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EXISTENCE OF FIXED POINT AND BEST PROXIMITY POINT OF p-CYCLIC ORBITAL CONTRACTION OF BOYD-WONG TYPE

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Abstract: Let $A_1, A_2, ..., A_p$ ($p \in \mathbb{N}$) be non empty subsets of a metric space (X, d). In this paper, a map $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$, called p-cyclic orbital contraction of Boyd-Wong type is introduced. Convergence of a unique fixed point and a best proximity point for this map are obtained in a uniformly convex Banach space settings. Moreover, the obtained best proximity point is the unique periodic point of the map.

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1. Introduction and Preliminaries

It is well known that a large number of attempts were made to weaken the contraction condition of Banach contraction theorem, which is of great importance in non-linear analysis, both of abstract and applied directions. One such generalization of the contraction map is given in [1] which is stated as follows:

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Definition 1. A function $\psi : [0,\infty) \to [0,\infty)$ is said to be an upper semi continuous from the right if $r_j \searrow r \geq 0$, then $\lim_j sup\psi(r_j) \leq \psi(r)$.

Theorem 2. ([1]) *Let* X *be a complete metric space and* $T : X \to X$ *be such that* $d(Tx,Ty) \leq \psi(d(x,y)), \quad x,\ y\ \in X$, *where* $\psi : [0,\infty) \to [0,\infty)$ *is upper semi continuous from the right and satisfies* $\psi(t) < t$ *for* $t > 0$ *and* $\psi(0) = 0$. *Then for any* $x \in X$, $\{T^n x\}$ *converges to a unique fixed point of* T *in* X.

As a generalization of Boyd-Wong's theorem, in [2], the following fixed point theorem is proved.

Theorem 3. *Let* $A_1, A_2, ..., A_p$ $(p \geq 2)$ *be non empty closed subsets of a complete metric space* (X,d). *Suppose* $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ *be a map satisfying the following conditions:*

1) $T(A_i) \subseteq A_{i+1}$ *where* $A_{p+i} = A_i$.
2) $d(Tx,Ty) \leq \psi(d(x,y)), x \in A_i, \quad y \in A_{i+1}, 1 \leq i \leq p$, *where* ψ *is a map as defined in Theorem 2. Then* T *has a unique fixed point in* $\cap_{i=1}^{p} A_i$.

Note that in $\cap_{i=1}^{p} A_i$, T is a self map and is a map defined by Boyd-Wong. In [3], Eldred and Veeramani further extended Boyd-Wong's result by introducing a notion of cyclic maps which is defined as follows:

Definition 4. Let A and B be non empty subsets of a metric space (X,d). A map $T : A \cup B \to A \cup B$ is said to be a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $x \in A \subseteq B$ is said to be a best proximity point, if $d(x,Tx) = dist(A,B)$, where $dist(A,B) = inf\{(x,y) : x \in A, y \in B\}$.

Note that if $dist(A,B) = 0$, then the obtained best proximity point is a fixed point. Hence best proximity point theorems are direct extensions of fixed point theorems.

Theorem 5. ([3]) *Let* A *and* B *be non empty, closed and convex subset of a uniformly convex Banach space. Let* $T : A \cup B \to A \cup B$ *be a cyclic map such that*

$$\|Tx - Ty\| \leq \psi(\|x - y\| - dist(A,B)) + dist(A,B), \quad x \in\ A,\ y \in\ B, \tag{1}$$

where ψ *is a map as defined in Theorem 2. Then there exists a unique best* *proximity point* $\xi \in A$. *Further, if* $x_0 \in A$, *and* $x_{n+1} = Tx_n$, *then* $\{x_{2n}\}$ *converges to this unique best proximity point.*

In [3], the following lemma is proved, which is used to prove the main results.

