International Journal of Applied Mathematics

Volume 31 No. 4 2018, 613-625

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version) doi: http://dx.doi.org/10.12732/ijam.v31i4.7

THE PROBLEM OF OBSTACLE FOR THE FUNDAMENTAL EIGENVALUE

Badahi Ould Mohamed 1 §, Y.O. Mohamed Abdelhaye²

1,2 Faculté des Sciences et Techniques
Université de Nouakchott Al-Aasriya
Nouakchott, MAURITANIA

²Faculty of Science and Arts at Al Qurayat
Jouf University, Kingdom of SAUDI ARABIA

Abstract: By using the tools of derivatives with respect to the domain and the Bernoulli's free boundary problem for Laplacian operator we give sufficient conditions in the case of placement obstacle problems for Laplacian Dirichlet eigenvalue.

AMS Subject Classification: 48A60, 49A80, 83Q15, 59K46, 95C41, 59N55 **Key Words:** first eigenvalue, local strict minimum, shape derivatives, obstacle, sufficient conditions, hard obstacle

1. Introduction

The study of problems of analysis of stability of the exterior and interior Bernoulli's free boundary problems form in [2] leads us to get interesting information about obstacle problem for the principal eigenvalue (the first eigenvalue of the Laplace operator with Dirichlet boundary conditions). And what is the obstacle problem?

The problem of obstacle for the fundamental eigenvalue is to locate the position of the placement of obstacles or wells so as to maximize or to minimize the first eigenvalue of the considered operator.

In [15], the authors studied this problem by considering the Laplace or

Received: April 15, 2018

© 2018 Academic Publications

[§]Correspondence author

Schrodinger operator defined within a fixed, bounded, open domain D with zero Dirichlet boundary conditions on the boundary. Inside this domain, they placed a ball which represents an obstacle or a well, the position of which is under their control, and their goal is to locate the optimal position of the piece under their control. And in their works (cf. [15]), one can find some interesting partial answers assuming convexity and/or symmetry properties for D. They also gave illustrative examples.

In this part of our work, considering an obstacle or a well not necessarily a ball, we study sufficient conditions to obtain the minimum or maximum value for the first eigenvalue of the Laplace or Schrodinger operator. The beginning of this study is that, we suppose that we have critical position of the obstacle in the domain. This means that the shape derivative of the fundamental eigenvalue which we shall explain in this paper, equals a constant on the boundary of the considered domain.

In the sequel, we focus our efforts on the minimization problem because the maximization one's uses the same techniques.

The problem of the optimal placement of the obstacle is stated as follows:

We assume that $\Omega = D \backslash K$, where $K \subset D$, represents an obstacle or a well which is a \mathcal{C}^2 -regular domain. The shape of K is fixed a priori and only its position changes by rigids motions (translations and rotations, but in this study we are going to focus on motions by translations). Let us suppose that Ω is a critical point, i.e. we have a position of K such that the derivative with respect to the domain of the first eigenvalue is equal to a constant on the boundary of K or D.

Our aim is to give sufficient conditions to characterize the shape of the obstacle K so as the fundamental eigenvalue of the Laplace or Schrodinger operator on Ω with Dirichlet condition on the boundary of the domain is minimum.

The obstacles we shall consider may be hard, by which we mean that the zero Dirichlet conditions are additionally imposed on the boundary of K or then may be soft, by which we mean that the operator we are going to consider is of the following form:

$$-\Delta + \alpha \chi_K I$$
,

where $\alpha \in \mathbb{R}$, and χ_K is the indicator function of the region K defined by:

$$\chi_K(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin K \end{cases} . \tag{1}$$

A hard obstacle corresponds to $\alpha = +\infty$. The term of well refers to the case in which the constant α is negative. These types of operators are defined in standard ways, and our sign convention of the fundamental eigenvalue with

a hard obstacle is positive and in the case of well it may be negative. For more information on these operators, see for instance [6].

Notation:

In this paper, we use the following notation:

$$V(x,0) = V(0) = V.$$

The placement obstacle problem for the fundamental eigenvalue:

In this article, we mention same techniques used in our paper [2]. We study the obstacles positions problem using the findings from [2] on the functional J, introduced by Alt and Caffarelli in [1].

We offer the following details. Suppose we have a critical point for the first eigenvalues of the Laplace operator, we will give the quadratic form associated with the first eigenvalues of the Laplace operator and finally we will end this article by giving the placement obstacle problem

2. Critical point

Let us define

 $\mathcal{O}_{\epsilon} = \{ w \subset D, \ w \text{ open set verifying the uniform cone property and } vol(w) = m_0 \}$.

