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COMMON FIXED POINT THEOREM IN b-MENGER SPACES WITH A FULLY CONVEX STRUCTURE

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Abstract: We prove, in b-Menger spaces [9] the existence of common fixed point for nonexpansive mappings in fully convex b-Menger space by using the normal structure property. We provide examples to analyze and illustrate our main results.

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1. Introduction

Let (M,d) be a metric space and suppose $A \subset M$. A mapping $f: A \to M$ is called nonexpansive if its Lipschitz constant k(f) does not exceed 1. Thus this class of mappings includes the contractions and strictly contractive mappings; moreover it contains all isometries (including the identity). Explicitly, $f: A \to M$ is nonexpansive if

$$d(fa, fb) \le d(a, b), \qquad a, b \in A.$$

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If M is a Banach space with norm $\|.\|$ and A is a nonempty subset of M, in this context a mapping $f: A \to A$ is nonexpansive if

$$||fa - fb|| \le ||a - b||, \quad a, b \in A.$$

The nonexpansive mapping f may frequently fail to have a fixed point in a setting which permits the existence of a decreasing sequence $\{A_i\}_{i\in\mathbb{N}}$ of nonempty, closed, convex and f-invariant $(f(A_i) \subset A_i, i \in \mathbb{N}))$ sets having empty intersection. However, Kirk [7] observed that the presence of a geometric property called 'normal structure' (M.S. Brodskii, D.P. Milman, On the center of a convex set, Dokl. Akad. Nauk. SSSR, **59** (1948), 837-840) guarantees that the nonexpansive f such that A is nonempty, weakly compact convex subset of a Banach space M, has a fixed point in M.

Since its publication in 1965, many have tried to extend it to metric spaces. But because of its strong connection to the linear convexity structure of linear spaces, it was hard to come up with a nice and flexible extension. For example, Takahashi [15] was may be the first one to give a metric analogue to Kirk's theorem. His approach was based on defining a convexity in metric spaces extremely similar to the linear convexity also known as Menger convexity [1]-[3], [8].

In 1987 Hadžić [6] offered an extension of Takahachi's structure to Menger spaces and proved fixed point theorem for nonexpansive mappings in probabilistic metric spaces with a convex structure.

Following Hadžić's approach, Ješić et al. [14] have introduced, strictly convex and normal structure in Menger spaces and proved fixed point theorem for nonexpansive mappings.

Recently, Mbarki et al. [9] introduced the probabilistic b-metric spaces (b-Menger spaces) as a generalization of probabilistic metric spaces (Menger spaces) and they studied some topological properties and showed the fixed point property for nonlinear contractions in these spaces.

This paper is organized as follows. In Section 2, we present some basic concepts and definitions on b-Menger spaces. In Section 3, we show some geometric and topological properties in convex b-Menger spaces and we define the fully convex b-Menger spaces. We finish this section by proving the main result in this paper, i.e., the existence of common fixed point for nonexpansive mappings in fully convex b-Menger space using the normal structure property. Finally, in Section 4, we construct a significant example of fully convex b-Menger spaces from the literature and we prove a common fixed point theorem in these spaces. Our results generalize some well-known results in the literature.

2. Preliminaries

In this section, we collect some basic definitions, notation and conventions of the theory of probabilistic b-metric spaces, which will be used throughout the paper.

Definition 1. A distance distribution function (briefly, a d.d.f.) is a nondecreasing function F defined on $\mathbb{R}^+ \cup \{\infty\}$ that satisfies f(0) = 0 and $f(\infty) = 1$, and is left continuous on $(0, \infty)$. The set of all d.d.f's will be noted by Δ^+ ; and the set of all F in Δ^+ for which $\lim_{t\to\infty} f(t) = 1$ by D^+ .

A simple example of distribution function is Heavyside function in D^+

$$H(t) = \begin{cases} 0 & \text{if} \quad t \le 0, \\ 1 & \text{if} \quad t > 0. \end{cases}$$

A commutative, associative and nondecreasing mapping $T:[0,1]^2 \to [0,1]$ is called a t-norm if and only if

- 1. T(a,1) = a, for all $a \in [0,1]$,
- 2. T(0,0) = 0.

As examples we mention the three typical examples of continuous t-norms as follows:

$$T_v(a,b) = ab$$
, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max\{a+b-1,0\}$.

We define the operator T^n recursively by $T^1(a_1, a_2) = T(a_1, a_2)$ and $T^n(a_1, a_2, ..., a_{n+1}) = T(T^{n-1}(a_1, a_2, ..., a_n), a_{n+1})$ for all $n \ge 1$ and $a_1, a_2, ..., a_{n+1} \in [0, 1]$.

Definition 2. [9] A b-Menger space is a quadruple (M, F, T, s) where M is a nonempty set, F is a function from $M \times M$ into Δ^+ , T is a t-norm, $s \ge 1$ is a real number, and the following conditions are satisfied: For all $p, r, q \in M$ and x, y > 0,

- 1. $F_{pp} = H$,
- $2. F_{pr} = H \Rightarrow p = r,$
- $3. F_{pr} = F_{rp},$
- 4. $F_{pr}(s(x+y)) \ge T(F_{pq}(x), F_{qr}(y))$.

It should be noted that a Menger space is a b-Menger space with s = 1.

Definition 3. Let $\{x_n\}$ be a sequence in a probabilistic semimetric space (M, F) (i.e., (i), (ii) and (iii) of Definition 2 are satisfied).