Lemma 6. *Let* A *and* B *be non empty and closed subsets of a uniformly convex Banach space* X. *Let* A *be convex. Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences in* A *and* $\{z_n\}$ *be sequence in* B *such that* $\lim_{n\to\infty} \|x_n - z_n\| = dist(A, B)$ *and* $\lim_{n\to\infty} \|y_n - z_n\| = dist(A, B)$, *then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

In [6], a notion of p-cyclic map is introduced. If (X, d) is a metric space, $A_1, A_2, ..., A_p$, $(p \geq 2)$ are non empty subsets of X, then a map $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is called a p-cyclic map if $T(A_i) \subseteq A_{i+1}$, where $A_{p+1} = A_1$. A point $x \in A_i$ is said to be a best proximity point of T in A_i if $d(x, Tx) = dist(A_i, A_{i+1})$. In [4], the following best proximity point theorem for a contraction of Boyd-Wong type, is obtained for a p-cyclic map.

Theorem 7. ([4]) *Let* $A_1, A_2, ..., A_p$, $(p \geq 2)$ *be non empty closed and convex subsets of a uniformly convex Banach space. Let* $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ *be a p-cyclic map satisfying the following condition, for* $x \in A_i$, $y \in A_{i+1}, 1 \leq i \leq p$,

$$\|Tx - Ty\| \leq \psi(\|x - y\| - dist(A_i, A_{i+1})) + dist(A_i, A_{i+1}), \tag{2}$$

where ψ *is a map as given in Theorem 2. Then for* $x \in A_i (1 \leq i \leq p)$, *the sequence* $\{T^{pn}x\}$ *converges to a point* $z \in A_i$ *such that* z *is a best proximity point of* T *in* A_i *and unique periodic point in* A_i. *Further,* $T^k z$ *is a best proximity point of* T *and also unique periodic point of* T *in* A_{i+k}.

In [5], a notion of cyclic orbital contraction is introduced in which the contraction condition need not be satisfied for all the points. Fixed points and best proximity points are obtained for such a map. In [10], the following notion of p-cyclic orbital non expansive map is introduced.

Definition 8. Let (X, d) be a metric space. Let $A_1, A_2, ..., A_p$, be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map such that for some $x \in A_i$, $(1 \leq i \leq p)$, the following inequality holds for each $k = 0, 1, 2, ..., (p-1)$

$$d(T^{pn+k}x, T^{k+1}y) \leq d(T^{pn+k-1}x, T^k y), \quad \forall \ y \in A_i \ \text{ and } \ \forall \ n \in \mathbb{N}. \tag{3}$$

Then the map T is called **p-cyclic orbital non expansive map.**

In [10], the following proposition is proved which is useful to prove the main results.

Proposition 9. *Let X be a strictly convex normed space. Let $A_1, A_2, ..., A_p$, be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital non expansive map such that for some $x \in A_i$, $(1 \leq i \leq p)$,the inequality (3) holds for each $k = 0, 1, 2, ..., (p-1)$. Suppose for each $k = 0, 1, 2, ..., (p-1)$ and $y \in A_i$,*

$$\lim_{n\to\infty} d(T^{pn+k-1}x, T^k y) = dist(A_{i+k-1}, A_{i+k})$$

and $\{T^{pn+k}x\}$ converges to $z_k \in A_{i+k}$. Then the following hold:

a) $dist(A_1, A_2) = dist(A_2, A_3) = ... = dist(A_{p-1}, A_p) = dist(A_p, A_1)$,
b) z_k is a best proximity point of T in A_{i+k} and $z_k = T^k z_0$ for $k = 1, 2, ..., p$,
c) z_k is the unique periodic point of T with period p in A_{i+k}.

2. Main Results

We introduce a notion of **p-cyclic orbital contraction of Boyd-Wong type**, which is defined as follows:

Definition 10. Let (X, d) be a metric space. Let $A_1, A_2, ..., A_p$ be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map such that for some $x \in A_i$, $(1 \leq i \leq p)$, for all $y \in A_i$ and for all $n \in \mathbb{N}$, the following inequality holds:

$$\begin{aligned} d(T^{pn+k}x, T^{k+1}y) &\leq \psi[d(T^{pn+k-1}x, T^k y) - dist(A_{i+k-1}, A_{i+k})] \\ &+ dist(A_{i+k-1}, A_{i+k}), \end{aligned} \tag{4}$$

where $\psi : [0, \infty) \to [0, \infty)$ is upper semi continuous from the right and satisfies $\psi(t) < t$ for $t > 0$ and $\psi(0) = 0$. Then T is called p-cyclic orbital contraction of Boyd-Wong type.