We assume that $\omega = D \setminus K$, where $K \subset D$ represents an obstacle which is a \mathcal{C}^2 -domain. The shape of D and the obstacle K are fixed a priori and only the position of K changes.

Proposition 2.1. (Case of hard obstacle) Let us consider the following problem:

$$\begin{cases}
-\Delta u_{\Omega} = \lambda_{\Omega} u_{\Omega} & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases},$$
(2)

where $\Omega = D \backslash K$ and K is a C^2 domain.

The first eigenvalue is defined by:

$$\lambda_{\Omega} = \inf \{ \int_{\Omega} |\nabla u|^2 dx : u \in W_0^{1,2}(\Omega) / \int_{\Omega} u^2 dx = 1 \}.$$

We assume that there is $\Omega \in \mathcal{O}_{\epsilon}$.

Then, we have

$$\lambda'(\Omega, V) = -\int_{\partial K} (\frac{\partial u}{\partial n})^2 V(0).n \ d\sigma.$$

And if Ω is a critical point for λ_{Ω} , then there is a Lagrange multiplier β_{Ω} such that:

$$-(\frac{\partial u}{\partial n})^2 = \beta_{\Omega} \text{ on } \partial K,$$

where n is the exterior unit normal vector to $\Omega = D \setminus K$ (n is the interior unit normal vector to K).

Proof. We give a sketch of proof with some hints giving the desired result. We use, in particular, the implicit functions theorem, also the shape derivative techniques, see for instance pioneer works of Schiffer [28] or [18], [25]. Some hints for the proof are:

Let us consider the problem

$$\begin{cases}
-\Delta u_{\Omega} = \lambda_{\Omega} u_{\Omega} & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}$$
(3)

Using the shape derivative, we get

$$\begin{cases}
-\Delta u' = \lambda' u + \lambda u' & \text{in } \mathcal{D}(\Omega) \\
u' = -\frac{\partial u}{\partial n} V(0).n & \text{on } \partial\Omega
\end{cases}$$
(4)

Multiplying by u the first equation of the above system, using the Green formula and finally replacing u' by its value on the boundary of Ω , we get:

$$\lambda'(\Omega, V) = -\int_{\partial K} (\frac{\partial u}{\partial n})^2 V(0) \cdot n d\sigma.$$

The Lagrange multiplier appears because of the constraint required on the volume of Ω .

Proposition 2.2. (The case of well) Let us consider the following problem:

$$\begin{cases}
-\Delta u_{\Omega} + \alpha \chi_K u_{\Omega} &= \lambda_{\Omega} u_{\Omega} & \text{in } D \\
u &= 0 & \text{on } D
\end{cases},$$
(5)

where $\Omega = D \backslash K$ and K is a C^2 domain.

The first eigenvalue is defined by:

$$\lambda_{\Omega} = \inf \{ \int_{D} |\nabla u|^{2} dx + \alpha \int_{D} \chi_{K} u^{2} dx : u \in W_{0}^{1,2}(\Omega) \int_{D} u^{2} dx = 1 \}.$$

Then, we have:

$$\lambda'(\Omega, V) = \alpha \int_{\partial K} u^2 V(0).nd\sigma$$
, where n is the exterior unit normal to K.

And if

 $\Omega \in \mathcal{O} = \{\omega \subset D, K \subset \omega \text{ w open set verifying the } \epsilon - \text{cone property},$

and
$$\int_{\omega} \chi_K dx = vol(K) = m_0$$

 $(m_0 \text{ being a fixed positive real number})$ is a critical point for λ_{Ω} , then there is a Lagrange multiplier γ_{Ω} such that:

$$\alpha u^2 = \gamma_0$$
 on ∂K .

If Ω is a critical point for λ_{Ω} and if, moreover, the volume of K may change, then u=0 on ∂K .

To prove this proposition, one uses the same techniques and the same way than in the proof of the case of the hard obstacle. But the equation satisfied by the shape derivative changes as follows:

$$\begin{cases}
-\Delta u' + \alpha \chi_K u' + \alpha \chi_K u V(0). n_e = \lambda' u + \lambda u' & \text{in } \mathcal{D}(D) \\
u' = 0 & \text{on } \partial D
\end{cases} .$$
(6)

3. Quadratic form associated with the obstacle placement problem

The quadratic shape is obtained by calculating the second derivative of $\lambda'(\Omega, V)$ against the domain. Let us take V given by $V(x;t) = v(x)n(x), \ v \in H^{\frac{1}{2}}(\partial\Omega)$ and n(x) is the exterior normal defined on $\partial\Omega$.

