

ON PÓLYA AND STEFFENSEN INTEGRAL INEQUALITIES
FOR TRIGONOMETRICALLY ρ -CONVEX FUNCTIONS

Nashat Faried¹, Mohamed S.S. Ali², Asmaa A. Badr^{2 §}

¹Department of Mathematics

Faculty of Science

Ain Shams University, Cairo, EGYPT

²Department of Mathematics

Faculty of Education

Ain Shams University, Cairo, EGYPT

Abstract: The aim of this article is to derive several Pólya and Steffensen type integral inequalities for trigonometrically ρ -convex functions.

AMS Subject Classification: 52A41, 26A51, 26B25

Key Words: integral inequalities, supporting functions, trigonometrically ρ -convex functions

1. Introduction

In 1908, Phragmén and Lindelöf (see for example Ali [1] and Levin [7, pp. 53]) introduced the function

$$h(\theta) = \limsup_{r \rightarrow \infty} \frac{\log |F(re^{i\theta})|}{r^\rho}$$

as the indicator function of an entire function $F(z)$ of order $0 < \rho < \infty$. For the function

$$F(z) = e^{A-iB} z^\rho$$

holomorphic in an angle $\{z = re^{i\theta} : \alpha < \theta < \beta\}$, $\beta - \alpha \leq 2\pi$, we have

Received: June 18, 2018

© 2018 Academic Publications

[§]Correspondence author

$$|F(re^{i\theta})| = e^{(A \cos \rho\theta + B \sin \rho\theta)r^\rho},$$

and its indicator is equal to

$$H(\theta) = A \cos \rho\theta + B \sin \rho\theta,$$

(such functions are called sinusoidal or ρ -trigonometric) which has the same value of $h(\theta)$ at the points θ_1 and θ_2 where $0 < \theta_2 - \theta_1 < \frac{\pi}{\rho}$, then for $[\theta_1, \theta_2] \subseteq [\alpha, \beta]$ we have

$$h(\theta) \leq H(\theta), \quad \theta_1 \leq \theta \leq \theta_2.$$

This property is called a trigonometric ρ -convexity.

Trigonometrically ρ -convex functions have numerous applications in the theory of entire functions and in the theory of cavitation diagrams for hydroprofiles Avhadief et al. [3], Levin [7] and Maergoiz [8].

In 1921, Pólya [11] showed that if the function $f(x)$ is differentiable and not identically a constant on $[a, b]$ with $f(a) = f(b) = 0$, then there exists at least one point ξ in (a, b) for which

$$|f'(\xi)| > \frac{4}{(b-a)^2} \int_a^b f(x) dx, \quad (1)$$

this inequality is called Pólya integral inequality. The relation (see for example Pólya et al. [12, pp. 80 and 267], Qi [13] and Klambauer [6, pp. 322-323])

$$\int_a^b f(x) dx < L \frac{(b-a)^2}{4}, \quad (2)$$

is known in the literature as Pólya type integral inequality, which holds for function $f(x)$ which is differentiable and not identically a constant in $[a, b]$ with $f(a) = f(b) = 0$ and

$$L = \sup_{x \in [a, b]} |f'(x)|.$$

In 1919, Steffensen [15] (see e.g. Niculescu et al. [10, pp. 190] and Qi [13]) proved that if two functions $f(t)$ and $\phi(t)$ are integrable on $[a, b]$ such that $f(t)$ never increases and $0 \leq \phi(t) \leq 1$. Putting for abbreviation

$$\lambda = \int_a^b \phi(t) dt. \quad (3)$$

Then,

$$\int_{b-\lambda}^b f(t) dt \leq \int_a^b f(t) \phi(t) dt \leq \int_a^{a+\lambda} f(t) dt. \quad (4)$$

If $\phi(t) = 0$ or $\phi(t) = 1$ or $f(t)$ is a constant for all t , the two limits in (4) coincide.

The object of the present paper is to establish certain integral inequalities for trigonometrically ρ -convex functions, which are resemble to Pólya integral inequality Pólya [11], Steffensen integral inequality Steffensen [15] and Hermite-Hadamard's inequality Mitrinović et al. [9]. Furthermore, we acquire that integral inequality for product n^{th} trigonometrically ρ -convex functions. We are interested in real finite functions on a finite or infinite interval I such that $I \subset \mathbf{R}$. The rest of this paper falls into three parts. After Introduction, Section 2 contains the basic concepts and result, which are needed in this paper. The goal of Section 3 is to introduce a certain inequalities trigonometrically ρ -convex functions.