Proposition 11. *Let (X, d) be a metric space and $A_1, A_2, ..., A_p$ be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital contraction* *map of Boyd-Wong type, with an* $x \in A_i$, $1 \leq i \leq p$, *satisfying* (4)*. Then the following hold:*

a) T *is a p - cyclic orbital non expansive map.*
b) $\lim_{n\to\infty} d(T^{pn+k}x, T^{pn+k+1}y) = dist(A_{i+k}, A_{i+k+1})$, $y \in A_i$, *for each* $k \in \{0, 1, 2, ..., p\}$
c) $\lim_{n\to\infty} d(T^{pn-1}x, T^{pn}y) = dist(A_{i-1}, A_i)$, $y \in A_i$.
d) $\lim_{n\to\infty} d(T^{pn+p}x, T^{pn+1}y) = dist(A_i, A_{i+1})$, $y \in A_i$.
e) $\lim_{n\to\infty} d(T^{pn-p}x, T^{pn+1}y) = dist(A_i, A_{i+1})$, $y \in A_i$.
f) $\lim_{n\to\infty} d(T^{pn}x, T^{pn+p+1}y) = dist(A_i, A_{i+1})$, $y \in A_i$.

Proof. Let $x \in A_i$, $1 \leq i \leq p$, satisfy (4) and $y \in A_i$.
a) Let $d(T^{pn+k-1}x, T^k y) > dist(A_{i+k-1}, A_{i+k})$.
Let $\mu_k = d(T^{pn+k}x, T^{k+1}y)$, $\mu_{k-1} = d(T^{pn+k-1}x, T^k y)$ and $d = dist(A_{i+k-1}, A_{i+k})$. To prove that $\mu_k \leq \mu_{k-1}$, let $\mu_{k-1} > d$. Then $(\mu_{k-1} - d) > 0$. Since $\psi(t) < t$, for $t > 0$, we have $\mu_k \leq \psi(\mu_{k-1} - d) + d < \mu_{k-1} - d + d = \mu_{k-1}$. Thus $\mu_k < \mu_{k-1}$.

If $\mu_k = d$, then $\psi(\mu_{k-1} - d) = 0$. Since $\psi(0) = 0$, we have $\mu_{k-1} = d$. Thus $\mu_k = d = \mu_{k-1}$. Hence it is proved that T is a p-cyclic orbital non expansive map.

b) Let $y \in A_i$ and $k \in \{0, 1, 2, ..., p\}$. Since T is a p-cyclic orbital non expansive map, the sequence $\{T^{pn+k}x, T^{pn+k+1}y\}$ is a non increasing sequence, bounded below by $dist(A_{i+k}, A_{i+k+1})$. Therefore, this sequence converges to $r \geq dist(A_{i+k}, A_{i+k+1})$, where $r = inf_n d(T^{pn+k}x, T^{pn+k+1}y)$. If $r = dist(A_{i+k}, A_{i+k+1})$, there is nothing to prove. Hence, let $r > dist(A_{i+k}, A_{i+k+1})$. Now

$$d(T^{p(n+1)+k}x, T^{p(n+1)+k+1}y) \leq d(T^{pn+k+1}x, T^{pn+k+2}y)$$
$$\leq \psi(d(T^{pn+k}x, T^{pn+k+1}y) - dist(A_{i+k}, A_{i+k+1})) + dist(A_{i+k}, A_{i+k+1}),$$

$$d(T^{p(n+1)+k}x, T^{p(n+1)+k+1}y) - dist(A_{i+k}, A_{i+k+1}) \leq$$
$$\psi(d(T^{pn+k}x, T^{pn+k+1}y) - dist(A_{i+k}, A_{i+k+1})).$$