So before going on, we need some hypotheses, let us suppose that:

(i) - Ω is a \mathcal{C}^2 -regular open domain;

(ii)
$$-(\frac{\partial u}{\partial n}) = c > 0$$
 (a positive constant).

Proposition 3.1. (Case of hard obstacle) Let us suppose that Ω is a critical point, then

$$\begin{split} Q(v) &= d^2 \lambda(\Omega; V; V) = d^2 J(\Omega; V; V) \\ &= -2\beta_{\Omega} \int_{\partial K} (N-1) H v^2 ds - 2\beta_{\Omega} \int_{\Omega} |\nabla \Lambda|^2 dx \\ &= -2\beta_{\Omega} \int_{\partial K} (N-1) H v^2 ds - 2\beta_{\Omega} \int_{\partial K} v L v ds. \end{split}$$

Here β_{Ω} is the Lagrange multiplier which is negative, Λ is the solution of the following boundary value problem

$$\begin{cases}
-\Delta \Lambda = 0 & \text{in } D \backslash K \\
\Lambda = v & \text{on } \partial K \\
\Lambda = 0 & \text{on } \partial D
\end{cases} \tag{7}$$

H is the mean curvature of ∂K and L is a pseudo-differential operator which is known as the Steklov-Poincaré or capacity or Dirichlet to Neumann operator (see e.g [12]), defined by $Lv = \frac{\partial \Lambda}{\partial n}$, n is the unit exterior normal of K. In fact Λ is the harmonic extension of v in Ω .

Proof of Proposition 3.1. We use the definition of the derivative with respect to the domain and we apply it to $\lambda'(\Omega, V)$. Then we get

$$\begin{split} Q(v) &= d^2 \lambda(\Omega, V, V) \\ &= \int_{\Omega \backslash K} (\operatorname{div}((-|\nabla u|^2)V(x,0)))' dx + \int_{\Omega \backslash K} \operatorname{div}(V(x,0)\operatorname{div}(-|\nabla u|^2)V(x,0)) dx, \\ Q(v) &= -[\int_{\partial K} 2\nabla u \nabla u' V(x,0).n + \operatorname{div}((|\nabla u|^2)V(x,0)V(x,0).n] ds. \end{split}$$

Note that

$$-\frac{\partial u}{\partial n} = c$$
 a.e. on ∂K ,

as we have u = 0 on ∂K , and: $\nabla u = \frac{\partial u}{\partial n} n = -cn$. Hence,

$$Q(v) = -\left[\int_{\partial K} -2cn \cdot \nabla u' V(x, 0) \cdot n + \operatorname{div}(|\nabla u|^{2} V(x, 0)) V(x, 0) \cdot n\right] ds.$$

We have $u^{'} = -\frac{\partial u}{\partial n}V.n = cV.n$ on ∂K . And since $-\frac{\partial u}{\partial n} = c$ a.e. on ∂K , and V.n = v, we get $u^{'} = cv$ on ∂K and $n.\nabla u^{'} = \frac{\partial u^{'}}{\partial n} = c\frac{\partial v}{\partial n} = cLv$, where L is a pseudo-differential operator, defined by $Lv = \frac{\partial \Lambda}{\partial n}$ and such that

$$\begin{cases}
-\Delta \Lambda &= 0 & \text{in } \Omega \backslash K \\
\Lambda &= 0 & \text{on } \partial K \\
\Lambda &= v & \text{on } \partial \Omega
\end{cases} , \tag{8}$$

 Λ is the extension of v in $\Omega \backslash K$.

Hence,

$$Q(v) = \int_{\partial K} (2c^2vLv - \operatorname{div}((|\nabla u|^2)vn)v)ds.$$

We have

$$\operatorname{div}(|\nabla u|^2 v n) = v \nabla(|\nabla u|^2) \cdot n = 2v |\nabla u| \nabla(|\nabla u|) \cdot n.$$

Since by hypothesis Ω is C^2 , then we use the formula of the motion of level set which is related to the mean curvature.

