International Journal of Applied Mathematics

Volume 31 No. 5 2018, 611-620
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v31i5.7

THE EXISTENCE OF A SOLUTION FOR SEMI-LINEAR ABSTRACT DIFFERENTIAL EQUATIONS WITH INFINITE $B-$CHAINS OF THE CHARACTERISTIC SHEAF

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Abstract: Initial-value problems for a semi-linear differential operator equations with singular linear part are considered. The existence of the infinite $B-$chains for the characteristic sheaf $\lambda A + B$ is assumed. In this case conditions for solvability have been obtained. The results are applied to a mixed problem for a partial differential equation.

AMS Subject Classification: 34G20
Key Words: infinite B-chains, singular operator sheaf, nonlinear differential equation, Frechet derivative

1. Introduction

We consider the initial-value problem for the semi-linear abstract differential equation

$$\frac{d(Au(t))}{dt} + Bu(t) = f(t,u), \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where A, B are closed linear operators from a Hilbert X space into a Hilbert space Y. In general the operator A is not invertible. Semi-linear equations

Received: June 15, 2018

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(1) with not invertible operator at the derivative in Banach spaces have been studied in works [1], [3], [4], [5]. The properties of the solution of initial-value problem (1), (2) are related to the properties of the characteristic $\lambda A+B$ sheaf of the equation's linear part (1). The operator sheaf $\lambda A+B$ is called regular when there is a complex number $\lambda_0 \in \mathbb{C}$, for which the resolvent $(\lambda_0 A+B)^{-1}$is defined and is bounded on the subspace $D_A \cap D_B$. Otherwise, the sheaf $\lambda A+B$ is called singular.

The works [3] and [4] are based on the assumption that the characteristic sheaf $\lambda A+B$ is regular. In [4] the case of a singular characteristic sheaf $\lambda A+B$ is considered.

Definition 1.1. A pair of subspaces $(X^{\acute{}}, Y^{\acute{}})$ is called an invariant relative to the sheaf $\lambda A+B$, if

$$A(D_A \cap X^{\acute{}}) \subset Y^{\acute{}}, B(D_B \cap X^{\acute{}}) \subset Y^{\acute{}}$$

and if at least one of the subspaces $X^{\acute{}}, Y^{\acute{}}$ is non-trivial.

Pairs of subspaces $(X_1, Y_1), (X_2, Y_2)$ reduce the sheaf $\lambda A+B$, if,

$$X = X_1 \dot{+} X_2, Y = Y_1 \dot{+} Y_2, A(D_A \cap X_i) \subset Y_i, B(D_B \cap X_i) \subset Y_i,\ i = 1, 2.$$

In [5] the existence of a singular and a regular pair of reduced subspaces $(X_s, Y_s), (X_r, Y_r)$ is assumed:

$$\begin{aligned} X = X_S \dot{+} X_R\ ,Y &= Y_S \dot{+} Y_R, \\ D_A &= (D_A \cap X_i) \dot{+} (D_A \cap X_R), \\ D_B &= (D_B \cap X_S) \dot{+} (D_B \cap X_R), \\ A(D_A \cap X_s) \subset Y_s\ \ &,\ \ B(D_B \cap X_s) \subset Y_s, \\ A(D_A \cap X_r) \subset Y_r\ \ &,\ \ B(D_B \cap X_r) \subset Y_r. \end{aligned} \tag{3}$$

The sheaf of operators $\lambda A_s + B_s\ = \lambda A + B\ |_{X_S}$induced from the subspace X_S into the subspace Y_S is singular; the sheaf $\lambda A_R + B_R : X_R \to Y_R$ induced from the subspaceX_R into the subspace Y_R is regular.

In [5] it was assumed that the structure of the singular component of the sheaf $\lambda A_S + B_S$ corresponds to a canonical form of the singular matrix sheaf according to L. Kroneker. Correspondingly, a canonical basis in the subspace X_S exists, which consists of the L. Kroneker finite singular chains.

In [6] the infinite chains for the pairs of operators A, B are introduced in connection with some problems of the perturbation theory for linear operator equations.