Letting $n \to \infty$, we get
$$lim_{n\to\infty}[d(T^{p(n+1)+k}x, T^{p(n+1)+k+1}y) - dist(A_{i+k}, A_{i+k+1})]$$
$$\leq limsup_{n\to\infty}\psi(d(T^{pn+k}x, T^{pn+k+1}y) - dist(A_{i+k}, A_{i+k+1}))$$
$$[r - dist(A_{i+k}, A_{i+k+1})] \leq \psi(r - dist(A_{i+k}, A_{i+k+1})).$$
Let $t = (r - dist(A_{i+k}, A_{i+k+1}))$. Then we get $t \leq \psi(t)$, which is the contradiction to definition of ψ that $\psi(t) < t$, for $t > 0$. Hence $r = dist(A_{i+k}, A_{i+k+1})$. Similarly c), d), e) and f) can be proved. □

The following proposition is useful to prove the main result, the proof of which follows from Lemma 6 and Proposition 11.

Proposition 12. *Let $A_1, A_2, ..., A_p$ be non empty, closed and convex subsets of a uniformly convex Banach space X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital contraction map of Boyd-Wong type with an $x \in A_i$, $1 \leq i \leq p$, satisfying (4). Then the following hold:*
a) $\|T^{pn}x - T^{pn+p}x\| \longrightarrow 0$
b) $\|T^{pn}x - T^{pn-p}x\| \longrightarrow 0$
c) $\|T^{pn+1}x - T^{pn+p+1}x\| \longrightarrow 0.$

Theorem 13. *Let (X, d) be a complete metric space. Let $A_1, A_2, ..., A_p$ be non empty and closed subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map such that for some $x \in A_i$, $1 \leq i \leq p$ the following is satisfied for all $y \in A_i$ and for all $n \in \mathbb{N}$:*

$$d(T^{pn+k}x, T^{k+1}y) \leq \psi[d(T^{pn+k-1}x, T^k y)] \tag{5}$$

where ψ is a map as given in 2. Then $\cap_{i=1}^{p} A_i$ is non empty and $\{T^{pn}x\}$ converges to $z_0 \in \cap_{i=1}^{p} A_i$ which is the unique fixed point of T.

Proof. When $dist(A_{i+k}, A_{i+k+1}) = 0$ in (4), we get (5). Let $x \in A_i$ $(1 \leq i \leq p)$ satisfy (5). Let us prove that, given $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$d(T^{pn}x, T^{pm}x) < \epsilon, \quad for \quad all \quad n, m \geq n_0, \tag{6}$$

by induction on m. Let $\epsilon > 0$ be given. Now

$$d(T^{pn}x, T^{pm}x) \leq d(T^{pn}x, T^{pm+1}x) + d(T^{pm+1}x, T^{pm}x).$$

By similar argument as in Proposition 11 b), it can be shown that by putting $dist(A_{i+k}, A_{i+k+1}) = 0$, $lim_{n\to\infty} d(T^{pm}x, T^{pm+1}x) = 0$. Hence there exists an $n_0 \in \mathbb{N}$ such that

$$d(T^{pm+1}x, T^{pm}x) < (\delta/p), 0 < \delta < (\epsilon/2), m \geq n_0. \tag{7}$$

Hence it is enough that if we prove that

$$d(T^{pn}x, T^{pm+1}x) < (\epsilon/2), \qquad m, n \geq n_0. \tag{8}$$

Fix $n_0 \in \mathbb{N}$ such that (7) holds. Now (8) is true for $m = n$. Assume that (8) is true for some $m, m \geq n_0$. We will prove that (8) is true for $(m + 1)$ in place of m. Now

$$\begin{array}{rcl} d(T^{pn}x, T^{p(m+1)+1}x) & \leq & d(T^{pn}x, T^{pm+1}x) + d(T^{pm+1}x, T^{pm+2}x) \\ & + & ... + d(T^{pm+p}x, T^{pm+p+1}x) \\ & < & (\epsilon/2) + (\delta/p)p \\ & < & \epsilon. \end{array}$$