- 1. A sequence $\{x_n\}$ in M is said to be convergent to x in M if, for every $\epsilon > 0$ and $\delta \in (0,1)$, there exists a positive integer $N(\epsilon,\delta)$ such that $F_{x_nx}(\epsilon) > 1 \delta$, whenever $n \geq N(\epsilon,\delta)$.
- 2. A sequence $\{x_n\}$ in M is called Cauchy sequence if, for every for every $\epsilon > 0$ and $\delta \in (0,1)$, there exists a positive integer $N(\epsilon,\delta)$ such that $F_{x_n x_m}(\epsilon) > 1 \delta$, whenever $n, m \geq N(\epsilon,\delta)$.

3. Results and discussion

3.1. Some topological properties of b-Menger space

Different kinds of topologies can be introduced in a b-Menger space (M, F, T, s). The strong topology is introduced by a strong neighborhood system $\wp = \bigcup_{p \in M} \wp_p$, where $\wp_p = \{N_p(t) : t > 0\}$, and

$$N_p(t) = \{q \in M: F_{pq}(t) > 1 - t\} \text{ for } t > 0 \text{ and } p \in M.$$

In [9] the following results are proved.

Lemma 4. Let (M, F, T, s) be a b-Menger space with T is continuous, then the family \Im consisting of \emptyset and all unions of elements of this strong neighborhood system determines a Hausdorff topology for M. Also, the function F is in general not continuous.

It is special interest (ε, λ) -topology on (M, F, T, s) which is introduced by a family of (ε, λ) -neighborhood $\{N_p(\varepsilon, \lambda)\}_{p \in M, \varepsilon > 0, \lambda \in [0,1]}$, where

$$N_p(\varepsilon,\lambda) = \{ q \in M : F_{pq}(\varepsilon) > 1 - \lambda \}.$$

Since $N_p(t,t) = N_p(t)$ for t > 0, and

$$N_p(\min\{\varepsilon,\lambda\}) \subset N_p(\varepsilon,\lambda)$$
 for every $\varepsilon > 0, \ \lambda \in [0,1]$ and $p \in M$

the strong neighborhood system is equivalent to the (ε, λ) -neighborhood system. Hence we reach the same conclusions as in Lemma 4, with (ε, λ) -topology in place of strong topology.

In this topology the function g is continuous in $p \in M$ if and only if for every sequence $p_n \to p$ it holds that $g(p_n) \to g(p)$.

Here and in the sequel, when we speak about a b-Menger space (M, F, T, s), we always assume that T is a continuous t-norm and $RanF \subset D^+$.

Definition 5. Let A be a subset nonempty of a b-Menger space (M, F, T, s), the closure \overline{A} of A is the set of all p in M such that, for any $\varepsilon > 0$, $\lambda \in (0, 1)$, there is a q in A such that

$$F_{pq}(\varepsilon) > 1 - \lambda.$$

As direct consequence of Definition 5 we have the following.

Lemma 6. Let A be a subset nonempty of a b-Menger space (M, F, T, s). Then

- 1. $p \in \overline{A}$ if and only if for any $\varepsilon > 0$, $\lambda \in (0,1)$, $N_p(\varepsilon,\lambda) \cap A \neq \emptyset$.
- 2. $p \in \overline{A}$ if and only if there exists a sequence $\{p_n\}$ in A such that $p_n \to p$.
- 3. A is a closed set if and only if $\overline{A} = A$.
- 4. \overline{A} is a closed set. Moreover, \overline{A} is the smallest closed set containing A.

The set $N_p[\varepsilon,\lambda]=\{q\in M: F_{pq}(\varepsilon)\geq 1-\lambda\}$, is called closed (ε,λ) -neighbourhood of a point $p\in M$.

Definition 7. We say that a b-Menger space has the property (P_1) if the closed (ε, λ) -neighbourhoods are closed sets.

Proposition 8. Let (M, F, T, s) be a b-Menger space, if for any $q \in M$, t > 0 and sequence $p_n \to p$ in M we have

$$F_{pq}(t) \ge \liminf_{n \to \infty} F_{p_n q}(t). \tag{3.1}$$

Then (M, F, T, s) has the property (P_1) .

Proof. Let $p \in \overline{N_q[\varepsilon, \lambda]}$, by Lemma 6 there exists a sequence $\{p_n\} \subseteq N_q[\varepsilon, \lambda]$ such that $p_n \to p$, and so

$$F_{n,n}(\varepsilon) \geq 1 - \lambda$$
 for all $n \in \mathbb{N}$,

we deduce that

$$F_{pq}(\varepsilon) \geq \liminf_{n} F_{p_nq}(\varepsilon)$$

 $> 1 - \lambda...$

Consequently, $p \in N_q[\varepsilon, \lambda]$ and hence $N_q[\varepsilon, \lambda]$ is closed set.

Example 9. Let $M = \mathbb{R}, i \in \mathbb{N}^* - \{1\}$. Define $F: M \times M \to \Delta^+$ by

$$F_{xy}(t) = H(t - |x - y|^i).$$

Firstly, from [9], $(M, F, T_M, 2^{i-1})$ is a *b*-Menger space. Next, let $p_n \to p$ in $(M, F, T_M, 2^{i-1})$. Clearly, $p_n \to p$ in $(M, |\cdot|)$. Now, let $q \in M$, t > 0, we can write

$$t - |p_j - q|^i \ge \inf_{n > j} (t - |p_n - q|^i), \ \forall j \in \mathbb{N}^*,$$

and passing to limit as $j \to \infty$ in the previous inequality, we get

$$|t - |p - q|^i \ge \sup_{i} \inf_{n \ge j} (t - |p_n - q|^i),$$

which implies

$$H(t - |p - q|^i) \ge H(\sup_{i} x_i). \tag{3.2}$$

where $x_j = \inf_{n \geq j} (t - |p_n - q|^i)$. Since $\{x_j\} \uparrow \sup_j x_j$. Using the fact that H is non-decreasing left continuous function, we obtain