Definition 2.1. A sequence of vectors $\{x_i\}_{i=1}^{\infty} \subset D_A \cap D_B$ is called infinite B-chains for the sheaf $\lambda A+B$ if the vectors $\{x_i\}_{i=1}^{\infty}$ satisfy the following recurrent correlations

$$Ax_1 = 0, \quad Bx_i = Ax_{i+1}, \quad i = 1, 2, 3, ..., \tag{4}$$

and if $\{Bx_i\}_{i=1}^{\infty}$are linearly independent vectors.

In this work the initial-value problem (1), (2) in the case of the singular characteristic sheaf $\lambda A + B$ with the infinite $B-$chains will be considered.

2. Example. The Initial-Value Problem for a Partial Derivative Equation with Infinite $B-$ Chains of the Characteristic Sheaf $\lambda A + B$

In the domain $t \geq 0$, -$\pi \leq x \leq \pi$ we shall consider a mixed problem

$$e^{-ix}\frac{\partial^2 u(t,x)}{\partial t \partial x} - ie^{-ix}\frac{\partial u(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x} = f(t,u), \tag{5}$$

$$u(t,-\pi) = u(t,\pi), \quad u(0,x) = u_0(x). \tag{6}$$

We shall assume that $f(t,u) : [0,\tau] \times C \to C$ is a continuous complex function.

Let us introduce the following operators in the space $C_{[-\pi.\pi]} = X = Y$:

$$\begin{aligned} &A = e^{-ix}\frac{d(.)}{dx} - ie^{-ix} \quad , \quad B = \frac{d(.)}{dx}, \\ &D_A = D_B = D = \{y \in C^1_{[-\pi,\pi]} : y(t,-\pi) = y(t,\pi)\}. \end{aligned} \tag{7}$$

Considering function $u(t,x)$ as a mapping $u(t) : [0,\tau] \to C^1_{[-\pi,\pi]}$, let us represent the initial-value problem (5), (6) as an equivalent to the Cauchy abstract problem (1), (2) in the spaces $C^1_{[-\pi,\pi]} = X = Y$.

The characteristic sheaf $\lambda A + B$ is singular in fact, the equation

$$(\lambda A + B)y = 0$$

can be rewritten in the form

$$\left(\left(\lambda e^{-ix} + 1\right) y\right)' = 0. \tag{8}$$

Clearly, in the case $|\lambda| \neq 1$ the equation (8) has a non-trivial solution - the function

$$y_0(x,\lambda) = (\lambda e^{-ix} + 1)^{-1}; \tag{9}$$

while in the case $|\lambda| = 1$, the coefficient by the derivative of the differential operator $\lambda A + B$ is irreversible.

The sequence of exponents $\left\{e^{ikx}\right\}_{k=1}^{\infty}$forms infinite $B-$chains for the sheaf $\lambda A + B$.

In fact, the exponents which are in the domain of the sheaf $\lambda A + B$: $e^{ikx} \in C^1_{[-\pi,\pi]}$, $e^{-ikx} = e^{ikx}$ satisfies the correlation (4):

$$A(e^{ix}) = 0, B(e^{ix}) = ie^{ix} = A(e^{i2x}), ..., B((e^{ikx}) = kie^{ikx} = A((e^{i(k+1)x}), ...;$$

the sequence of functions $\{Bx_i\}_{i=1}^{\infty} = \left\{e^{ikx}\right\}_{k=1}^{\infty}$ forms the system of linearly independent functions.

In the next section the initial-value problem (1), (2) in a general case of a characteristic sheaf with the infinite $B-$chains will be considered.

3. The Theorem of Existence of a Solution of the Initial-Value Problem (1), (2) with the Infinite $B-$Chains for a Characteristic Sheaf

The initial-value problem (1), (2) in the Hilbert spaces X, Y is considered in this work. We assume that the decompositions (3), $D_A \subset D_B$, hold true; the regular component of the sheaf $\lambda A_R + B_R$ has the bounded inverse operator $A_R^{-1} \in L\left(Y_R, X_R\right)$,

Hence, $B_R A_R^{-1} \in L\left(Y_R\right)$.

Furthermore, let us assume that the following orthogonal decompositions hold true for the singular pair of the subspaces (X_S, Y_S) (3)

$$X_S = \ker A \cap \ker B \oplus X_1, \quad , \quad Y_S = \ker A^* \cap \ker B^* \oplus Y_1, \tag{10}$$

where the subspace X_1 is the closing of the linear span of the finite set of the infinite $B-$chains, $Y_1 = B(X_1)$.