2. Definitions and preliminary results

In this section, we present the basic definitions and results which will be used later. For more details about the basic definitions and results concerning the class of trigonometrically ρ -convex function, see (Ali [1]- Green [5], Levin [7] - Maergoiz [8] and Roberts et al. [14]).

Definition 1. (see Ali [1]) A function $f : I \rightarrow \mathbf{R}$ is said to be **Trigonometrically ρ -Convex Function** if for any arbitrary closed subinterval $[u, v]$ of I such that $0 < \rho(v - u) < \pi$, the graph of $f(x)$ for $x \in [u, v]$ lies nowhere above the ρ -trigonometric function, determined by the equation

$$M(x) = M(x; u, v, f) = A \cos \rho x + B \sin \rho x,$$

where A and B are chosen such that $M(u) = f(u)$ and $M(v) = f(v)$. Equivalently, if for all $x \in [u, v]$

$$f(x) \leq M(x) = \frac{f(u) \sin \rho(v - x) + f(v) \sin \rho(x - u)}{\sin \rho(v - u)}. \quad (5)$$

The trigonometrically ρ -convex functions possess many of properties analogous to those of convex functions. For example: [1] If $f : I \rightarrow \mathbf{R}$ is trigonometrically ρ -convex function, then for any $u, v \in I$ such that $0 < \rho(v - u) < \pi$, the inequality

$$f(x) \geq M(x; u, v, f),$$

holds outside the interval $[u, v]$.

Definition 2. (see Levin [7]) A function

$$T_u(x) = A \cos \rho x + B \sin \rho x,$$

is said to be **supporting function** for $f(x)$ at the point $u \in I$ if

$$T_u(u) = f(u) \text{ and } T_u(x) \leq f(x), \quad \forall x \in I. \quad (6)$$

Theorem 3. (Ali [1]) A function $f : I \rightarrow \mathbf{R}$ is trigonometrically ρ -convex function on I if and only if there exists a supporting function for $f(x)$ at each point $x \in I$.

Proposition 4. (Ali [1]) If $f : I \rightarrow \mathbf{R}$ is differentiable trigonometrically ρ -convex function, then the supporting function for $f(x)$ at the point $u \in I$ has the form

$$T_u(x) = f(u) \cos \rho(x - u) + \frac{f'(u)}{\rho} \sin \rho(x - u), \quad \forall x \in I. \quad (7)$$

Remark 5. (Ali [1]) For a trigonometrically ρ -convex function $f : I \rightarrow \mathbf{R}$, if $f(x)$ is not differentiable at the point u , then the supporting function has the form

$$T_u(x) = f(u) \cos \rho(x - u) + K_{u,f} \sin \rho(x - u), \quad \forall x \in I, \quad (8)$$

where

$$K_{u,f} \in \left[\frac{f'_-(u)}{\rho}, \frac{f'_+(u)}{\rho} \right]. \quad (9)$$

Theorem 6. (Maergoiz [8]) A trigonometrically ρ -convex function $f : I \rightarrow \mathbf{R}$ has finite right and left derivatives $f'_+(x)$, $f'_-(x)$ at every point $x \in I$ and $f'_-(x) \leq f'_+(x)$.

Theorem 7. (Levin [7]) Let $f : I \rightarrow \mathbf{R}$ be a two times continuously differentiable function. Then f is trigonometrically ρ -convex on I if and only if

$$f''(x) + \rho^2 f(x) \geq 0, \quad \forall x \in I.$$

3. Main Results

The aim of section is to establish various Pólya and Steffensen type integral inequalities for trigonometrically ρ -convex functions. This section splits into two subsections. The goal of the first one is to introduce Pólya integral inequalities for trigonometrically ρ -convex and drive an inequality, which like to Hermite-Hadamard's inequality for trigonometrically ρ -convex Ali [2]. Whereas, the purpose of the second is to give Steffensen integral inequalities for trigonometrically ρ -convex. In addition to obtain that integral inequalities for product n^{th} trigonometrically ρ -convex functions.