$$H(\sup_{j} x_j) = \lim_{j} H(x_j) = \sup_{j} H(x_j). \tag{3.3}$$

On the other hand,

$$H(x_j) \geq H(\inf_{n \geq j} (t - |p_n - q|^i))$$

$$\geq H(t - \sup_{n \geq j} |p_n - q|^i)$$

$$\geq \inf_{n \geq j} H(t - |p_n - q|^i).$$

It follows by (3.2) and (3.3) that

$$H(t - |p - q|^{i}) \geq H(\sup_{j} x_{j})$$

$$\geq \sup_{j} H(x_{j})$$

$$\geq \sup_{j} \inf_{n \geq j} H(t - |p_{n} - q|^{i}).$$

Therefore, condition (3.1) is satisfied, then $(M, F, T_M, 2^{i-1})$ has the property (P_1) .

Definition 10. A subset A of a b-Menger space is called compact if every open cover of A has a finite subcover.

Consequently, a subset A of a b-Menger space is compact if and only if, from any family of closed subsets of A whose intersection is empty, we can extract a finite subfamily whose intersection is empty.

Remark 11. Since (M, F, T, s) is a Hausdorff topological space, then every compact subset A is closed set.

3.2. Some geometrical properties and fully convex b-Menger space

Egbert [5], in 1968 introduced a probabilistic generalization of the notion of diameter of nonempty set in metric space.

Definition 12. Let (M, F, T, s) be a b-Menger space and $A \subset M$. The probabilistic diameter of set A is given by

$$D_A(t) = \sup_{\varepsilon < t} \inf_{a,b \in A} F_{ab}(\varepsilon),$$

and the diameter of the set A is defined by

$$D_A = \sup_{t>0} \sup_{\varepsilon < t} \inf_{a,b \in A} F_{ab}(\varepsilon).$$

If $D_A = 1$ the set A will be called probabilistic bounded.

Example 13. Let (M,d,s) be a b-metric space. Define $F: M \times M \to \Delta^+$ by

$$F_{pq}(t) = H(t - d(p, q)).$$

It is easy to check that for a nonempty subset A of M we have

$$D_A(t) = H(t - diam(A)),$$

where

$$diam(A) = \sup\{d(p,q): p, q \in A\}.$$

Indeed, let t > 0, if H(t - diam(A)) = 1 then diam(A) < t, we deduce that

$$H(\varepsilon - diam(A)) = 1$$

for every $diam(A) < \varepsilon < t$, and since,

$$\inf_{a,b \in A} H(\varepsilon - d(p,q)) \ge H(\varepsilon - diam(A)),$$

for every $\varepsilon < t$. Consequently, we get

$$D_{A}(t) = \sup_{\varepsilon < t} \inf_{a,b \in A} H(\varepsilon - d(p,q))$$

$$\geq \sup_{\varepsilon < t} H(\varepsilon - diam(A))$$

$$\geq \sup_{diam(A) < \varepsilon < t} H(\varepsilon - diam(A))$$

$$\geq 1,$$

and hence, $D_A(t) \geq H(t - diam(A))$. On the other hand, suppose that H(t - diam(A)) = 0, then $diam(A) \geq t$. Now, let $\varepsilon < t$, so for $\zeta = diam(A) - \varepsilon$ there exist $a, b \in A$ such that $d(a, b) > diam(A) - \zeta > 0$, which implies $H(\varepsilon - d(a, b)) = 0$, and so $\inf_{a,b \in A} H(\varepsilon - d(a,b)) = 0$. Since the choice of $\varepsilon < t$ was arbitrary, this is true for all $\varepsilon < t$, which gives

$$D_A(t) = \sup_{\varepsilon < t} \inf_{a,b \in A} H(\varepsilon - d(a,b)) = 0.$$

Hence

$$D_A(t) \le H(t - diam(A)).$$

We conclude that $D_A(t) = H(t - diam(A))$. So, A is probabilistic bounded in (M, F, T_M, s) if and only if A is bounded in (M, d, s).

From the definition of functionals sup and inf, we have the following lemma.

Lemma 14. Let (M, F, T, s) be a b-Menger space. A set $A \subset M$ is probabilistic bounded if and only if for each $\lambda \in (0,1)$ there exists t > 0 such that $F_{ab}(t) > 1 - \lambda$ for all $a, b \in A$.

Lemma 15. Every compact subset A of a b-Menger space (M, F, T, s) is probabilistic bounded.

Proof. Let A be a compact subset of a b-Menger space M. Let fix ε and $\lambda \in (0,1)$. Since T^2 is continuous in (1,1,1) there exists $\delta \in (0,1)$ such that $T^2(1-\delta,1-\delta,1-\delta)>1-\lambda$. Now, we will consider an (ε,δ) -cover $\{N_a(\varepsilon,\delta):a\in A\}$. Since A is compact, there exist $a_1,a_2,...,a_n\in A$ such that $A\subset\bigcup_{i=1}^n N_{a_i}(\varepsilon,\lambda)$. Since $Ran(F)\subset D^+$ then for all i,j there exists $\varepsilon_{ij}>\varepsilon$ such that $F_{a_ia_j}(\varepsilon_{ij})>1-\delta$. If we take $t=(2s^2+s)\rho$ where $\rho=\max\{\varepsilon_{ij}\}$, let $a,b\in A$. Then there exist $i,j\in\{1,...,n\}$ such that $a\in N_{a_i}(\varepsilon,\delta)$ and $b\in N_{a_j}(\varepsilon,\delta)$. Thus we have $F_{aa_i}(\varepsilon)>1-\delta$ and $F_{ba_j}(\varepsilon)>1-\delta$. It follows that

$$\begin{split} F_{ab}(t) &= F_{ab}((2s^2+s)\rho) & \geq & T^2(F_{aa_i}(\rho), F_{a_ia_j}(\rho), F_{a_jb}(\rho)) \\ & \geq & T^2(F_{aa_i}(\varepsilon_{ij}), F_{a_ia_j}(\varepsilon_{ij}), F_{a_jb}(\varepsilon_{ij})) \\ & \geq & T^2(F_{aa_i}(\varepsilon), F_{a_ia_j}(\varepsilon_{ij}), F_{a_jb}(\varepsilon)) \\ & \geq & T^2(1-\delta, 1-\delta, 1-\delta) > 1-\lambda. \end{split}$$