The first vector for each of the $B-$chains belongs to the subspace $\ker A$.

We shall denote the closing of the linear span of the other vectors by X_2.

From $A_R^{-1} \in L\left(Y_R, X_R\right)$ follows the decomposition $X_S = \ker A \dotplus X_2$.

We introduce two pairs of the mutually complementary projectors

$$\begin{aligned} K &: X \to \ker A, \qquad P = (1 - K) : X \to X_2 \dotplus X_R, K^2 = K; \\ L &: Y \to \overline{\mathrm{Im} A}, \qquad Q : Y \to \ker A^*, \;\; L^2 = L. \end{aligned}$$

Definition 3.1. The continuous vector-function $u(t) \in C\left([0, \tau_0], X\right)$ with the values from $D_A \cap D_B$ is called the solution of the equation (1) in the range $0 \le t \le \tau_0$, if $Au(t) \in C^1\left([0, \tau_0], Y\right)$ and the function $u(t)$ satisfies the equation (1) for all $t \in [0, \tau_0]$.

Let us designate a closed sphere in the Hilbert space X with the center $x_0 \in X$ as $B(X, x_0, r) = \{x \in X : \|x - x_0\| \le r\}$.

Theorem 3.1. *Let the decompositions (3),(10) hold true for the equation (1). Let* $\dim(\ker A^* \cap \ker B^*) \le \dim\ker A$ *and vector* $u_0 \in D_A$ *satisfies the condition* $Qf(0,u_0) = 0$.

Assume that the function $f(t,u)$ *is continuous in the set* $[0,\tau]\times B(X,u_0,r)$, *the component* Lf *of the vector-function* $f(t,u)$ *satisfies the Lipschitz condition*

$$\left\|Lf(t,u^1) - Lf(t,u^2)\right\| \le a\left\|u^1 - u^2\right\|, \quad \forall u^1,u^2 \in B(X,u_0,r), \quad t\in[0,\tau], \tag{11}$$

and the component Qf *has a continuous Frechet derivative* $\dfrac{\partial Qf(t,u)}{\partial u}$.

If

$$\operatorname{rang}\frac{\partial}{\partial Ku}Qf(0,u)\mid_{u=u_0} = \dim(\ker A^*\cap\ker B^*),$$

then the problem (1),(2) has the solution $u(t)$ *on the non-trivial interval* $[0,\tau_0]$. *Moreover the solution is unique if*

$$\dim\ker A = \dim(\ker A^*\cap\ker B^*).$$

Proof. Let us denote

$$u_K = Ku, u_p = Pu, A_L = LA, B_L = LB.$$

From the construction projector of K follows that $K(X) = \ker A$.

This means that, $AK = 0$.

From the other side of the definition of infinite $B-$chains it follows that $B(\ker A) \subset \mathrm{Im}A$.

Hence $QBK = 0$.

Therefore, equation (1) is equivalent to the following two equations

$$Qf(t,u_k+u_p) = 0, \tag{12}$$

$$\frac{d(A_Lu_p(t))}{dt} + B_Lu_p(t) = Lf(t,u_k+u_p) - B_Lu_k(t). \tag{13}$$

From the conditions of Theorem 3.1 it follows that there exist subspace $X_{10}\subset \ker A$ and the projector $K_1 : X \to X_{10}$ such that $\dim X_{10} = \dim Y_1$, $K_1K = K$, and the linear operator $\dfrac{\partial}{\partial K_1u}Qf(0,u_0) : X \to X_{10}$ is invertible.

Let us denote $K_2 = K - K_1$, $X_{11} = K_2(X)$, $u_{K_1} = K_1u_K$, $u_{K_2} = K_2u_K$. If we substitute $u_K = u_{K_1} + u_{K_2}$ in the equation (12) and apply the theorem of implicit function [2], we get the function

$$u_{K_1} = \Psi(t,u_{K_2},u_p).$$

This function has a continuous Frechet derivative $\dfrac{\partial\Psi}{\partial u_p}$ and defined on the set

$$\Omega = \{(t,u_{K_2},u_p)\} = [0,\tau_1]\times B(X_{11}\dotplus X_2\dotplus X_R,(K_2+P)u_0,r_1), \quad r_{1\le}r, \tau_{1\le}\tau.$$

