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PROPERTIES OF A CERTAIN SUBCLASS OF STARLIKE FUNCTIONS DEFINED BY A GENERALIZED OPERATOR

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Abstract: In this paper, we introduce the class $S^{n,s}_{\alpha,\lambda}(\beta)$, consisting of analytic functions defined by a generalized operator. We derive coefficient inequalities, growth and distortion theorem, extreme points and Fekete-Szegö problem for functions in this class.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by $f(0) = 0$ and $f'(0)-1 = 0$ in the form of

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

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An analytic function is said to be in the class of starlike functions of order β in $\mathbb{U}$, denoted by $S^*(\beta)$, if it satisfies the following condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (z \in \mathbb{U}; \quad 0 \leq \beta < 1). \tag{2}$$

Note that $S^*(\beta) \subseteq S*(0) =: S$, where S is the well-known class of starlike functions with respect to the origin in $\mathbb{U}$.

Applications of some linear integral and differential operators play a vital role in geometric function theory. Salagean [11] defined and studied the derivative operator denoted as $D^n f(z)$. Then, Al-Oboudi [1] generalized the Salagean operator. Srivastava and Attiya [12] introduced a convolution operator $J_{s,b}(f)(z)$ that is defined by the Hadamard product in terms of the Hurwitz-Lerch Zeta function, $\phi(z,s;b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s}$. Liu [7], then generalized the operator, $J_{s,b}(f)(z)$. Several interesting subclasses of analytic functions defined by associating the above mentioned operators are introduced and investigated in literature. (See for example [3, 2, 10, 9]). For further extensions of similar studies, related to operators of generalized fractional calculus, see for example, Kiryakova [5], [6] and many references therein.

Recently, Yunus et. al [13] introduced the operator $\vartheta_{\alpha,\lambda}^{n,s} f(z) : \mathcal{A} \to \mathcal{A}$ defined by:

$$\vartheta_{\alpha,\lambda}^{n,s} f(z) = z + \sum_{k=2}^{\infty} (1 - \alpha(1-\lambda)(1-k))^n \left(\frac{1+b}{k+b}\right)^s a_k z^k, \tag{3}$$

where $0 \leq \lambda < 1, \quad 0 < \alpha \leq 1, \quad b \in \mathbb{C}/\mathbb{Z}_0^-, \quad s \in \mathbb{C}, \quad n \in \mathbb{N}_0$. Observe that when:

(i) $n = 0$, $\vartheta_{\alpha,\lambda}^{0,s} f(z)$ is the Srivastava–Attiya operator [12];

(ii) $s = 0, \lambda = 0$, $\vartheta_{\alpha,0}^{n,0} f(z)$ is the Al-Oboudi operator [1];

(iii) $s = 0, \lambda = 0, \alpha = 1$, $\vartheta_{\alpha,\lambda}^{n,s} f(z)$ is the Salagean differential operator [11].

In this paper, by applying the operator $\vartheta_{\alpha,\lambda}^{n,s} f(z)$, we introduce a subclass of A denoted by $S_{\alpha,\lambda}^{n,s}(\beta)$.

Namely, we define $f \in A$ to be in the class $S_{\alpha,\lambda}^{n,s}(\beta)$, if $\vartheta_{\alpha,\lambda}^{n,s} f(z)$ is in the class $S^*(\beta)$, that is, if

$$Re\left\{\frac{z(\vartheta_{\alpha,\lambda}^{n,s} f(z))'}{\vartheta_{\alpha,\lambda}^{n,s} f(z)}\right\} > \beta \tag{4}$$

$$(z \in \mathbb{U}; 0 \le \beta < 1; 0 \le \lambda < 1, 0 < \alpha \le 1, b \in \mathbb{C}/\mathbb{Z}_0^-, s \in \mathbb{C}, n \in \mathbb{N}_0).$$

The properties of the above class such as coefficient estimates, growth and distortion theorems, extreme point and Fekete-Szegö problem are investigated.

2. Coefficient estimates

We obtain a sufficient condition for function $f(z) \in \mathcal{A}$ to be in the class $S^{n,s}_{\alpha,\lambda}(\beta)$.

Theorem 1. *Let $f(z) \in \mathcal{A}$ be given by (1). If $\le \beta < 1,\ 0 \le \lambda < 1,\ 0 < \alpha \le 1,\ b \in \mathbb{C}/\mathbb{Z}_0^-,\ s \in \mathbb{C},\ n \in \mathbb{N}_0$ and*

$$\sum_{k=2}^{\infty}(k-\beta)|(1-\alpha(1-\lambda)(1-k))^n|\left|\left(\frac{1+b}{k+b}\right)^s\right||a_k| \le 1-\beta, \tag{5}$$

then $f(z) \in S^{n,s}_{\alpha,\lambda}(\beta)$.