Therefore $F_{ab}(t) > 1 - \lambda$ for all $a, b \in A$. Hence, by Lemma 14, we obtain that A is probabilistic bounded set.

Remark 16. Note that " $RanF \subset D^+$ " is a necessary condition in Lemma 15. For example, consider $M = \{p, q\}$ and $F_{pq}(t) = \frac{1}{2}H(t - |p - q|) + \frac{1}{2}\varepsilon_{\infty}(t)$, where

$$\varepsilon_{\infty}(t) = \begin{cases} 1 & \text{if} \quad t = \infty, \\ 0 & \text{if} \quad otherwise. \end{cases}$$

Clearly, M is a compact space and is only semi-bounded.

We adopt the following Hadžić's convex structure concept.

Definition 17. [6] Let (M, F, T, s) be a b-Menger space. A mapping $W: M \times M \times [0,1] \to M$, is said to be a convex structure on M if for every $(x,y) \in M \times M$ holds W(x,y,0) = y, W(x,y,1) = x and for all $x,y,z \in M$, $\lambda \in (0,1,)$ and t > 0

$$F_{W(x,y,\lambda)z}(2t) \ge T(F_{xz}(\frac{t}{\lambda}); F_{yz}(\frac{t}{1-\lambda})).$$

Proposition 18. Let (X, F, T, s) be a b-Menger space with a convex

structure W. If $x \in X$ and $\lambda \in [0,1]$, then

$$W(x, x, \lambda) = x.$$

Proof. Let $x \in X$, then W(x, x, 0) = x and W(x, x, 1) = x. Now, let $\lambda \in (0, 1)$, we have

$$F_{xW(x,x;\lambda)}(t) \ge T(F_{xx}(\frac{t}{2\lambda}); F_{xx}(\frac{t}{2(1-\lambda)})),$$

for all t > 0. Hence, $F_{xW(x,x;\lambda)}(t) = 1$, for all t > 0, hence $W(x,x,\lambda) = x$. \square

Definition 19. Let (M, F, T, s) be a b-Menger space with a convex structure W. A subset $A \subset M$ is said to be convex set if for every $a, b \in A$ and $\lambda \in [0, 1]$ it follows that $W(a, b, \lambda) \in A$.

Definition 20. Let (M, F, T, s) be a b-Menger space with a convex structure W. We denote by (C_1) the condition: For every $\lambda \in [0, 1]$, t > 0 and $a, b, c \in M$ we have

$$F_{W(a,b,\lambda)c}(t) \ge \min\{F_{ca}(t), F_{cb}(t)\}.$$

Lemma 21. Let (M, F, T, s) be a convex b-Menger space with a convex structure W satisfying (C_1) . Then the closed (ε, λ) -neighbourhoods $N_x[\varepsilon, \lambda]$ are convex sets.

Proof. Let $a, b \in N_x[\varepsilon, \lambda]$ be arbitrary points, then

$$F_{ax}(\varepsilon) \ge 1 - \lambda$$
 and $F_{bx}(\varepsilon) \ge 1 - \lambda$.

Now, using (C_1) with $\delta \in (0,1)$, we obtain

$$F_{W(a,b,\delta)x}(\varepsilon) \ge \min\{F_{ax}(\varepsilon), F_{bx}(\varepsilon)\} \ge \min\{1-\lambda, 1-\lambda\} = 1-\lambda,$$

we deduce $W(a, b, \delta) \in N_x[\varepsilon, \lambda]$. Also, W(a, b, 0) = b and W(a, b, 1) = a belong to $N_x[\varepsilon, \lambda]$, and hence the proof is completed.

It is easy to check the following lemma.

Lemma 22. Let (M, F, T, s) be a b-Menger space and $\{A_{\mu}\}$ for $\mu \in \Sigma$ be a family of convex subsets of M. Then the intersection $A = \bigcap_{\mu \in \Sigma} A_{\mu}$ is a convex set.

Definition 23. Let (M, F, T, s) be a b-Menger space with a convex structure W and $A \subseteq M$. The closed convex shell of set A denoted by co(A), is the intersection of all closed, convex sets that contain A.

Definition 24. A convex b-Menger space (M, F, T, s) with a convex structure $W: M \times M \times [0, 1] \to M$ will be called fully convex if, for arbitrary $a, b \in M$, $a \neq b$ there exists $\lambda \in (0, 1)$ such that $W(a, b, \lambda) \notin \{a, b\}$.