Now changing $u_p = A_L^{-1}\upsilon$ and substitute function $u_{K_1} = \Psi(t, u_{K_2}, u_p)$ in the equation (13) we obtain

$$\frac{d\upsilon}{dt} + B_L A_L^{-1} u_p(t) = (t, u_{K_2}, \nu) - B_L \Psi(t, u_{K_2}, \nu) - B_L u_{K_2}, \tag{14}$$

where

$$\begin{aligned} \varphi(t, u_{K_2}, \nu) &= Lf(t,u),\ \Psi(t, u_{K_2}, \nu) = \Psi(t, u_{K_2}, A_L^{-1}\upsilon), \\ u &= \Psi(t, u_{K_2}, A_L^{-1}\upsilon) + u_{K_2} + A_L^{-1}\upsilon. \end{aligned}$$

Let us introduce the set

$$G = [0, \tau_2] \times B(X_{11} \dotplus X_2 \dotplus X_R, (K_2 + P)u_0, r_1), \quad r_2 \le r, \tau_{2 \le} \tau_1.$$

From the conditions of Theorem 3.1 it follows that the function $\Psi(t, u_{K_2}, \nu)$ satisfies the Lipschitz condition on G. We apply the Picard theorem [6] to the equation (14) with the initial condition $\nu(0) = A_L u_p(0)$. We obtain a continuous differentiable solution $\upsilon(t, u_{K_2})$ on some set

$$\Phi = \{(t, u_{k_2})\} = [0, \tau_0] \times B(X_{11}, K_2 u_0, r_3).$$

The obtained function defines a continuous differentiable function $u_2(t) = A_L^{-1}\upsilon(t, u_{K_2})$, which is the component of the solution $u(t)$.

The component u_{K_2} is being chosen arbitrarily from the class $C([0, \tau_0]; B(X_{11}, K_2 u_0, r_3)$ with the initial condition $K_2 u(0) = K_2 u_0$.

Finally, the function $u = \Psi(t, u_{K_2}, u_p) + u_{K_2} + u_p$ is the solution of the problem (1), (2). □

In the mixed problem (5), (6) we consider the restriction of the operators A, B (7) in a Hilbert spaces $X' = Y' = L^2_{[-\pi,\pi]}$.

Introduce the following subspaces

$$X_S = \overline{\mathrm{Lin}\{e^{ikx}\}_{k=1}^{\infty}}, Y_S = \overline{\mathrm{Lin}\{e^{ikx}\}_{k=0}^{\infty}}, X_R = \overline{\mathrm{Lin}\{e^{ikx}\}_{k=0}^{\infty}},$$

$$Y_R = \overline{\mathrm{Lin}\{e^{ikx}\}_{k=1}^{\infty}}, \ker A = \{e^{ix}\}, X_1 = X_S = Y_1,$$

$$X_2 = \overline{\mathrm{Lin}\{e^{ikx}\}_{k=2}^{\infty}}, \ker A^* \cap \ker B^* = \mathrm{Lin}\{1\}.$$

It may be shown that for $|\lambda| > 1$ the operator $(\lambda A_R + B_R)$ is invertible bounded and defined on all subspace Y_R.

Note that for $|\lambda| > 1$ the function $y_0(x, \lambda)$ (9) is expanded into the following series:

$$y_0(x, \lambda) = \left(\lambda e^{-ix} + 1\right)^{-1} = \frac{1}{\lambda} e^{ix} \left(1 + \frac{1}{\lambda} e^{ix}\right)^{-1}$$

$$= \frac{1}{\lambda}e^{-ix} - \frac{1}{\lambda^2}e^{-i2x} + ...(-1)^{k-1}\frac{1}{\lambda^k}e^{-ikx} + ...,$$

and for $|\lambda| < 1$ the function $y_0(x, \lambda)$ (9) expanded as follows:

$$y_0(x, \lambda) = \left(\lambda e^{-ix} + 1\right)^{-1} = 1 - \lambda e^{-ix} + ...(-1)^k e^{-ikx} + ... \quad .$$

The sequence of functions $\{e^{ikx}\}_{k=0}^{\infty}$ forms the infinite $A-$chains [6]:

$$B(1) = 0, A(1) = -ie^{-ix} = B(e^{-ix}), ... - i(k+1)e^{-i(k+1)x} = B(e^{-i(k+1)x}), ...$$

Hence, if $|\lambda| < 1$ then the complex number λ is the singular point of the sheaf $\lambda A_R + B_R$.