In fact, $\partial K = \{x \in \mathbb{R}^N; \ u(x) = 0\}$ and we have

$$-(N-1)H = \operatorname{div}(\frac{\nabla u}{|\nabla u|}) = \frac{\Delta u}{|\nabla u|} - \frac{\nabla u \cdot \nabla(|\nabla u|)}{|\nabla u|^2},$$

where H is the mean curvature of $\partial\Omega$. Furthermore: since u=0 on ∂K , we have $\Delta u=0$ on ∂K .

Finally we get

$$(N-1)H = \frac{-cn}{|\nabla u|^2} \cdot \nabla(|\nabla u|), \quad \text{i.e.},$$

$$-(N-1)H|\nabla u|^2 = cn \cdot \nabla(|\nabla u|),$$

$$c\nabla(|\nabla u|^2) = -(N-1)H|\nabla u|^2n,$$

$$\nabla(|\nabla u|^2) \cdot n = \frac{-(N-1)}{c}H|\nabla u|^2, \quad \text{hence}$$

$$\operatorname{div}(|\nabla u|^2vn) = v(\frac{-2(N-1)}{c}H)|\nabla u|^3, \quad \text{then}$$

$$Q(v) = \int_{\partial K} (2c^2vLv + 2c^2(N-1)Hv^2ds)$$

$$= \int_{\partial K} (-2\beta_{\Omega}vLv - 2\beta_{\Omega}(N-1)Hv^2ds).$$

And by the Green formula, we get

$$\int_{\partial K} vLvds = \int_{\Omega \backslash K} |\nabla \Lambda|^2 dx.$$

Remark 3.1. Let us note that since the Lagrange multiplier β_{Ω} is negative, we have to take into account this information on the research of local strict minimum.

In fact, $Q_1 = -\beta_{\Omega} Q(v)$, where Q(v) is the quadratic form computed in the free boundary part of this paper.

To have information on the positiveness of the quadratic form Q_1 , it suffices the positiveness of Q and after on cane conclude easily as follows: if Q > 0 then $Q_1 > 0$.

The sufficient conditions for the strict local minimum are of the same conditions in the case of free boundary problem is obtained in [2].

Proposition 3.2. (Case of well) Let us suppose that Ω is a critical point, then for any $v \in H^{\frac{1}{2}}(\partial\Omega)$, we have

$$\begin{split} Q(v) &= d^2 \lambda(\Omega; V; V) \\ &= 2\alpha \int_{\partial K} (v^2 u \frac{\partial u}{\partial n} + u' u v) d\sigma = 0. \end{split}$$

Proof of Proposition 3.2. One can use the same techniques as in the previous Proposition 3.1.

Remark 3.1. In fact, $u' = \frac{\partial u}{\partial n} v$ on ∂K .

No information can be obtained from only the calculation of the second derivative. We have a degeneracy situation. We think that it would be a good challenge to study this situation.

4. Sufficient conditions for the minimum

To provide sufficient conditions for a local minimum is of basic worth. We first present the results obtained in our previous paper [2].

Let A be an operator defined in the following sense:

$$A: H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega)$$

$$A = L + (N-1)(||H^{-}||_{\infty} + H)I,$$

where I is the identity operator and L is the pseudo-differential operator defined as in Proposition 3.1, and $H^- = \max(0, -H)$.

Remark 4.1. By hypothesis $\partial\Omega$ is of the class \mathcal{C}^2 , then the mean curvature H is a continuous function on $\partial\Omega$.

Let us set $\alpha(x) = (N-1)(||H^-||_{\infty} + H(x)), \ \forall \ x \in \partial\Omega$. We remark that α is continuous and $\forall \ x \in \partial\Omega, \ \alpha(x) \geq 0$ (and $\alpha(x) > 0$ on a sufficiently large set).

- **Lemma 4.1.** 1 The operator A is a bijection from $H^{\frac{1}{2}}(\partial\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$ and it is continuous.
- 2 The inverse operator A^{-1} is compact and self adjoint from $H^{-\frac{1}{2}}(\partial\Omega)$ into $H^{\frac{1}{2}}(\partial\Omega)$.

Proof. See in [2].

Remark 4.2. Since the inverse operator: $(\alpha I + L)^{-1}$ is compact, self adjoint, then there exists a Hilbert basis $(\phi_n)_{(n \in \mathbb{N})} \subset H^{\frac{1}{2}}(\partial\Omega)$ and a decreasing sequence of eigenvalues μ_n which goes to 0.