3.1. Pólya integral inequalities

Theorem 8. *Let $f : [a, b] \rightarrow \mathbf{R}$ be differentiable trigonometrically ρ -convex function such that $0 < b - a < \frac{\pi}{\rho}$ and not identically constant on $[a, b]$ with $f(a) = f(b) = 0$ and*

$$m = \inf_{x \in [a, b]} |f'(x)|.$$

Then

$$\int_a^b f(x)dx \geq \frac{4m}{\rho^2} \sin^2 \rho \left(\frac{b-a}{4} \right). \quad (10)$$

Proof. Since $f(x)$ is trigonometrically ρ -convex function, then from Definition 2 we have

$$T_u(x) \leq f(x), \quad \forall x \in [a, b], \quad (11)$$

where $T_u(x)$ is the supporting function for $f(x)$ at the point $u \in [a, b]$. Consequently,

$$\begin{aligned} \int_a^b f(x)dx &\geq \int_a^b T_u(x)dx \\ &= \int_a^{\frac{a+b}{2}} T_a(x)dx + \int_{\frac{a+b}{2}}^b T_b(x)dx. \end{aligned} \quad (12)$$

As $f(x)$ is differentiable, $f(a) = f(b) = 0$ and by using Proposition 4, then the supporting functions $T_a(x)$ and $T_b(x)$ for $f(x)$ at the points $a \in [a, \frac{a+b}{2}]$ and $b \in [\frac{a+b}{2}, b]$ have the forms

$$T_a(x) = \frac{f'(a)}{\rho} \sin \rho(x - a), \quad (13)$$

and

$$T_b(x) = \frac{f'(b)}{\rho} \sin \rho(x - b), \tag{14}$$

respectively. Substitute from (13) and (14) in (12) it follows that

$$\begin{aligned} \int_a^b f(x)dx &\geq \int_a^{\frac{a+b}{2}} \frac{f'(a)}{\rho} \sin \rho(x - a)dx \\ &\quad + \int_{\frac{a+b}{2}}^b \frac{f'(b)}{\rho} \sin \rho(x - b)dx \\ &= \frac{f'(a)}{\rho^2} [1 - \cos \rho(\frac{b-a}{2})] + \frac{-f'(b)}{\rho^2} [1 - \cos \rho(\frac{a-b}{2})] \\ &= \frac{2 \sin^2 \rho(\frac{b-a}{4})}{\rho^2} f'(a) + \frac{2 \sin^2 \rho(\frac{b-a}{4})}{\rho^2} (-f'(b)) \\ &\geq m \frac{2 \sin^2 \rho(\frac{b-a}{4})}{\rho^2} + m \frac{2 \sin^2 \rho(\frac{b-a}{4})}{\rho^2} \\ &= \frac{4m}{\rho^2} \sin^2 \rho(\frac{b-a}{4}). \end{aligned}$$

This completes the proof. □

Theorem 9. Let $f : [a, b] \rightarrow \mathbf{R}$ be trigonometrically ρ -convex function and not identically constant on $[a, b]$ with $f(a) = f(b) = 0$ and $0 < b - a < \frac{\pi}{\rho}$, then

$$\int_a^b f(x)dx \leq \frac{2}{\rho} f(\frac{a+b}{2}) \tan \rho(\frac{b-a}{4}). \tag{15}$$

Proof. As $f(x)$ is trigonometrically ρ -convex function, then by using Definition 1 and $f(a) = f(b) = 0$, we obtain

$$\begin{aligned} f(x) &\leq M(x; a, \frac{a+b}{2}, f) \\ &= \frac{f(\frac{a+b}{2})}{\sin \rho(\frac{b-a}{2})} \sin \rho(x - a), \quad \forall x \in [a, \frac{a+b}{2}]. \end{aligned} \tag{16}$$

And

$$\begin{aligned} f(x) &\leq M(x; \frac{a+b}{2}, b, f) \\ &= \frac{f(\frac{a+b}{2})}{\sin \rho(\frac{b-a}{2})} \sin \rho(b - x), \quad \forall x \in [\frac{a+b}{2}, b]. \end{aligned} \tag{17}$$

Consequently, if we integrate (16) over $x \in [a, \frac{a+b}{2}]$ and (17) over $x \in [\frac{a+b}{2}, b]$, we get