Example 25. Let
$$n \in \mathbb{N}^* - \{1\}$$
. Define $F : \mathbb{R} \times \mathbb{R} \to \Delta^+$ by $F_{ab}(t) = H(t - |a - b|^n), \qquad t > 0.$

From [9], $(\mathbb{R}, F, T_M, 2^{n-1})$ is a b-Menger space and in general it is not a Menger space. Define $W : \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ as

$$W(a, b; \lambda) = \lambda a + (1 - \lambda)b.$$

We prove that W is a fully convex structure. Indeed, let $a, b \in \mathbb{R}$. It is clear that W(a, b; 1) = a and W(a, b; 0) = b. Now, let $\lambda \in (0, 1)$. Suppose that

$$Min(H(\frac{t}{\lambda} - |a-c|^n), H(\frac{t}{1-\lambda} - |b-c|^n)) = 1,$$

then

$$\frac{t}{\lambda} > |a - c|^n$$
 and $\frac{t}{1 - \lambda} > |b - c|^n$,

and so

$$|\lambda a + (1 - \lambda)b - c|^{n} = |\lambda(a - c) + (1 - \lambda)(b - c)|^{n}$$

$$\leq |\lambda|a - c| + (1 - \lambda)|b - c|^{n}$$

$$\leq \lambda|a - c|^{n} + (1 - \lambda)|b - c|^{n}$$

$$(By \ convexity \ of \ function \ f(x) = x^{n})$$

$$< \lambda \frac{t}{\lambda} + (1 - \lambda) \frac{t}{1 - \lambda}$$

$$= 2t.$$

Therefore,

$$F_{W(a,b,\lambda)c}(2t) \ge T(F_{ac}(\frac{t}{\lambda}); F_{bc}(\frac{t}{1-\lambda})).$$

Thus, W is a convex structure. On the other hand, let $a, b \in \mathbb{R}$ such that $a \neq b$. Suppose that for all $\lambda \in (0,1)$, $W(a,b;\lambda) \in \{a,b\}$. If $W(a,b;\lambda) = a$ we obtain

$$(1 - \lambda)a = (1 - \lambda)b,$$

which is a contradiction. On the other hand, if $W(a, b; \lambda) = b$, we also obtain a contradiction. Hence W is a fully convex structure.

3.3. Normal structure and fixed point theorem

We require the following definitions in our further discussion.

Definition 26. We say that $a \in B$ is a diametral point of B if

$$\sup_{\varepsilon < t} \inf_{b \in B} F_{ab}(\varepsilon) = D_B(t),$$

holds for all t > 0.

Remark 27. A point $a \in B$ is non-diametral of B if there exists r > 0 such that

$$\sup_{\varepsilon < r} \inf_{b \in B} F_{ab}(\varepsilon) > D_B(r).$$

Definition 28. Let (M, F, T, s) be a b-Menger space, we say that it possesses a normal structure if, for every closed, probabilistic bounded and convex set $B \subset M$, which consists of at least two different points, there exists a point $b \in B$ which is non-diametral.

Definition 29. Let (M, F, T, s) be a b-Menger space with a convex structure W. We denote by (C_2) the condition: For every $\delta \in (0, 1)$, $a, b, c \in M$ and $a \neq b$, there exists $t_0 > 0$ such that

$$F_{W(a,b,\delta)c}(t_0) > min\{F_{ca}(t_0), F_{cb}(t_0)\}.$$

Definition 30. The probabilistic *b*-metric F has the property (P_2) if for all compact subset A of M, $p \in A$, there exists $q_0 \in A$ such that

$$\inf_{q \in A} F_{pq} = F_{pq_0}.$$

Before stating our main result, we need the following lemmas.

Lemma 31. Let (M, F, T, s) be a fully convex b-Menger space with a convex structure W satisfying (C_2) and suppose that the mapping F has the property (P_2) . Let $A \subset M$ be nonempty, convex and compact subset of M. Then A possesses a normal structure.

Proof. To the contrary, assume that A has not a normal structure. So, by the Definition 28, there exist a closed, probabilistic bounded and convex subset

 $B \subset A$ and two different points a_1 and a_2 in B then B not have a non-diametral point which gives by the use of Definition 26 that

$$\sup_{\varepsilon < t} \inf_{b \in B} F_{ab}(\varepsilon) = D_B(t),$$

for every $a \in B$. In other hand, by the Definition 24 and the fact that M is fully convex, there exists $\delta \in (0,1)$ such that $W(a_1,a_2,\delta) \notin \{a_1,a_2\}$. But B is a convex set it follows that $W(a_1,a_2,\delta) \in B$. Hence

$$D_B(t) = \sup_{\varepsilon < t} \inf_{b \in B} F_{W(a_1, a_2, \delta)b}(\varepsilon).$$

Thus the infimum of the above equality is attained since B is closed subset of the compact A and F possesses the property (P_2) . Let $a_3 \in B$ such point. So, we get

$$D_B(t) = \sup_{\varepsilon < t} F_{a_3 W(a_1, a_2, \delta)}(\varepsilon).$$

In one side, since the mapping $F_{a_3W(a_1,a_2,\delta)}(.)$ is nondecreasing and left continuous we obtain

$$D_B(t) = F_{a_3W(a_1, a_2, \delta)}(t).$$

In the other side, with the fact that the structure W satisfies the condition (C_2) and the use of the infimum and supremum properties we have that, there exists $t_0 > 0$ such that

$$D_B(t_0) = F_{a_3W(a_1,a_2,\delta)}(t_0)$$

> $\min\{F_{a_3a_1}(t_0), F_{a_3a_2}(t_0)\}$

which is a contradiction because we have $D_B \leq F_{ab}$ for all $a, b \in B$.

Lemma 32. (Zorn's Lemma) Let M be a nonempty partially ordered set in which every chain has a lower bound. Then M has a minimal element.

We say that a subset D of A is invariant under $f, g: A \to A$ if $f(D) \cap g(D) \subset D$. The (f,g)-invariant subsets of A which are nonempty, closed and convex are of particular interest since the search for common fixed points of f and g may be confined to such sets. Also, the intersection of any family of nonempty, closed, convex (f,g)-invariant subsets of A is itself closed, convex and (f,g)-invariant, although it may be empty.

