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FINITE DIFFERENCE FORMULATION OF LIGHTHILL WHITHAM RICHARDS MACROSCOPIC MODEL FOR TRAFFIC FLOW PREDICTION

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Abstract: Mathematical modelling of traffic flow for accurate prediction of future traffic conditions plays a vital role in real time traffic control systems. Modelling of traffic at macroscopic level using Lighthill Whitham Richards (LWR) model is computationally less intensive than microscopic car following models. With advanced computing facilities, numerical solution of the LWR model is preferred than traditional analytical approaches. In the present study, an attempt has been made to solve numerically the LWR model using "Forward-Time Backward-Space (FTBS)" finite difference scheme. Actual traffic flow data collected from a roadway in Vellore were used to frame the initial and boundary conditions and for corroboration of the results. The results were promising as the predicted flow values by the FTBS scheme were matched closely with the actuals with mean absolute percentage error (MAPE) of 12.854. This showed the suitability of the proposed numerical scheme for traffic flow prediction.

AMS Subject Classification: 35L60, 74S20, 97M50

Key Words: numerical method; partial differential equations; traffic flow; mathematical modelling; LWR macroscopic model; finite difference method

1. Introduction

Traffic flow, the number of vehicles passing a particular point on a roadway in a

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given time interval is of great interest to traffic engineers since the appearance of traffic bottlenecks. Modelling of traffic flow help to predict the future traffic conditions which in turn help the traffic engineers to divert the traffic in critical areas and maximize the overall throughput of traffic along a given stretch of road. In general, the traffic flow models can be categorized into two types: microscopic and macroscopic. The microscopic traffic flow models concerned with simulating the single vehicle-driver units and analyze the microscopic properties like the position and velocity of each individual vehicle. The popular microscopic models are car-following models [1] and cellular automata models [2]. Car-following models mainly describe how vehicles follow one another on a roadway by maintaining sufficient time gap or distance. In cellular automata models, the roadway is considered as a string of cells which may be either empty if there are no vehicles or occupied by a vehicle. The problem with microscopic models is that they are computationally intensive as they deal with individual vehicles and sometimes it becomes complex to describe the interaction among the vehicles especially in heterogeneous traffic conditions as exists in India as there are wide variety of vehicles with varying static and dynamic characteristics. In such cases, it is desirable to use macroscopic traffic flow models as they are computationally less intensive as they work with aggregate variables and do not describe the traffic situation on the level of independent vehicles. Lighthill, Whitham [3] and Richards [4] proposed a simple continuum macroscopic traffic flow model based on the hydrodynamic theory of fluids to describe the traffic characteristics which is popularly called as LWR model. It involves a non-linear, first-order hyperbolic partial differential equation (PDE) in space and time which can be solved either by analytical [5, 6] or numerical methods [7, 8]. With high performance computing facilities now-a-days, numerical solution of the PDE has been reported in many studies [7, 8]. However the major drawback of the reported studies is that the corroboration of results were mostly done by simulation or using hypothetical data. Use of real world data were not attempted much. The present study aims to develop a model for real time prediction of traffic flow through numerical discretization of LWR macroscopic model. The finite difference scheme, namely, "Forward-Time Backward-Space (FTBS)" was used to solve the PDE with real world data as input and the results also were corroborated using the actual data from the field.

2. Field Data Collection and Extraction

The traffic flow data which can be used as initial and boundary conditions and for corroboration of the results was collected from one of the busy roads in Vellore city located in Tamil Nadu, India. The study stretch was located in Palar bridge which is about 700m in length separating Vellore and Katpadi town. The traffic data was collected by placing the video cameras over the tripod at the entry and exit points of the bridge and continuously recorded for a period of 100 minutes from 7.45 am to 9.25 am on a typical working day. As a pilot study, only two-wheelers were considered and the recorded videos were played to count the number of two-wheelers at each one minute interval at entry and exit. The first one minute flow at entry and exit was used to frame the initial condition. Flow at each one minute interval at the entry was treated as the boundary condition in the numerical scheme. The FTBS scheme predicts the exit flows which were then compared with the actual flow values obtained from the video to check the accuracy of the proposed prediction scheme.

3. Development of Prediction Scheme

In this section, the basic equations of the LWR model are first detailed followed by the numerical solution through finite difference formulation for solving the PDE.