Proposition 4.1. Let Ω the critical shape for λ_{Ω} (The first eigenvalue of the Laplace operator with Dirichlet boundary conditions).

 Ω is a local strict minimum of λ_{Ω} if and only if

$$(N-1)||H^-||_{\infty} < \frac{1}{\mu_0}.$$

Proof. Since $v \in H^{\frac{1}{2}}(\partial\Omega)$, then

$$v = \sum_{n=0}^{\infty} v_n \phi_n,$$

$$(L + (N-1)(H + ||H^-||_{\infty})I)^{-1} \phi_n = \mu_n \phi_n, \text{ then}$$

$$(L + (N-1)H)I)\phi_n = (\frac{1}{\mu_n} - (N-1)||H^-||_{\infty})\phi_n.$$

Let us set $\lambda_n = \frac{1}{\mu_n} - (N-1)||H^-||_{\infty}$.

So (λ_n) is an increasing sequence going to infinity. Then we have

$$\frac{Q(v)}{-2\beta_{\Omega}} = \langle (L + (N-1)HI)v, v \rangle$$
 (9)

$$= \sum_{n=0}^{\infty} \lambda_n |v_n|^2. \tag{10}$$

Suppose that

$$(N-1)||H^-||_{\infty} < \frac{1}{\mu_0},$$

this gives

$$\lambda_0 = \frac{1}{\mu_0} - (N-1)||H^-||_{\infty} > 0.$$

Since $\lambda_0 > 0$, then as a result λ_n growing and we get

$$Q(v) \ge c^2 \lambda_0 ||v||_{H^{\frac{1}{2}}(\partial\Omega)}^2,$$

with $c^2 = -2\beta_{\Omega}$.

We are now in the case where it is possible to use the main result in [11], which can be formulated as follows: there is a positive constant C and a positive function w and $\lim_{\eta \longrightarrow 0} w(\eta) = 0$ such that

$$|j''(t) - j''(0)| \le Cw(\eta),$$
 (11)

where $j''(t) = d^2 \lambda(\Omega_t, V, V)$, $j(t) = \lambda(\Omega_t)$, and $j''(0) = d^2 \lambda(\Omega, V, V) = Q(v)$, $j(0) = \lambda(\Omega)$.

We replace in (13) and get

$$|d^2\lambda(\Omega_t, V, V) - d^2\lambda(\Omega, V, V)| \le Cw(\eta). \tag{12}$$

Using the Taylor formula with integral rest, we get:

$$\lambda(\Omega_t) = \lambda(\Omega) + \int_0^1 (1 - t) d^2 \lambda(\Omega_t, V, V) dt.$$

Using the inequality (12), we have the following:

if $\lambda_0 > 0$, then Ω is a local strict minimum for the functional λ_{Ω} .

if Ω is a convex domain, then $\lambda_0 = \frac{1}{\mu_0} > 0$. Hence Ω is a local strict minimum for the functional λ_{Ω} .

5. Conclusions

Let Ω be a critical form of the first eigenvalue of the Laplace operator with Dirichlet boundary conditions. By Proposition 4.1 we conclude that:

- if $(N-1)||H^-||_{\infty} < \frac{1}{\mu_0}$, then Ω is a local strict minimum for the first eigenvalue λ_{Ω} of the Laplace operator;
- if Ω is a convex domain, then $\frac{1}{\mu_0} > 0$. Hence Ω is a local strict minimum for the first eigenvalue λ_{Ω} of the Laplace operator.

Acknowledgments

The author would like to thank the referees for their fruitfull remarks and comments.

References

- [1] H.W. Alt and L.A. Caffarelli, Existence and regularity for a minimum problem with free boundary, *J. Reine Angew. Math.*, **325** (1981), 105-144.
- [2] O.M. Badahi, I. Ly, D. Seck, Analysis of stability of the exterior and interior Bernoulli's free boundary problems, *Applied Mathematical Sciences*, **10**, No 12 (2016), 591-612.
- [3] O.M. Badahi, The variation of the first eigenvalue of the Laplace operator and the problem of locating an obstacle. *Internat. J. of Pure and Appl. Math.*, **116**, No 2 (2017), 361-372.
- [4] M. Berger and B. Gostiaux, Géométrie différentielle: vaiètés, courbes et surfaces, Puf Mathematiques (2013).
- [5] A. Beurling, On free boundary for the Laplace equation, Seminars on Analytic Functions I, Institute Advance Studies Seminars (1957), Princeton, 248-263.
- [6] E.B. Davies, Spectral Theory and Differential Operators, Cambridge Stud. Adv. Math. 42, Cambridge University Press, Cambridge (1995).

