y^{(i)}(t+l) := \frac{1}{i!} \frac{d^i \psi_{t+l}}{d\varepsilon^i}(0). \tag{19}$$

Zero order approximation coefficients refer to the sequence solution of the reduced problem (5)-(6), thus $x^{(0)}(t)$ verify (7) and hypothesis H1 allows computing $y^{(0)}(t)$ from (8). For first order coefficients, we obtain the forward recurrence

$$x^{(1)}(0) = \alpha^{(1)}, \quad x^{(1)}(1) = f\left(\alpha(0), y^{(0)}(0), 0, 0\right), x^{(1)}(t+1) = f\left(0, y^{(0)}(t), 0, t\right), \quad 1 \le t \le N,$$
(20)

and the difference equation

$$y^{(1)}(t) = [D_2 g_t^{(0)}]^{-1} [y^{(1)}(t+1) - D_1 g_t^{(0)} x^{(1)}(t) - D_3 g_t^{(0)}],$$

$$y^{(1)}(N) = \beta^{(1)}$$
(21)

which may be solved backwards with the final value. For second order development, we determine $x^{(0)}(t)$ by straightforward computation from the difference equation

$$x^{(2)}(0) = \alpha^{(2)},$$

$$x^{(2)}(t+1) = D_1 f_t^{(0)} x^{(1)}(t) + D_2 f_t^{(0)} y^{(1)}(t) + D_3 f_t^{(0)}, \ t \in I_{N-1},$$
(22)

then we can calculate $y^{(0)}(t)$ backwards from the recurrence

$$y^{(2)}(t) = [D_2 g_t^{(0)}]^{-1} [y^{(2)}(t+1) - D_1 g_t^{(0)} x^{(2)}(t) - \frac{1}{2!} D_3^2 g_t^{(0)} - \frac{1}{2!} D_1^2 g_t^{(0)} (x^{(1)}(t))^2 - \frac{1}{2!} D_2^2 g_t^{(0)} (y^{(1)}(t))^2 - D_1 D_3 g_t^{(0)} x^{(1)}(t) - D_2 D_3 g_t^{(0)} y^{(1)}(t) - D_1 D_2 g_t^{(0)} x^{(1)}(t) y^{(1)}(t)],$$

$$y^{(2)}(N) = \beta^{(2)}.$$
(23)

In general, to find the n order coefficients, first we perform the forward iteration

$$x^{(n)}(0) = \alpha^{(n)},$$

$$x^{(n)}(t+1) = \sum_{0} \cdots \sum_{n-1} \frac{D_{1}^{p_{1}} D_{2}^{p_{2}} D_{3}^{p_{3}} f_{t}^{(0)} \prod_{i=1}^{n-1} (x^{(i)}(t))^{q_{i1}} (y^{(i)}(t))^{q_{i2}} \delta_{i}}{\prod_{i=1}^{n-1} \prod_{i=1}^{3} q_{ij}!},$$
(24)

then after we solve backwards the difference equation

$$y^{(n)}(t) = [D_2 g_t^{(0)}]^{-1} [y^{(n)}(t+1) - D_1 g_t x^{(n)}(t) - \sum_0 \sum_1 \dots \sum_n \frac{D_1^{p_1} D_2^{p_2} D_3^{p_3} g_t \times \prod_{i=1}^n (x^{(i)}(t))^{q_{i1}} (y^{(i)}(t))^{q_{i2}} (\delta_i)^{q_{i3}}}{\prod_{i=1}^n \prod_{j=1}^3 q_{ij}!}], \qquad (25)$$

$$y^{(n)}(N) = \beta^{(n)}.$$

The following theorem illustrates the effectiveness of the suggested algorithm.

Theorem 3. If H1 holds, there exists $\epsilon > 0$, such that for all $|\epsilon| < \epsilon$, the boundary value problem (1)-(2) has a unique solution which satisfies (3); the coefficients $x^{(0)}(t)$, $y^{(0)}(t)$, $x^{(1)}(t)$, $y^{(1)}(t)$, $x^{(2)}(t)$, $y^{(2)}(t)$, $x^{(n)}(t)$, $y^{(n)}(t)$, are found following the ordered iterative process (7), (8), (20), (21), (22), (23), (24), (25), respectively.