3.1. Basic Equations of the LWR Model

The LWR model is a simple continuum equation for traffic flow, where an analogy is made between the vehicular flow and the flow of a compressible fluid. The two basic aspects of this model are (a) traffic flow is conserved, that is, the total number of vehicles is conserved; and (b) there is a relationship between speed and density (number of vehicles present in the section per unit length), or between flow and density. The conservation equation is given by

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0,\tag{1}$$

where k and q represent the traffic density and flow rate respectively and the independent variables x and t represent space and time respectively. The LWR model is usually accompanied by the fundamental traffic flow relationship given by

$$q = uk, (2)$$

where u is the space mean speed of vehicles travelling along the test bed under consideration. A speed-density relationship (i.e., an equation linking the space mean speed u and density k) is also needed to solve the above PDE. One of the simplest relationships relating speed and density is Greenshield's linear model [9] which is given by

$$u = F(1 - \frac{k}{K}),\tag{3}$$

where F is the free flow speed and K is the jam density. Equations (1)-(3) form a set of basic equations describing the LWR model.

3.2. Finite Difference Formulation

The Equations (1) to (3) can be solved numerically using a finite difference formulation of the time (t) and space (x) derivatives. To do so, the solution domain in the x-t space is first discretized using a rectangular grid with grid spacing given by Δx and Δt in the x and t directions respectively as shown in Figure 1.

For ease of implementation, the grid spacings are chosen to be uniform throughout. As illustrated, the space domain (x) is divided into J number of intervals and time to N intervals depending on the duration of the analysis. Thus, $x_j = j.\Delta x$ for j = 0, 1, ..., J and $t_n = n$. Δt for n = 0, 1, ..., N. The intersections of the grid lines are called grid point, where the density and flow are denoted by k_j^n and q_j^n , respectively. That is, k_j^n and q_j^n are density and flow defined in space-time domain, where j represents space and n represents time.

The finite difference formulation used in the present study is a 'forward-time backward-space', or FTBS scheme. In this scheme, the time derivative is approximated using the current grid point and the corresponding grid point in the next time level (forward-time), while the space derivative is approximated using the current grid point and the corresponding grid point in the previous space step (backward space). Thus, Equations (1), (2) and (3) may be written in discretized form as

$$\frac{q_j^n - q_{j-1}^n}{\Delta x} + \frac{k_j^{n+1} - k_j^n}{\Delta t} = 0,$$
(4)

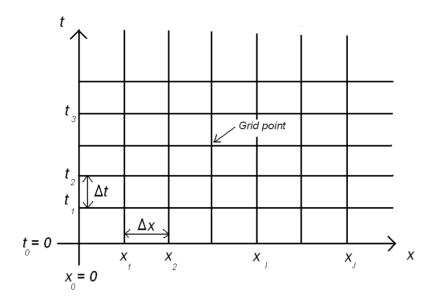


Figure 1: Space and Time Discretization

$$q_i^n = u_i^n k_i^n, (5)$$

$$u_j^n = F(1 - \frac{k_j^n}{K}). \tag{6}$$

Equation (5) after substituting for u_j^n from equation (6) results in,

$$q_j^n = k_j^n \cdot F - \frac{F}{K} k_j^n k_j^n. \tag{7}$$

Equation (4) after substituting for q_j^n from equation (7) results in,

$$\frac{k_{j}^{n}F}{\Delta x} - \frac{\frac{F}{K}k_{j}^{n}k_{j}^{n}}{\Delta x} - \frac{k_{j-1}^{n}F}{\Delta x} + \frac{\frac{F}{K}k_{j-1}^{n}k_{j-1}^{n}}{\Delta x} + \frac{k_{j}^{n+1}}{\Delta t} - \frac{k_{j}^{n}}{\Delta t} = 0,$$

$$\frac{k_{j}^{n}F}{\Delta x} - \frac{\frac{F}{K}k_{j}^{n}k_{j}^{n}}{\Delta x} - \frac{k_{j-1}^{n}F}{\Delta x} + \frac{\frac{F}{K}k_{j-1}^{n}k_{j-1}^{n}}{\Delta x} = \frac{k_{j}^{n}}{\Delta t} - \frac{k_{j}^{n+1}}{\Delta t},$$

$$\Delta t \left[\frac{k_j^n F}{\Delta x} - \frac{\frac{F}{K} k_j^n k_j^n}{\Delta x} - \frac{k_{j-1}^n F}{\Delta x} + \frac{\frac{F}{K} k_{j-1}^n k_{j-1}^n}{\Delta x} \right] = k_j^n - k_j^{n+1},$$

$$\frac{\Delta t}{\Delta x}k_j^n F - \frac{\Delta t}{\Delta x}\frac{F}{K}k_j^n k_j^n - \frac{\Delta t}{\Delta x}k_{j-1}^n F + \frac{\Delta t}{\Delta x}\frac{F}{K}k_{j-1}^n k_{j-1}^n - k_j^n = -k_j^{n+1},$$