- [7] B. Chen, Total Mean Curvature and Submanifolds of Finite Type, Maths. Michigan State Univ., World Scientific (2014).
- [8] M. Dambrine, On variations of the shape Hessian and sufficient conditions for the stability of critical shapes, *Rev. Real Acad. Cien. Exactas Fis. Nat.*, *Ser. A Math.*, **96**, No 1 (2002), 95-121.
- [9] M. Dambrine, Hessienne de forme et stabilité des formes critiques, PhD Thesis, École Normale Supérieure de Cachan (2000).
- [10] D. Daners and B. Kawohl, An isoperimetric inequality related to a Bernoulli problem, *Calculus of Variations and PDE's* **39** (2010), 547-555.
- [11] M. Dambrine and M. Pierre, About stability of equilibrium shapes, *Math. Model. Numer. Anal.*, **34**, No 4 (2000), 811-834.
- [12] R. Dautray and J.L. Lions, Analyse mathématique et Calcul numérique pour les sciences et les techniques, Vol. 2, Masson (1984).
- [13] J. Descloux, On the two dimensional magnetic shaping problem without surface tensions, *Report* Nr. 07.90, École Polytechnique Fédérale de Lausanne (1990).
- [14] M. Flucher and M. Rumpf, Bernoulli's free boundary problem, qualitative theory and numerical approximation, *J. Reine Angew. Math.*, **486** (1997), 165-204.
- [15] E.M. Harrell II, P. Kroger, K. Kurata, On the placement of an obstale nor well so as to optimize the fundamental eigenvalue, *SIAM J. Math. Anal.*, **33**, No 1 (2001), 240-259.
- [16] A. Henrot, Subsolutions and Supersolutions in a free boundary problem, *Ark. Mat.*, **32** (1994), 79-98.
- [17] A. Henrot and M. Pierre, About crtical points of the energy in a electromagnetic shaping problem, In: *Boundary Control and Boundary Variation* (Sophia Antipolis, 1990), Springer, Berlin (1992), 238-252.
- [18] A. Henrot and M. Pierre, Variations et optimisation de formes, une analyse géométrique, Mathématiques et Applications, # 48, Springer (2005).
- [19] A. Henrot, M. Pierre, M. Rihani, Finite dimensional reduction for the positivity of some second shape devivatives, *Methods and Applications of Analysis*, **10** (2003), 457-476.

- [20] A. Henrot, M. Pierre, M. Rihani, Positivity of the shape Hessian and instability of some equilibrium shapes, *Mediterr. J. Math.*, 1 (2004), 195-214.
- [21] A. Henrot and H. Shahgholian, Existence of classical solutions to a free boundary problem for the p-Laplace operator: the exterior convex case, J. Reine Angew., 521 (2000), 85-97.
- [22] A. Henrot and H. Shahgholian, Existence of classical solutions to a free boundary problem for the p-Laplace operator II: the interior convex case, *Indiana Univ. Math. J.*, **49**, No 1 (2000), 301-323.
- [23] A. Henrot and D. Seck, Retour à une ancienne approche, *Publ. Mathématiques de Besançon Analyse non linéaire*, **15** (1995/97), 29-40.
- [24] Y.O. Mohamed Abdelhaye, Badahi Ould Mohamed and M.V. Ould Moustapha, Wave kernel for the Schrödinger operator with a Liouville potential, Far East J. of Math. Sci., 102, No 7 (2017), 1523-1532.
- [25] F. Murat, J. Simon, Derivation de fonctionnelles par à un domaine géométrique, Numer. Prépublications de Paris 6.
- [26] I. Ly, D. Seck, Isoperimetric inequality for an interior free boundary problem with p-Laplacian operator, *Electron. J. Differ. Equ.*, **2004** (2004), Art. # 109, 1-12.
- [27] D. Seck, Etude d'un problème à frontière libre de type Bernoulli, Thèse d'Université Franche-Comté numéro d'ordre 505 (1996).
- [28] M. Schiffer, Hadamard's formula and variation of domain functions. *Amer. J. Math.*, **68** (1946), 417.
- [29] J. Simon, Differential with respect to the domain in boundary value problems. *Numer. Funct. Anal. and Optimiz.*, **2**, No 7-8 (1980), 649-687.