Hence $\{T^{pn}x\}$ is a Cauchy sequence and hence converges to a limit say, $\xi \in A_i$. By putting $dist(A_{i-1}, A_i) = 0$, in Proposition 11, c), it can be proved that $\lim_{n\to\infty} d(T^{pn-1}x, T^{pn}x) = 0$. Now

$$d(\xi, T\xi) = \lim_n d(T^{pn}x, T\xi) \leq \lim_n d(T^{pn-1}x, \xi) = \lim_n d(T^{pn-1}x, T^{pn}x) = 0.$$

Since T is p-cyclic, $\xi \in \cap_{i=1}^{p} A_i$ and is a fixed point. To prove that ξ is unique, consider η be such that $\eta = T\eta$. It can shown by similar argument as in Proposition 11, b), $d(T^{pn}x, T^{pn+1}\eta) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence $d(\xi, \eta) = d(T^{pn}x, T^{pn+1}\eta) \longrightarrow 0 \;\; as \;\; n \to \infty$. Hence $\eta = \xi$. ☐

Remark 14. *It is easy to see that Theorem* 3 *is a corollary of Theorem* 13.

Theorem 15. *Let $A_1, A_2, ..., A_p$ be non empty, closed and convex subsets of a uniformly convex Banach space X, $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital contraction map of Boyd-Wong type. Then for every $x \in A_i$ $(1 \leq i \leq p)$ satisfying equation (4).$\{T^{pn}x\}$ converges to a unique $z_i \in A_i$ which is a best proximity point of T in A_i and it is also a unique periodic point of T in A_i. Moreover, $T^j z_i = z_{i+j}$ is a best proximity point and unique periodic point of T in A_{i+j}, for $j = 1, 2, ..., (p-1)$.*

Proof. If $dist(A_i, A_{i+1}) = 0$, in (4), then by Theorem 13 we have a unique fixed point of T in $\cap_{i=1}^{p} A_i$. Hence assume $dist(A_i, A_{i+1}) > 0$. We claim that, for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$,

$$\|T^{pm}x - T^{pn+1}x\| \leq dist(A_i, A_{i+1}) + \epsilon. \tag{9}$$

Suppose not, then there exists an $\epsilon_0 > 0$ such that for all $k \in \mathbb{N}$, there exists $m_k > n_k \geq k$, for which

$$\|T^{pm_k}x - T^{pn_k+1}x\| > dist(A_i, A_{i+1}) + \epsilon_0. \tag{10}$$

By choosing m_k to be the least integer greater than n_k to satisfy the above inequality, we have,

$$\|T^{pm_k-p}x - T^{pn_k+1}x\| \leq dist(A_i, A_{i+1}) + \epsilon_0. \tag{11}$$

Now for each k, $dist(A_i, A_{i+1}) + \epsilon_0 \leq \|T^{pm_k}x - T^{pn_k+1}x\|$
$\leq \|T^{pm_k}x - T^{pm_k-p}x\| + \|T^{pm_k-p}x - T^{pn_k+1}x\|$
$< \|T^{pm_k}x - T^{pm_k-p}x\| + dist(A_i, A_{i+1}) + \epsilon_0$, by (8).By letting $k \longrightarrow \infty$ and by Proposition 12, b), we have
$dist(A_i, A_{i+1}) + \epsilon_0 \leq \lim_{k\longrightarrow\infty} \|T^{pm_k}x - T^{pn_k+1}x\| \leq dist(A_i, A_{i+1}) + \epsilon_0$. Now,
$\|T^{pm_k}x - T^{pn_k+1}x\| \leq \|T^{pm_k}x - T^{pm_k+p}x\|$
$+\|T^{pm_k+p}x - T^{pn_k+p+1}x\| + \|T^{pn_k+p+1}x - T^{pn_k+1}x\|$. Now by using p-cyclic orbital non expansiveness of T, $(p-1)$ times to $\|T^{pm_k+p}x - T^{pn_k+p+1}x\|$, we get,