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^{\frac{a+b}{2}} f(x)dx + \int_{\frac{a+b}{2}}^b f(x)dx \\ &\leq \frac{f(\frac{a+b}{2})}{\sin \rho(\frac{b-a}{2})} \int_a^{\frac{a+b}{2}} \sin \rho(x-a)dx \\ &\quad + \frac{f(\frac{a+b}{2})}{\sin \rho(\frac{b-a}{2})} \int_{\frac{a+b}{2}}^b \sin \rho(b-x)dx, \\ &= \frac{f(\frac{a+b}{2})}{\rho \sin \rho(\frac{b-a}{2})} [1 - \cos \rho(\frac{b-a}{2})] \\ &\quad + \frac{f(\frac{a+b}{2})}{\rho \sin \rho(\frac{b-a}{2})} [1 - \cos \rho(\frac{b-a}{2})] \\ &= \frac{1}{\rho} f(\frac{a+b}{2}) \tan \rho(\frac{b-a}{4}) + \frac{1}{\rho} f(\frac{a+b}{2}) \tan \rho(\frac{b-a}{4}) \\ &= \frac{2}{\rho} f(\frac{a+b}{2}) \tan \rho(\frac{b-a}{4}). \end{aligned}$$

Then, the theorem is proved. □

Theorem 10. Assume that $f : [a, b] \rightarrow \mathbf{R}$ be trigonometrically ρ -convex function such that $0 < b - a < \frac{\pi}{\rho}$ and not identically constant on $[a, b]$ with $f(a) = f(b) = 0$ and

$$w = \inf_{x \in [a, b]} |K_{x, f}|.$$

Then,

$$\frac{4w}{\rho} \sin^2 \rho(\frac{b-a}{4}) \leq \int_a^b f(x)dx \leq \frac{2}{\rho} f(\frac{a+b}{2}) \tan \rho(\frac{b-a}{4}), \tag{18}$$

where $K_{x, f} \in [\frac{f'_-(x)}{\rho}, \frac{f'_+(x)}{\rho}]$.

Proof. From Theorems 8, 9 and Remark 5 we deduced (18). □

The inequality (18) is similar to

$$\frac{2}{\rho} f(\frac{a+b}{2}) \sin \rho(\frac{b-a}{2}) \leq \int_a^b f(x)dx \leq \frac{1}{\rho} [f(a) + f(b)] \tan \rho(\frac{b-a}{2}).$$

This relation is in Ali [2] as Hermite-Hadamard's inequality for trigonometrically ρ -convex function, which holds for trigonometrically ρ -convex function $f : I \rightarrow \mathbf{R}$ and $a, b \in I$ with $a < b$ such that $0 < b - a < \frac{\pi}{\rho}$.

3.2. Steffensen integral inequalities

Theorem 11. Assume that $f : [a, b] \rightarrow \mathbf{R}$ is non-negative trigonometrically ρ -convex function where $0 < b - a \leq \frac{\pi}{2\rho}$, with $f(a) = 0$ and $g : [a, b] \rightarrow [0, \infty)$ is continuous non-increasing function. Then

$$\int_a^b f(x)g(x)dx \leq \frac{f(b)}{\sin \rho(b-a)} \int_a^{a+\lambda} g(x)dx, \quad (19)$$

where

$$\lambda = \int_a^b \sin \rho(x-a)dx.$$

Proof. Since $f(x)$ is trigonometrically ρ -convex function and $f(a) = 0$, then from Definition 1 the ρ -trigonometric function $M(x; a, b, f)$ can be written in the form

$$M(x; a, b, f) = \frac{f(b)}{\sin \rho(b-a)} \sin \rho(x-a). \quad (20)$$

Consequently, from (5) we have

$$f(x) \leq \frac{f(b)}{\sin \rho(b-a)} \sin \rho(x-a), \quad \forall x \in [a, b]. \quad (21)$$

As $g(x) \geq 0$, then from (21), we get:

$$\int_a^b f(x)g(x)dx \leq \frac{f(b)}{\sin \rho(b-a)} \int_a^b \sin \rho(x-a)g(x)dx. \quad (22)$$

Since,

$$0 < b - a \leq \frac{\pi}{2\rho},$$

then

$$0 < \sin \rho(x-a) \leq 1 \quad \forall x \in [a, b].$$

By integrating the above inequality over x , we get

$$0 < \int_a^b \sin \rho(x-a)dx \leq (b-a). \quad (23)$$

Consequently, we have

$$0 < \lambda \leq b - a. \tag{24}$$

Assume that

$$h(x) = \int_a^{a+\lambda} g(x)dx - \int_a^b \sin \rho(x - a)g(x)dx. \tag{25}$$

Now, we prove that $h(x) \geq 0$. From (24), we observe that

$$\begin{aligned} h(x) &= \int_a^{a+\lambda} g(x)dx - \int_a^{a+\lambda} \sin \rho(x - a)g(x)dx \\ &\quad - \int_{a+\lambda}^b \sin \rho(x - a)g(x)dx \\ &= \int_a^{a+\lambda} (1 - \sin \rho(x - a))g(x)dx - \int_{a+\lambda}^b \sin \rho(x - a)g(x)dx. \end{aligned}$$