Proof. Let $F\left(\tilde{\mathcal{X}}\right) = \left(\varepsilon, \mathcal{F}\left(\tilde{\mathcal{X}}\right)\right)$ where $\mathcal{D}F$ denotes its jacobian matrix, $\tilde{\mathcal{X}} = (\varepsilon, \mathcal{X}), \ |\varepsilon| \leq \delta < 1$. We have from hypothesis H1

$$\det \mathcal{D}F\left(\tilde{\mathcal{X}}^{(0)}\right) = \prod_{t=0}^{N-1} \mathcal{D}_{2}g\left(x^{(0)}(t), y^{(0)}(t), 0, t\right) \neq 0$$

which justifies that at

$$\tilde{\mathcal{X}}^{(0)} = \left(0, x^{(0)}(0), y^{(0)}(0), x^{(0)}(1), y^{(0)}(1), \dots, x^{(0)}(N), y^{(0)}(N)\right),\,$$

the inverse of $\mathcal{D}F$ exists and (5)-(6) has a unique solution. We can choose $\xi > 0$ such that, if $\|\tilde{\mathcal{X}} - \tilde{\mathcal{X}}^{(0)}\| < \xi$, we have

$$\|\mathcal{D}F\left(\tilde{\mathcal{X}}\right) - \mathcal{D}F\left(\tilde{\mathcal{X}}^{(0)}\right)\| < \frac{1}{2} \|\left(\mathcal{D}F\left(\tilde{\mathcal{X}}^{(0)}\right)\right)^{-1}\|^{-1},\tag{26}$$

since $\mathcal{D}F$ is continuous. Let $\epsilon = \frac{\xi}{2} \| \left(\mathcal{D}F \left(\tilde{\mathcal{X}}^{(0)} \right) \right)^{-1} \|^{-1}$, the mapping

$$\Omega_{\tau}(\tilde{\mathcal{X}}) = \tilde{\mathcal{X}} - \left(\mathcal{D}F\left(\tilde{\mathcal{X}}^{(0)}\right)\right)^{-1} \left(F(\tilde{\mathcal{X}}) - \tau\right)$$

is a contraction from $B\left(\tilde{\mathcal{X}}^{(0)},\xi\right)$ to itself, when $|\varepsilon|<\epsilon$ and $\|\tau\|<\epsilon$. Then Ω_{τ} has a unique fixed point $\tilde{\mathcal{X}}$. Therefore, for τ fixed, $\|\tau\|<\epsilon$, there exists a unique $\tilde{\mathcal{X}}$ such that $\|\tilde{\mathcal{X}}-\tilde{\mathcal{X}}^{(0)}\|<\xi$, and $\tau=\mathrm{F}(\tilde{\mathcal{X}})$, i.e., F is 1-to-1 from $\mathrm{F}^{-1}\left(B(0,\epsilon)\right)$ into $B(0,\epsilon)$. Suppose that $|\varepsilon|<\epsilon$, therefore, we have

 $(\varepsilon,0,\cdots,0)\in B(0,\epsilon)$, there exists a unique $(\varepsilon,\Gamma(\varepsilon))$ in $B\left(\tilde{\mathcal{X}}^{(0)},\xi\right)$, such that $(\varepsilon,0,\cdots,0)=F(\varepsilon,\Gamma(\varepsilon))$, where $\Gamma(\varepsilon)=(\phi_0(\varepsilon),\psi_0(\varepsilon),\cdots,\phi_N(\varepsilon),\psi_N(\varepsilon))$. We proved that $|\varepsilon|<\epsilon$, there exists a unique $\phi(\varepsilon)$ such that $\mathcal{F}(\varepsilon,\Gamma(\varepsilon))=0$, then BVP (1)-(2) has a unique solution. In addition, as are F and F⁻¹, the function Γ is $C^n(-\epsilon,\epsilon)$, and Lemma 2 gives its derivatives.

Iterative problems given above are defined for all order if f and g are smooth functions and the asymptotic developments for the boundary conditions are convergent.

H2 Assume that $\|\alpha_k^{(i)}\| \leq \frac{A}{\delta^i}$, $\|\beta^{(i)}\| \leq \frac{B}{\delta^i}$, A and B are constants.

Theorem 4. If assumptions H1 and H2 hold, and f is a smooth function, then there exists $\epsilon > 0$, for all $|\varepsilon| < \epsilon$, the boundary value problem (1)–(2) has a unique solution which satisfies

$$x(t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n x^{(n)}(t), \quad y(t,\varepsilon)) = \sum_{n=0}^{\infty} \varepsilon^n y^{(n)}(t),$$

where $x^{(0)}(t)$, $y^{(0)}(t)$, $x^{(1)}(t)$, $y^{(1)}(t)$, $x^{(2)}(t)$, $y^{(2)}(t)$, $x^{(n)}(t)$, $y^{(n)}(t)$, are the solutions of the problems (7), (8), (20), (21), (22), (23), (24), (25), respectively.

3. Conclusion

We have addressed a two-time-scale nonlinear discrete system. A BVP has been analyzed using the perturbation method and an iterative algorithm is given to find asymptotic solutions at any order. The same technique can be helpful for the IVP since it has the advantage to suppress time scales generating error accumulation, and provides reliable results with few iterations.

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