Lemma 33. Let (M, F, T, s) be a b-Menger space and let $A \subset M$ be a nonempty, convex and compact subset of M. Let f and g be two self-mappings

on A such that A is (f,g)-invariant. Denote by Γ a collection of all nonempty subset of A, closed, convex and (f,g)-invariant. Then Γ has a minimal element A_0 . Moreover, if $A_1 = Co(f(A_0) \cap g(A_0))$, then $A_1 = A_0$.

Proof. We have A is a compact set, then by Remark 11 the set A is closed. In other hand A is (f,g)-invariant, also A is convex. So, $A \in \Gamma$ which implies that $\Gamma \neq \emptyset$. Now, we order this collection of Γ with inclusion: For $D_1, D_2 \in A$, $D_1 \leq D_2$ provided $D_2 \subset D_1$, then (Γ, \leq) is a partially ordered set. By compactness, any arbitrary chain of the collection Γ such $\{A_i, i \in I\}$ the subset $\bigcap_{i \in I} A_i \neq \emptyset$, hence an upper bound relative to the order relation \leq . Finally, using Zorn's Lemma we deduce that there exists a maximal element of Γ relative to \leq denoted A_0 . Moreover, we have $A_1 = Co(f(A_0) \cap g(A_0))$ and A_0 is (f,g)-invariant, then

$$A_1 = Co(f(A_0) \cap g(A_0)) = \overline{Co(f(A_0) \cap g(A_0))}$$

$$\subseteq \overline{Co(A_0)} = \overline{A_0} = A_0.$$

Thus $A_1 \subseteq A_0$. Furthermore, $A_1 \subseteq A_0 \Rightarrow f(A_1) \cap g(A_1) \subseteq f(A_0) \cap g(A_0) \subseteq Co(f(A_0) \cap g(A_0)) = A_1$ which claim that A_1 is (f,g)-invariant. So, $A_1 \in \Gamma$ and $A_0 \subseteq A_1$. Therefore $A_0 = A_1$, which achieve the proof.

Lemma 34. Let (M, F, T, s) be a convex b-Menger space has the property (P_1) with the convex structure satisfying the condition (C_1) , assume that there exists $\zeta \in (0.1)$ such that $1 - \zeta = \sup_{\varepsilon < r} \inf_{b \in A_0} F_{ab}(\varepsilon)$ for some r > 0 and some $a \in A_0$ and consider $U = (\bigcap_{b \in A_0} N_b[r, \zeta]) \bigcap A_0$ and $V = (\bigcap_{b \in f(A_0) \cap g(A_0)} N_b[r, \zeta]) \bigcap A_0$, with $U \neq \emptyset$. Then U = V.

Proof. Since $f(A_0) \cap g(A_0) \subseteq A_0$, we get immediately $V \subseteq U$. But to have $U \subseteq V$, we will take an arbitrary c in V. So, by the definition of the set V we get $c \in N_b[r,\zeta]$, for every $b \in f(A_0) \cap g(A_0)$. Then $F_{bc}(r) \geq 1 - \zeta$, $\forall b \in f(A_0) \cap g(A_0)$, which implies that also $b \in N_c[r,\zeta]$, for all $b \in f(A_0) \cap g(A_0)$. Therefore

$$f(A_0) \cap g(A_0) \subseteq N_c[r,\zeta].$$

By Lemma 21, the Definition 7 and F satisfies (P_1) -condition it follows that the set $N_c[r,\zeta]$ is closed and convex, thus

$$A_1 = \overline{Co(f(A_0) \cap g(A_0))} \subseteq N_c[r, \zeta].$$

Now, we use the Lemma 33 in order to get $A_0 \subseteq N_c[r,\zeta]$ since $A_0 = A_1$. So, every $b \in A_0$ belongs in $N_c[r,\zeta]$ it means that $c \in N_b[r,\zeta]$. Hence $c \in U$ and therefore U = V.

Definition 35. [14] Let (M, F, T, s) be a b-Menger space and let f be a self-mapping on M. We say that f is a nonexpansive mapping if

$$F_{fafb}(t) \ge F_{ab}(t),$$

for all $a, b \in M$ and t > 0.

Now, we are ready to state and prove our main theorem.

Theorem 36. Let (M, F, T, s) be a fully convex b-Menger space has the property (P_1) with a convex structure W satisfying (C_1) and (C_2) , suppose that F has the property (P_2) , and let $A \subset M$ be a nonempty, convex and compact subset of M. Let f and g be two self-mappings on A such that A is (f,g)-invariant and satisfying

$$F_{fagb}(t) \ge F_{ab}(t),\tag{3.4}$$

for all $a, b \in A$, $a \neq b$ and for every t > 0. Then f and g have at least one common fixed point on A.

Proof. To prove this result, we will proceed by absurd. So, let A_0 the minimal element of Γ , such that Γ is the collection of all nonempty subsets of A, closed, convex and (f,g)-invariant (see Lemma 33). Indeed, we suppose that A_0 has at least two different points. Since the convex structure W satisfies (C_2) , F verifies (P_2) and A is nonempty, convex and compact subset of M, thus using Lemma 31 we obtain that A possesses a normal structure. Furthermore, by Lemma 33, we have that A_0 is closed subset of the compact set A then A_0 is also a compact set, so using Lemma 15 we conclude that A_0 is a probabilistic bounded set. Then, by what we obtained previously and Definition 28 it follows that there exists some non-diametral point $a_0 \in A_0$ which means by Remark 27 that

$$\exists r > 0, \sup_{\varepsilon < r} \inf_{b \in A_0} F_{a_0 b}(\varepsilon) > D_{A_0}(r). \tag{3.5}$$