In fact that $g(x)$ is non-increasing implies $g(x) \geq g(a + \lambda)$ for all $x \in [a, a + \lambda]$, then we get

$$\begin{aligned} h(x) &\geq g(a + \lambda) \int_a^{a+\lambda} (1 - \sin \rho(x - a))dx \\ &\quad - \int_{a+\lambda}^b \sin \rho(x - a)g(x)dx \\ &= g(a + \lambda)(\lambda - \int_a^{a+\lambda} \sin \rho(x - a)dx) \\ &\quad - \int_{a+\lambda}^b \sin \rho(x - a)g(x)dx \\ &= g(a + \lambda) \int_{a+\lambda}^b \sin \rho(x - a)dx - \int_{a+\lambda}^b \sin \rho(x - a)g(x)dx \\ &= \int_{a+\lambda}^b (g(a + \lambda) - g(x)) \sin \rho(x - a)dx. \end{aligned} \tag{26}$$

Using the fact that $g(x)$ is non-increasing implies $g(x) \leq g(a + \lambda)$ for all $x \in [a + \lambda, b]$. Hence, we get

$$\int_{a+\lambda}^b (g(a + \lambda) - g(x)) \sin \rho(x - a)dx \geq 0,$$

thus, from (26) implies

$$h(x) \geq 0, \tag{27}$$

Using (25) and (27) one obtains

$$\int_a^b \sin \rho(x-a)g(x)dx \leq \int_a^{a+\lambda} g(x)dx. \quad (28)$$

As $f(x) \geq 0$, then from (22) and multiplying the inequality (28) by $\frac{f(b)}{\sin \rho(b-a)} \geq 0$ we conclude that

$$\begin{aligned} \int_a^b f(x)g(x)dx &\leq \frac{f(b)}{\sin \rho(b-a)} \int_a^b \sin \rho(x-a)g(x)dx \\ &\leq \frac{f(b)}{\sin \rho(b-a)} \int_a^{a+\lambda} g(x)dx. \end{aligned}$$

Therefore, the theorem is hold. \square

Theorem 12. Assume that $f : [a, b] \rightarrow \mathbf{R}$ is trigonometrically ρ -convex function, where $0 < b-a \leq \frac{\pi}{2\rho}$ with $f(a) = 0$ and $f'_-(a) \geq 0$ and if $g : [a, b] \rightarrow [0, \infty)$ is continuous non-increasing function. Then

$$K_{a,f} \int_{b-\lambda}^b g(x)dx \leq \int_a^b f(x)g(x)dx, \quad (29)$$

where

$$\lambda = \int_a^b \sin \rho(x-a)dx.$$

Proof. By utilising Remark 5 and $f'_-(a) \geq 0$, then from (9) we get

$$K_{a,f} \geq 0. \quad (30)$$

As $f(x)$ is trigonometrically ρ -convex function, then by using Definition 2 we have

$$T_a(x) \leq f(x), \quad \forall x \in [a, b]. \quad (31)$$

Since $f(a) = 0$, then by using Remark 5 the supporting function $T_a(x)$ for $f(x)$ at the point $a \in [a, b]$ can be written in the form

$$T_a(x) = K_{a,f} \sin \rho(x-a). \quad (32)$$

Using (31) and (32), one obtains

$$K_{a,f} \sin \rho(x-a) \leq f(x), \quad \forall x \in [a, b]. \quad (33)$$

We have $g(x)$ is non-negative, then from (33) we get that

$$K_{a,f} \int_a^b \sin \rho(x - a)g(x)dx \leq \int_a^b f(x)g(x)dx. \tag{34}$$

Let

$$\psi(x) = \int_a^b \sin \rho(x - a)g(x)dx - \int_{b-\lambda}^b g(x)dx. \tag{35}$$

Thus, by (24) implies

$$\psi(x) = \int_a^{b-\lambda} \sin \rho(x - a)g(x)dx - \int_{b-\lambda}^b [1 - \sin \rho(x - a)]g(x)dx. \tag{36}$$