$\|T^{pm_k+p}x - T^{pn_k+p+1}x\| \leq \|T^{pm_k+1}x - T^{pn_k+2}x\|$
$\leq \psi(\|T^{pm_k}x - T^{pn_k+1}x\| - dist(A_i, A_{i+1})) + dist(A_i, A_{i+1})$.

Let us denote $dist(A_i, A_{i+1})$ by d and $\|T^{m_k}x - T^{pn_k+1}x\|$ by μ_k. Since $\mu_k - d \downarrow \epsilon_0$, $limsup_{k\to\infty}\psi(\mu_k - d) \leq \psi(\epsilon_0)$. By Proposition 12, (a) and (c), we have
$lim_{k\to\infty}\|T^{pm_k}x - T^{pm_k+p}x\| = 0$ and
$lim_{k\to\infty}\|T^{pn_k+p+1}x - T^{pn_k+1}x\| = 0$.

Thus $lim_{k\to\infty}\|T^{pm_k}x - T^{pn_k+1}x\| \leq \psi(\epsilon_0) + dist(A_i, A_{i+1})$.
$dist(A_i, A_{i+1}) + \epsilon_0 \leq \psi(\epsilon_0) + dist(A_i, A_{i+1})$.
Therefore $\epsilon_0 \leq \psi(\epsilon_0)$, a contradiction. Hence the claim follows. Now by Proposition 11, b),
$\|T^{pn}x - T^{pn+1}x\| \longrightarrow dist(A_i, A_{i+1})$.
Combining this with the claim, by Lemma 6, we have the following: For every $\epsilon > 0$ there exists an $n_1 \in \mathbb{N}$, such that $\|T^{pm}x - T^{pn}x\| \leq \epsilon, \ m > n > n_1$. Hence $\{T^{pn}x\}$ is a Cauchy sequence in A_i and converges to a $z_i \in A_i$. By Proposition 9, b) and c), z_i is a best proximity point of T in A_i and also unique periodic point of T in A_i. Further, $T^j z_i = z_{i+j}$ is a best proximity point in A_{i+j}.

Now we prove that, for any $y \in A_i$, $y \neq x$ satisfying the inequality (4), the sequence $\{T^{pn}y\}$ converges to the same $z_i \in A_i$.

Let $\{T^{pn}y\}$ converge to $\eta \in A_i$. By what we have proved, $\|\eta - T\eta\| = dist(A_i, A_{i+1})$. Also, $T^p\eta = \eta$ and $\|T^{p+1}\eta = T\eta\|$. Claim: $\|z_i - T\eta\| = dist(A_i, A_{i+1})$. Suppose $\|z_i - T\eta\| > dist(A_i, A_{i+1})$. Then

$$\|z_i - T\eta\| - dist(A_i, A_{i+1}) > 0,$$

$$\|Tz_i - T^2\eta\| \leq \psi(\|z_i - T\eta\| - dist(A_i, A_{i+1})) + dist(A_i, A_{i+1})$$

$$\begin{aligned} &< \|z_i - T\eta\| - dist(A_i, A_{i+1}) + dist(A_i, A_{i+1}) \\ &= \|T^p z_i - T^{p+1}\eta\| \\ &\leq \|Tz_i - T^2\eta\|. \end{aligned}$$

Thus we arrive at a contradiction. Therefore $\|z_i - T\eta\| = dist(A_i, A_{i+1}) = \|\eta - T\eta\|$. Since the underlying space is uniformly convex Banach space and the sets are convex, we have $\eta = z_i$. $\square$

Remark 16. *It is easy to see that Theorems* 5 *and* 7 *are corollaries to Theorem* 15*.*

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