Proof. Suppose that condition (5) is satisfied for $\beta \in [0,1)$. For $f(z) \in \mathcal{A}$, we define the function $G(z)$ by

$$G(z) := \frac{z(\vartheta^{n,s}_{\alpha,\lambda}f(z))'}{\vartheta^{n,s}_{\alpha,\lambda}f(z)} - \beta. \tag{6}$$

In order to prove that $f(z) \in S^{n,s}_{\alpha,\lambda}(\beta)$, it suffices to show that

$$\left|\frac{G(z)-1}{G(z)+1}\right| < 1, \quad (z \in \mathbb{U}).$$

Making use of (3), we have

$$\begin{aligned}
|F(z)| &= \left|\frac{G(z)-1}{G(z)+1}\right| \\
&= \left|\frac{z(\vartheta^{n,s}_{\alpha,\lambda}f(z))' - (1+\beta)\vartheta^{n,s}_{\alpha,\lambda}f(z)}{z(\vartheta^{n,s}_{\alpha,\lambda}f(z))' + (1-\beta)\vartheta^{n,s}_{\alpha,\lambda}f(z)}\right| \\
&= \left|\frac{\begin{array}{l} z+\sum_{k=2}^{\infty}k(1-\alpha(1-\lambda)(1-k))^n(\frac{1+b}{k+b})^s a_k z^k \\ -(1+\beta)(z+\sum_{k=2}^{\infty}(1-\alpha(1-\lambda)(1-k))^n(\frac{1+b}{k+b})^s a_k z^k) \end{array}}{\begin{array}{l} z+\sum_{k=2}^{\infty}k(1-\alpha(1-\lambda)(1-k))^n(\frac{1+b}{k+b})^s a_k z^k \\ +(1-\beta)(z+\sum_{k=2}^{\infty}(1-\alpha(1-\lambda)(1-k))^n(\frac{1+b}{k+b})^s a_k z^k) \end{array}}\right|
\end{aligned}$$

$$= \left| \frac{\beta z + \sum_{k=2}^{\infty}(1+\beta-k)(1-\alpha(1-\lambda)(1-k))^n \left(\frac{1+b}{k+b}\right)^s a_k z^k}{(2-\beta)z + \sum_{k=2}^{\infty}(1-\beta+k)(1-\alpha(1-\lambda)(1-k))^n \left(\frac{1+b}{k+b}\right)^s a_k z^k} \right|$$

$$\leq \frac{\beta|z| + \sum_{k=2}^{\infty}(1+\beta-k)(1-\alpha(1-\lambda)(1-k))^n \left|\left(\frac{1+b}{k+b}\right)^s\right| |a_k||z^k|}{(2-\beta)|z| + \sum_{k=2}^{\infty}(1-\beta+k)(1-\alpha(1-\lambda)(1-k))^n \left|\left(\frac{1+b}{k+b}\right)^s\right| |a_k||z^k|}$$

$$< \frac{\beta + \sum_{k=2}^{\infty}(1+\beta-k)(1-\alpha(1-\lambda)(1-k))^n \left|\left(\frac{1+b}{k+b}\right)^s\right| |a_k|}{(2-\beta) + \sum_{k=2}^{\infty}(1-\beta+k)(1-\alpha(1-\lambda)(1-k))^n \left|\left(\frac{1+b}{k+b}\right)^s\right| |a_k|}$$

$$\leq 1,$$

provided that

$$\beta + \sum_{k=2}^{\infty}(1+\beta-k)(1-\alpha(1-\lambda)(1-k))^n \left|\left(\frac{1+b}{k+b}\right)^s\right| |a_k|$$
$$\leq (2-\beta) + \sum_{k=2}^{\infty}(1-\beta+k)(1-\alpha(1-\lambda)(1-k))^n \left|\left(\frac{1+b}{k+b}\right)^s\right| |a_k|,$$

which is equivalent to hypothesis (5) in Theorem 1. So this completes the proof of Theorem 1. □

The next theorem aims to provide coefficient inequalities so that the function $f(z)$ belongs to $S^{n,s}_{\alpha,\lambda}(\beta)$.

Theorem 2. *Let $\beta \in [0,1)$. If $f(z) \in S^{n,s}_{\alpha,\lambda}(\beta)$, then*

$$|a_k| \leq \frac{2(1-\beta)}{(k-1)|(1-\alpha(1-\lambda)(1-k))^n|} \left|\left(\frac{k+b}{1+b}\right)^s\right| \prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)}{j-1}\right), \quad (7)$$

with $k \in \mathbb{N}/\{1\}$.

The result is sharp.

Proof. Let $f(z) \in S^{n,s}_{\alpha,\lambda}(\beta)$. Set

$$p(z) = \frac{1}{1-\beta}\left\{\frac{z(\vartheta^{n,s}_{\alpha,\lambda} f(z))'}{\vartheta^{n,s}_{\alpha,\lambda} f(z)} - \beta\right\} = 1 + c_1 z + c_2 z^2 + \ldots \quad (8)$$