In fact, since $g(x)$ is non-increasing, then $-g(x) \geq -g(b - \lambda)$ for any $b - \lambda \leq x \leq b$. Hence, we get

$$\begin{aligned} \psi(x) &\geq \int_a^{b-\lambda} \sin \rho(x - a)g(x)dx \\ &\quad - g(b - \lambda) \int_{b-\lambda}^b [1 - \sin \rho(x - a)]dx \\ &= \int_a^{b-\lambda} \sin \rho(x - a)g(x)dx \\ &\quad - g(b - \lambda) \left[\lambda - \int_{b-\lambda}^b \sin \rho(x - a)dx \right] \\ &= \int_a^{b-\lambda} \sin \rho(x - a)g(x)dx - g(b - \lambda) \int_a^{b-\lambda} \sin \rho(x - a)dx \\ &= \int_a^{b-\lambda} [g(x) - g(b - \lambda)] \sin \rho(x - a)dx. \end{aligned}$$

Similarly, from the fact that $g(x)$ non-increasing implies

$g(x) \geq g(b - \lambda)$ for all $a \leq x \leq b - \lambda$, then we conclude that,

$$\psi(x) \geq 0.$$

Consequently, we get

$$\int_{b-\lambda}^b g(x)dx \leq \int_a^b \sin \rho(x - a)g(x)dx. \tag{37}$$

If we multiply (37) by $K_{a,f} \geq 0$ and from (34), then the desired result (29) holds, this completes proof. □

Theorem 13. Let $f_1(x), f_2(x), \dots, f_n(x)$ be trigonometrically ρ -convex functions, non-negative, non-increasing and defined in $a \leq x \leq b$ where $0 < b - a \leq \frac{\pi}{2\rho}$ for which

$$f_i(a) = 0, \quad i = 1, 2, 3, \dots, n$$

and let $g : [a, b] \rightarrow [0, \infty)$ be continuous and non-increasing function. Then

$$\begin{aligned} & \int_a^b f_1(x)f_2(x)\dots f_n(x)g(x)dx \\ & \leq \frac{f_1(b)}{\sin \rho(b - a)} \left[\prod_{r=1}^{n-1} \frac{f_{r+1}(a + \lambda_r)}{\sin \rho \lambda_r} \right] \int_a^{a+\lambda_n} g(x)dx, \end{aligned} \tag{38}$$

where,

$$\lambda_{r+1} = \int_a^{a+\lambda_r} \sin \rho(x - a), \quad r = 1, 2, \dots, n - 1 \tag{39}$$

$$\lambda_1 = \int_a^b \sin \rho(x - a)dx. \tag{40}$$

Proof. Since $f_2(x), f_3(x), \dots, f_n(x)$ and $g(x)$ are non-negative and non-increasing functions, then the function $f_2(x).f_3(x)\dots f_n(x).g(x)$ is non-negative and non-increasing. Therefore, from the fact that $f_1(x)$ is trigonometrically ρ -convex functions, non-negative and non-increasing with $f_1(a) = 0$ and by using Theorem 11 we get

$$\begin{aligned} \int_a^b f_1(x)f_2(x)\dots f_n(x)g(x)dx & \leq \frac{f_1(b)}{\sin \rho(b - a)} \\ & \times \int_a^{a+\lambda_1} f_2(x)\dots f_n(x)g(x)dx. \end{aligned}$$

Using (23) and (24), we have

$$0 < \lambda_1 \leq b - a.$$

Again, since $f_3(x), \dots, f_n(x)$ and $g(x)$ are non-negative and non-increasing functions, $f_2(x)$ is trigonometrically ρ -convex functions, non-negative and non-increasing with $f_2(a) = 0$ and by using Theorem 11, we get

$$\int_a^{a+\lambda_1} f_2(x)\dots f_n(x)g(x)dx \leq \frac{f_2(a + \lambda_1)}{\sin \rho \lambda_1} \int_a^{a+\lambda_2} f_3(x)\dots g(x)dx,$$

where

$$0 < \lambda_2 \leq \lambda_1.$$

Repeating the above argument and using Theorem 11 each time, then we obtain

$$\begin{aligned} \int_a^b f_1(x)f_2(x)\dots f_n(x)g(x)dx &\leq \frac{f_1(b)f_2(a + \lambda_1)\dots f_n(a + \lambda_{n-1})}{\sin \rho(b - a) \sin \rho\lambda_1 \dots \sin \rho\lambda_{n-1}} \\ &\quad \times \int_a^{a+\lambda_n} g(x)dx \\ &= \frac{f_1(b)}{\sin \rho(b - a)} \left[\prod_{r=1}^{n-1} \frac{f_{r+1}(a + \lambda_r)}{\sin \rho\lambda_r} \right] \\ &\quad \times \int_a^{a+\lambda_n} g(x)dx, \end{aligned}$$