In other hand, applying Definition 30 and the fact that A_0 is a compact subset of M and F having the property (P_2) , we have

$$\exists b_0 \in A_0, \sup_{\varepsilon < t} \inf_{b \in A_0} F_{a_0 b}(\varepsilon) = F_{a_0 b_0}(t), \text{ for all } t > 0$$
(3.6)

Then, from (3.6) we deduce

$$F_{a_0b_0}(r) > D_{A_0}(r). (3.7)$$

Also, we have that there exists r' < r such that $F_{a_0b_0}(r') > D_{A_0}(r)$. Indeed, suppose to the contrary, for all r' < r that

$$F_{a_0b_0}(r') \le D_{A_0}(r).$$

But $F_{a_0b_0}$ is left continuous, so, letting $r' \to r$, we get

$$F_{a_0b_0}(r) \leq D_{A_0}(r),$$

which lead to a contradiction with (3.7). So, there exists r' < r such that

$$F_{a_0b_0}(r') > D_{A_0}(r).$$
 (3.8)

Let ζ a real number such that $1 - \zeta = \sup_{\varepsilon < r'} \inf_{b \in A_0} F_{a_0 b}(\varepsilon)$, then

$$\sup_{\varepsilon < r'} F_{a_0 b}(\varepsilon) \ge 1 - \zeta \quad \text{for all} \quad b \in A_0$$

by monotonicity and left continuity of the function F_{a_0b} we get $F_{a_0b}(r') \geq 1 - \zeta$ for all $b \in A_0$, hence $a_0 \in N_b[r', \zeta]$ for all $b \in A_0$ therefore $a_0 \in U$ and $U \neq \emptyset$. It follows from Lemma 34, that U = V. The set U is closed and convex as an intersection of closed and convex sets. It remains to show that $f(U) \cap g(U) \subseteq U$ therefore $U \in \Gamma$ to infer that $U = A_0$. For that, let $c \in U$ and $b \in f(A_0) \cap g(A_0)$. Then there exists $a \in A_0$ such that b = f(a) and b = g(a). Taking t = r' in the inequality (3.4), then

$$F_{f(c)b}(r') = F_{f(c)q(a)}(r') \ge F_{ca}(r') \ge 1 - \zeta.$$

Thus, the definition of the set V implies $f(c) \in V$. It follows that $f(U) \subseteq V$ and since V = U, we have that $f(U) \subseteq U$. Also, since b = f(a) we get

$$F_{g(c)b}(r) = F_{g(c)f(a)}(r) \ge F_{ca}(r) \ge 1 - \zeta,$$

and by the definition of the set U we get $g(c) \in V$. So, $g(U) \subseteq V$ but V = U, then $g(U) \subseteq U$ and therefore $U \in \Gamma$ hence $U = A_0$. This implies that

$$U \subset \bigcap_{b \in U} N_b[r', \zeta].$$

From last we have that for every $a, b \in U$ it holds that $F_{ab}(r') \ge 1 - \zeta$, then $\inf_{ab \in U} F_{ab}(r') \ge 1 - \zeta$. From the last and (3.8), it follows that

$$D_U(r) = \sup_{\varepsilon < r} \inf_{ab \in U} F_{ab}(\varepsilon) \ge \sup_{r' \le \varepsilon < r} \inf_{ab \in U} F_{ab}(\varepsilon) \ge 1 - \zeta > D_{A_0}(r).$$

This is a contradiction with $U = A_0$. Then A_0 consists of only one point \overline{x} and since $f(A_0) \cap g(A_0) \subseteq A_0$ this means that $f(\overline{x}) = g(\overline{x}) = \overline{x}$ and \overline{x} is a common fixed point of the mappings f and g. This completes the proof.

If we take f = g in Theorem 36, we get the following.

Corollary 37. Let (M, F, T, s) be a fully convex b-Menger space has the property (P_1) with a convex structure W satisfying (C_1) and (C_2) , suppose that F has the property (P_2) . Let $A \subset M$ be a nonempty, convex and compact subset of M. Then any nonexpansive mapping f of A into A has a fixed point.

4. Applications

We require the following lemma in our further discussion.

Lemma 38. ([12]) Let $(M, \| \|)$ be a Banach space. The following conditions are equivalent

- 1. M is strictly convex.
- 2. If $1 and <math>x, y \in M$, $x \neq y$, then $\|\lambda x + (1 \lambda)y\|^p < \lambda \|x\|^p + (1 \lambda)\|y\|^p$ for every $0 < \lambda < 1$.

Example 39. Let M be a Banach space strictly convex and $1 . Define <math>F: M \times M \to \Delta^+$ and $W: M \times M \times [0,1] \to M$ by

$$F_{ab}(t) = H(t - ||a - b||^p),$$

and

$$W(a, b; \lambda) = \lambda a + (1 - \lambda)b.$$

From [9], $(M, F, T_M, 2^{p-1})$ is a b-Menger space (but is not in general a standard probabilistic metric space). Proceeding as in Example 25, we prove that W is a fully convex structure. Consider the function $f: \mathbb{R} \to \mathbb{R}$ given by

$$fx = \begin{cases} x & \text{if } x \in A, \\ 2x & \text{otherwise.} \end{cases}$$

where A be a nonempty, convex and compact subset of M. It is very essay to see that $N_a(t) = B(a, t^{-p})$ is the open ball of radius t^{-p} centred at a in $(M, \| . \|)$. So, a subset A is compact in $(M, F, T_M, 2^{p-1})$ if and only if A is compact in $(M, \| . \|)$. We claim that F has the property (P_2) . Indeed, let B be a compact in $(M, F, T_M, 2^{p-1})$ and $b \in B$. This implies that B is compact