We shall be obtain a representation for $A_R^{-1} \in L(Y_R, X_R)$. Let us show that for any function $f \in Y_R$ there exists a unique function $y \in X_R \cap Y_R$, so that $A_R y = f$.

Consider the equation $A_R y = f$:

$$e^{-ix}y' - e^{-ix}y = f.$$

This equation can be written in the form

$$\left(e^{-ix}y\right)' = f. \tag{15}$$

From (15) it follows that:

$$y = e^{ix}\int_{-\pi}^{\pi} f(s)ds + Ce^{ix}. \tag{16}$$

Let us prove that the function $Y(16)$ belong to the domain of the operator A.

Really, $y(-\pi) = -C$; on the other side, using $f \in Y_R$ we obtain:

$$y(\pi) = -\int_{-\pi}^{\pi} f(s)ds - C = -(f, 1) - C = -C.$$

We shall find the value C so that the function y (16) shall belongs to the subspace X_R.

Calculate the scalar product (y, e^{ix}):

$$(y, e^{ix}) = \int_{-\pi}^{\pi}\int_{-\pi}^{x} f(s)dsdx + C\int_{-\pi}^{\pi} dx = \int_{-\pi}^{\pi}\int_{-\pi}^{x} f(s)dsdx + 2\pi C.$$

Hence, for the function y (16) belongs to the subspace X_R, it is necessary that C has the form

$$C = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{x} f(s) ds dx.$$

Now we calculate the scalar product (y, e^{ix}) for $k > 1$, using the fact that $f \in Y_R$:

$$\begin{aligned} (y, e^{ikx}) &= \int_{-\pi}^{\pi} e^{-i(k-1)x} \int_{-\pi}^{\pi} f(s) ds dx + C \int_{-\pi}^{\pi} e^{-i(k-1)x} dx \\ &= -\frac{i}{k-1} e^{-i(k-1)x} \int_{-\pi}^{\pi} f(s) ds \,|_{-\pi}^{\pi} + \frac{i}{k-1} \int_{-\pi}^{\pi} e^{-i(k-1)x} f(x) dx \\ &= \frac{i}{k-1}((-1)^k (f, 1) + (f, e^{i(k-1)x})) = 0. \end{aligned}$$

Thus, the representation for the operator A_R^{-1} has the form:

$$A_R^{-1} y = e^{ix} \left(\int_{-\pi}^{\pi} f(s) ds + \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{x} f(s) ds dx \right).$$

Calculate the projectors Q and K:

$$Qy = \frac{1}{2\pi} \int_{-\pi}^{\pi} y dx, \qquad Ky = \frac{1}{2\pi} \int_{-\pi}^{\pi} y e^{-ix} dx . e^{ix}.$$

Then the conditions of Theorem 3.1 for the mixed problem (5), (6) have the form:

1) $u_0 \in D_A \Longleftrightarrow u_0(-\pi) = u_o(\pi)$;

2) $Qf(0, u_0) = 0 \Longleftrightarrow \int_{-\pi}^{\pi} f(0, u_0) dx = 0$;

3) $\frac{\partial f(t, u)}{\partial u}$ is continuous;

4) $\int_{-\pi}^{\pi} \frac{\partial f(0, u_0)}{\partial u} dx = 0.$

For example, $f(t, u) = u^2 - 1$.

Hence, there exists a single solution of the problem (5), (6) for any function $u_0(x)$ that satisfies the following conditions:

1. $u_0(-\pi) = u_0(\pi)$;
2. $\int\limits_{-\pi}^{\pi} u_0^2(x)dx = 2\pi$;
3. $\int\limits_{-\pi}^{\pi} u_0(x)dx \neq 0$.

For example, $u_0(x) = e^{ix} + 1$.

4. Conclusion

In this work conditions for solvability of the Cauchy problem for some singular differential operator equations are received. In our case the singular characteristic sheaf $\lambda A + B$ has only infinite $B-$chains. If the characteristic sheaf $\lambda A + B$ has as infinite $B-$chains as finite singular chains, then the conditions for solvability will be other form. This problem will be considered in the next work.

The results have been applied to the investigation of mixed problems for partial differential equations. These mixed problems may be received in the investigation of the waveguide.

Acknowledgements

This research is funded by the Deanship of Research in Zarqa University, Jordan.

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