where

$$0 < \lambda_n \leq \lambda_{n-1} \leq \dots \leq \lambda_1 \leq b - a.$$

□

Theorem 14. Let $f_1(x), f_2(x), \dots, f_n(x)$ be trigonometrically ρ -convex functions, non-negative, non-increasing and defined in $a \leq x \leq b$ where $0 < b - a \leq \frac{\pi}{2\rho}$ for which

$$f_i(a) = 0, \quad i = 1, 2, 3, \dots, n.$$

Then

$$\int_a^b f_1(x)f_2(x)\dots f_n(x)dx \leq \frac{2f_1(b) \sin^2 \rho(\frac{\lambda_{n-1}}{2})}{\rho \sin \rho(b - a)} \left[\prod_{r=1}^{n-1} \frac{f_{r+1}(a + \lambda_r)}{\sin \rho\lambda_r} \right], \quad (41)$$

where λ_r, λ_1 defined in (39) and (40).

Proof. Using Theorem 13 and from (38) we have by putting $g(x) = 1$ that

$$\begin{aligned} \int_a^b f_1(x)f_2(x)\dots f_n(x)dx &\leq \frac{f_1(b)}{\sin \rho(b - a)} \left[\prod_{r=1}^{n-1} \frac{f_{r+1}(a + \lambda_r)}{\sin \rho\lambda_r} \right] \\ &\quad \times \int_a^{a+\lambda_n} dx \\ &= \frac{f_1(b)}{\sin \rho(b - a)} \end{aligned}$$

$$\times \left[\prod_{r=1}^{n-1} \frac{f_{r+1}(a + \lambda_r)}{\sin \rho \lambda_r} \right] \lambda_n. \tag{42}$$

Now, if we take $r = n - 1$ in (39), then we get

$$\begin{aligned} \lambda_n &= \int_a^{a+\lambda_{n-1}} \sin \rho(x - a) dx \\ &= \frac{1}{\rho} [-\cos \rho(x - a)]_a^{a+\lambda_{n-1}} \\ &= \frac{1}{\rho} [1 - \cos \rho \lambda_{n-1}] \\ &= \frac{2}{\rho} \sin^2 \rho \frac{\lambda_{n-1}}{2}. \end{aligned} \tag{43}$$

From (42) and (43) we get the inequality (41). □

Theorem 15. *Let $f_1(x), f_2(x), \dots, f_n(x)$ be trigonometrically ρ -convex functions, non-negative, non-increasing and defined in $a \leq x \leq b$, where $0 < b - a \leq \frac{\pi}{2\rho}$ for which*

$$\begin{aligned} f_1(a) = 0, \quad f_{r+1}(b - \lambda_r) = 0, \quad \forall r = 1, 2, 3, \dots, n - 1, \\ f'_1(a) \geq 0, \quad f'_{r+1-}(b - \lambda_r) \geq 0, \quad \forall r = 1, 2, 3, \dots, n - 1, \end{aligned} \tag{44}$$

and let $g : [a, b] \rightarrow [0, \infty)$ be continuous and non-increasing function. Then

$$\int_a^b f_1(x) f_2(x) \dots f_n(x) g(x) dx \geq K_{a, f_1} \left[\prod_{r=1}^{n-1} K_{b-\lambda_r, f_{r+1}} \right] \int_{b-\lambda_n}^b g(x) dx, \tag{45}$$

where

$$\lambda_{r+1} = \int_{b-\lambda_r}^b \sin \rho(x - (b - \lambda_r)) dx, \quad r = 1, 2, \dots, n - 1, \tag{46}$$

$$(i.e.) \lambda_1 = \int_a^b \sin \rho(x - a) dx. \tag{47}$$

Proof. By using (9) and (44), then we get

$$K_{a, f_1} \geq 0 \tag{48}$$

$$K_{b-\lambda_r, f_{r+1}} \geq 0 \quad \forall r = 1, 2, \dots, n - 1. \tag{49}$$