Changing sign on both the sides results in,

$$k_{j}^{n+1} = k_{j}^{n} - \frac{\Delta t}{\Delta x} k_{j}^{n} F + \frac{\Delta t}{\Delta x} \frac{F}{K} k_{j}^{n} k_{j}^{n} + \frac{\Delta t}{\Delta x} k_{j-1}^{n} F - \frac{\Delta t}{\Delta x} \frac{F}{K} k_{j-1}^{n} k_{j-1}^{n},$$

$$k_{j}^{n+1} = k_{j}^{n} \left[1 - \frac{\Delta t}{\Delta x} F + \frac{\Delta t}{\Delta x} \frac{F}{K} k_{j}^{n}\right] + k_{j-1}^{n} \left[\frac{\Delta t}{\Delta x} F - \frac{\Delta t}{\Delta x} \frac{F}{K} k_{j-1}^{n}\right],$$

$$k_{j}^{n+1} = k_{j}^{n} \left[1 - \frac{\Delta t}{\Delta x} F \left(1 - \frac{k_{j}^{n}}{K}\right)\right] + k_{j-1}^{n} \left[\frac{\Delta t}{\Delta x} F \left(1 - \frac{k_{j-1}^{n}}{K}\right)\right]. \tag{8}$$

Using Equation (8), the density at various time intervals can be predicted based on the previous time interval data. Once this prediction is carried out for the entire discretized section, the flow at various grid points can be determined by using equation (7). The estimation of density at various time intervals using equation (8) requires some initial and boundary conditions to start. The details of finding this initial and boundary conditions using the field data from entry and exit is explained below.

The first one minute flow value at entry and exit were converted to density using equation (7) and were linearly interpolated to get the density at various discretized grid points. This forms the initial condition for solving the PDE. Similarly the flow values at entry at every one minute time interval were converted to density and were linearly interpolated for every time interval of Δt to provide the required boundary condition for the numerical scheme. The values of Δx , Δt , F and K in equation (8) were carefully chosen as 50 m, 2 sec, 53 km/hr and 680 vehicles/km respectively in order to ensure the stability of the numerical scheme.

4. Corroboration of the Prediction Scheme

The efficacy of the FTBS scheme for the prediction of traffic flow was tested using the field data and the results are presented here. The Mean Absolute Percentage Error (MAPE) is used as a measure of prediction accuracy and is calculated using

$$MAPE = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{Flow_{pred} - Flow_{obser}}{Flow_{obser}} \right) X100, \tag{9}$$

where, n is 100, the total number of one minute time intervals during the observation period. The results are shown in Figure 2.

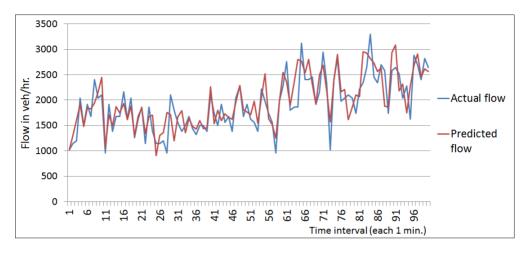


Figure 2: Plot of actual versus predicted flows at exit

It can be seen that the predicted flows matches closely with the observed flows with a MAPE of 12.854. In general, any forecast with a MAPE value of less than 10% can be considered highly accurate, 11% - 20% is good, 21% - 50% is reasonable and 51% or more is inaccurate. Based on this, it can be said that the results are good and thus the proposed FTBS scheme can be used to predict the traffic flow through space-time discretization of the LWR model.

5. Conclusion

Prediction of future traffic conditions is very essential for advanced traffic control systems to operate effectively. Traffic flow can be modelled either at macroscopic or microscopic level. Modelling of traffic at macroscopic level using models such as LWR is computationally faster than the microscopic simulation based approach. With advanced computing facilities, numerical solution of the LWR model is usually preferred than traditional analytical approaches. In the present study, an attempt has been made to solve numerically the LWR model using 'FTBS' finite difference formulation scheme. Actual traffic flow data from the field were used to frame the initial and boundary conditions and for corroboration of the results. The predicted flow values by the 'FTBS' scheme at the exit location were matched closely with the actuals and this showed the suitability of the proposed numerical scheme for traffic flow prediction.

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