in $(M, \| . \|)$ and since $a \mapsto \|a - b\|^p$ is continuous, then there exists $a_0 \in B$ such that

$$||a_0 - b||^p = \sup_{a \in B} ||a - b||^p.$$

We can check easily that

$$\inf_{a \in B} F_{ab} = F_{a_0b}.$$

Which prove our claims. By following the same lines of proof as in Example 9, we show that $(M, F, T_M, 2^{p-1})$ has the property (P_1) . Next, we prove that $(M, F, T_M, 2^{p-1})$ satisfies the conditions (C_2) and (C_1) . Indeed, let $a, b, c \in M$ and $\lambda \in [0, 1]$. By Lemma 38 we get

$$||W(a,b,\lambda) - c||^p \le \lambda ||a - c||^p + (1 - \lambda)||b - c||^p$$

SO

$$||W(a,b,\lambda) - c||^p \le \max\{||a - c||^p, ||b - c||^p\}.$$

Therefor

$$||W(a,b,\lambda) - c||^p \le ||a - c||^p \text{ or } ||W(a,b,\lambda) - c||^p \le ||b - c||^p,$$
 (4.1)

then

$$F_{W(a,b,\lambda)c} \geq F_{ac}$$
 or $F_{W(a,b,\lambda)c} \geq F_{bc}$.

Hence, condition (C_2) holds. Now, we prove that

$$||W(a,b,\lambda) - c||^p < ||a - c||^p \text{ or } ||W(a,b,\lambda) - c||^p < ||b - c||^p.$$
(4.2)

for all $a, b, c \in M$, $\lambda \in (0, 1)$ and $a \neq b$. Suppose that (4.2) does not hold, which in view of (4.1) gives

$$||W(a,b,\lambda) - c||^p = ||a - c||^p \text{ and } ||W(a,b,\lambda) - c||^p = ||b - c||^p,$$
 (4.3)

Taking ||b - c|| = r, we get

$$\frac{\|W(a,b,\lambda) - c\|}{r} = \|\lambda \frac{(a-c)}{r} + (1-\lambda) \frac{(b-c)}{r}\| = 1 \quad , \tag{4.4}$$

and

$$\frac{\|b - c\|}{r} = \frac{\|a - c\|}{r} = 1$$

Since (M, ||.||) is strictly convex

$$\|\lambda \frac{(a-c)}{r} + (1-\lambda) \frac{(b-c)}{r}\| < 1,$$

which a contradiction with (4.4). Therefore, (4.2) holds. Now, if

$$||W(a,b,\lambda) - c||^p < ||a - c||^p,$$

we take $t_0 > 0$ such that

$$||W(a,b,\lambda) - c||^p < t_0 < ||a - c||^p.$$

On the other hand, if

$$||W(a,b,\lambda)-c||^p < ||b-c||^p,$$

we take $t_0 > 0$ such that

$$||W(a,b,\lambda) - c||^p < t_0 < ||b - c||^p.$$

We conclude that condition (C_1) holds. Taking g is the identity function on M. Clearly, all the conditions of Theorem 36 are satisfied and every point of the set A is a common fixed point of f and g.

5. Conclusion

The geometric condition of normal structure plays a crucial role in the fixed point theory. In this paper, we use this condition to prove the existence of fixed point (common fixed point) for nonexpansive mappings in the framework of b-Menger spaces. Our main theorems extend and unify the existing results in the recent literature. Example is constructed to support our result.

References

- [1] L.M. Blumenthal, Distance Geometries: A Study of the Development of Abstract Metrics. University of Missouri Studies, (1938).
- [2] L.M. Blumenthal, *Theory and Applications of Distance Geometries*, Claremont Press, Oxford, (1953).
- [3] L.M. Blumenthal, K. Menger, *Studies in Geometry*, W.H.Freeman and Co., San Francisco, (1970).

- [4] M.S. Brodskii, D.P. Milman, On the center of a convex set, Dokl. Akad. Nauk. SSSR, 59 (1948), 837-840.
- [5] R.J. Egbert, Products and quotients of probabilistic metric spaces, *Pacific J. Math*, **24** (1968), 437-455.
- [6] O. Hadžić, Common fixed point theorems in probabilistic metric spaces with a convex structure, Zb. Rad. Prirod-Mat.Fak. Ser. Mat., 18 (1987), 165-178.
- [7] W.A. Kirk, A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly*, **72** (1965), 1004-1006.
- [8] K. Menger, Untersuchungenüber allgemeine Metrik, *Math. Ann.*, **100** (1928), 75-163.
- [9] A. Mbarki, R. Oubrahim, Probabilistic b-metric spaces and nonlinear contractions, Fixed Point Theory Appl., 2017 (2017), 15-29.
- [10] K. Menger, Statistical metrics, Proc. Nat. Acad. Sci., 28 (1942), 535-7.
- [11] K. Menger, Géométrie générale, Mémor. Sci. Math., Gauthier-Villars, Paris, 124, (1954).
- [12] S. Prus, Geometrical background of metric fixed point theory. In: W.A. Kirk, B. Sims (Eds), Handbook of Metric Fixed Point Theory, Springer, Dordrecht (2001).
- [13] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathimatics, **5** (1983).
- [14] J.N. Siniśa, R. M. Nikolić, N.A. Babaćev, A common fixed point theorem in strictly convex Menger PM-spaces, *Filomat*, **28**, No 4 (2014), 735-743.
- [15] T. Wataru, A convexity in metric spaces and nonexpansive mappings, I. Kodai Math. Sem. Rep., 22 (1970), 142-149.