Since $f_2(x), f_3(x), \dots, f_n(x)$ and $g(x)$ are non-negative and non increasing functions, then the function $f_2(x).f_3(x)\dots f_n(x).g(x)$ is non-negative and non-increasing. Therefore, using that $f_1(x)$ is trigonometrically ρ -convex functions, non-negative and non-increasing with $f_1(a) = 0$, $f'_{1-}(a) \geq 0$ and by using Theorem 12 we conclude that

$$\int_a^b f_1(x)f_2(x)\dots f_n(x)g(x)dx \geq K_{a,f_1} \int_{b-\lambda_1}^b f_2(x)\dots f_n(x)g(x)dx,$$

where, from (23) and (24)

$$0 < \lambda_1 \leq b - a.$$

Again, since $f_3(x), \dots, f_n(x)$ and $g(x)$ are non-negative and non-increasing functions, $f_2(x)$ is trigonometrically ρ -convex functions, non-negative and non-increasing with $f_2(b - \lambda_1) = 0$, $f'_{2-}(b - \lambda_1) \geq 0$ and by using Theorem 12, then

$$\int_{b-\lambda_1}^b f_2(x)\dots f_n(x)g(x)dx \geq K_{b-\lambda_1,f_2} \int_{b-\lambda_2}^b f_3(x)\dots f_n(x)g(x)dx,$$

where

$$0 < \lambda_2 \leq \lambda_1.$$

Repeating the above argument and using Theorem 12 each time, then, from (48) and (49). We have

$$\begin{aligned} \int_a^b f_1(x)f_2(x)\dots f_n(x)g(x)dx &\geq K_{a,f_1}K_{b-\lambda_1,f_2}\dots K_{b-\lambda_{n-1},f_n} \\ &\quad \times \int_{b-\lambda_n}^b g(x)dx \\ &= K_{a,f_1} \left[\prod_{r=1}^{n-1} K_{b-\lambda_r,f_{r+1}} \right] \int_{b-\lambda_n}^b g(x)dx, \end{aligned}$$

where

$$0 < \lambda_n \leq \lambda_{n-1} \leq \dots \leq \lambda_1 \leq b - a.$$

□

References

- [1] M.S.S. Ali, On certain properties of trigonometrically ρ -convex functions, *J. of Advances in Pure Mathematics*, **8**, No 2 (2012), 337-340.

- [2] M.S.S. Ali, On Hadamard's inequality for trigonometrically ρ -convex functions, *RGMI A Res. Rep. Coll.* (2018), Art. # 8, 7 pp.; <http://rgmia.org/papers/v21/v21a08.pdf>.
- [3] F.G. Avhadief and D.V. Maklakkov, A theory of pressure envelopes for hydrofoils, *J. of Ship Technology Research*, **42** (1995), 81-102.
- [4] E.F. Beckenbach, Convex functions, *Bull. of the Amer. Math. Soc.*, **54**, No 5 (1948), 439-460.
- [5] J.W. Green, Support, convergence, and differentiability properties of generalized convex functions, *Proc. of the Amer. Math. Soc.*, **4**, No 3 (1953), 391-39.
- [6] G. Klambauer, *Problems and Propositions in Analysis*, Lecture Notes in Pure and Appl. Math. 49, Marcel Dekker, New York and Basel (1979).
- [7] B.Ya. Levin, *Lectures on Entire Functions*, American Mathematical Society (1996).
- [8] L.S. Maergoiz, *Asymptotic Characteristics of Entire Functions and Their Applications in Mathematics and Biophysics*, Kluwer Acad. Publishers, New York (2003).
- [9] D.S. Mitrinović and I.B. Lacković, Hermite and convexity, *Aequationes Mathematicae*, **28**, No 3 (1985), 229-232.
- [10] C.P. Niculescu and L.E. Person, *Convex Functions and Their Applications A Contemporary Approach*, Springer Science+Business Media, Inc. (2006).
- [11] G. Pólya, Ein mittelwertsatz für Funktionen mehrerer Veränderlichen, *Tôhoku Math. J.*, **19** (1921), 1-3.
- [12] G. Pólya and G. Szegő, *Problems and Theorems in Analysis*, Vol. I, Classics in Mathematics, Springer-Verlag, Berlin (1998).
- [13] F. Qi, Pólya type integral inequalities: Origin, variants, proofs, refinements, generalizations, equivalences, and applications, *RGMI A, Res. Rep. Coll.*, **16** (2013), Article # 20, 32 pp.; at <http://rgmia.org/v16.php>.
- [14] A.W. Roberts and D.E. Varberg, *Convex Functions*, Academic Press, New York (1973).

- [15] J.F. Steffensen, On certain inequalities and methods of approximation, *Candinavisk Aktuarietids Krift*, **51**, No 3 (1919), 274-297.