Then $p(z)$ is analytic with

$$p(0) = 1 \quad \text{and} \quad Re\{p(z)\} > 0, \quad (z \in \mathbb{U}).$$

Rearranging (8), we get

$$z(\vartheta^{n,s}_{\alpha,\lambda} f(z))' = [(1-\beta)p(z) + \beta]\vartheta^{n,s}_{\alpha,\lambda},$$

and by virtue of (3), we get

$$\begin{aligned}&(k-1)(1-\alpha(1-\lambda)(1-k))^n \left(\frac{1+b}{k+b}\right)^s a_k \\ &= (1-\beta)c_{k-1} + \sum_{j=2}^{k-1}(1-\beta)c_{k-j}(1-\alpha(1-\lambda)(1-j))^n \left(\frac{1+b}{j+b}\right)^s a_j.\end{aligned} \tag{9}$$

By using Caratheodory's lemma [4], $|c_k| \leq 2$, so we have

$$\begin{aligned}&(k-1)(1-\alpha(1-\lambda)(1-k))^n \left|\left(\frac{1+b}{k+b}\right)^s\right| |a_k| \\ &\leq 2(1-\beta)\left\{1 + \sum_{j=2}^{k-1}(1-\alpha(1-\lambda)(1-j))^n \left|\left(\frac{1+b}{j+b}\right)^s\right| |a_j|\right\}.\end{aligned} \tag{10}$$

Using the principle of mathematical induction, we will prove that the inequality (7) holds true for $k \in \mathbb{N}/\{1\}$. Define formally the summation, $\sum_{j=n}^{m} d_j$ and product, $\prod_{j=n}^{m} h_j$ as follows:

$$\sum_{j=n}^{m} d_j = \begin{cases} 0, & \text{if } n > m, \\ d_n + \sum_{j=n+1}^{m} d_j, & \text{if } m \geq n, \end{cases}$$

and

$$\prod_{j=n}^{m} h_j = \begin{cases} 1, & \text{if } n > m, \\ h_n \prod_{j=n+1}^{m} h_j, & \text{if } m \geq n, \end{cases}$$

respectively.

If $k = 2$ in (10), then we have

$$(1+\alpha(1-\lambda))^n \left|\left(\frac{1+b}{2+b}\right)^s\right| |a_2| \leq 2(1-\beta)$$

$$|a_2| \leq \frac{2(1-\beta)}{(1+\alpha(1-\lambda))^n} \left|\left(\frac{2+b}{1+b}\right)^s\right|.$$

Therefore for $k=2$ the statement is true.

Assume that (7) is true for $k \leq m$. Then, for $k=m+1$, from (7) and (10), we obtain

$$
\begin{aligned}
&(m+1-1)(1-\alpha(1-\lambda)(1-(m+1))))^n \left| \left(\frac{1+b}{k+1+b} \right)^2 \right| |a_{m+1}| \\
&\leq 2(1-\beta) \left\{ 1 + \sum_{j=2}^{m-1} (1-\alpha(1-\lambda)(1-j))^n \left| \left(\frac{1+b}{j+b} \right)^s \right| |a_j| \right\} \\
&\leq 2(1-\beta) \left\{ 1 + \sum_{j=2}^{m} \frac{2(1-\beta)}{j-1} \prod_{j=2}^{j-1} \left(1 + \frac{2(1-\beta)}{j-1} \right) \right\} \\
&= 2(1-\beta) \prod_{j=2}^{j-1} \left(1 + \frac{2(1-\beta)}{j-1} \right),
\end{aligned}
$$

that is

$$
|a_{m+1}| \leq \frac{2(1-\beta)}{(m+1-1)(1-\alpha(1-\lambda)(1-(m+1)))^n} \left| \left(\frac{m+1+b}{1+b} \right)^s \right| \times \prod_{j=2}^{m+1-1} \left(1 + \frac{2(1-\beta)}{j-1} \right). \quad (11)
$$

Therefore (7) holds true for $k=m+1$. Hence, by principle of mathematical induction, it is true for all $k \in \mathbb{N}/\{1\}$.

The result is sharp for the function $f(z)$ given by

$$
f(z) = z + \frac{2(1-\beta)}{(k-1)(1-\alpha(1-\lambda)(1-k))^n} \left| \left(\frac{k+b}{1+b} \right)^s \right| \times \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\beta)}{j-1} \right) z^k. \quad (12)
$$

□

3. Growth and distortion inequalities

Distortion inequalities for the functions in this class are given by the following theorem.

Theorem 3. *Let* $f(z) \in S^{n,s}_{\alpha,\lambda}(\beta)$, $0 \leq \beta < 1$ *and* $|z| = r < 1$. *Then,*

$$r - 2(1-\beta)r^2 \sum_{k=2}^{\infty} \frac{1}{(k-1)(1-\alpha(1-\lambda)(1-k))^n} \left| \left(\frac{k+b}{1+b} \right)^s \right| \times \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\beta)}{j-1} \right)$$

$$\leq |f(z)| \leq$$

$$r \quad + 2(1-\beta)r^2 \sum_{k=2}^{\infty} \frac{1}{(k-1)(1-\alpha(1-\lambda)(1-k))^n} \left| \left(\frac{k+b}{1+b} \right)^s \right| \times \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\beta)}{j-1} \right)$$

and

$$1 - 2(1-\beta)r \sum_{k=2}^{\infty} \frac{k}{(k-1)(1-\alpha(1-\lambda)(1-k))^n} \left| \left(\frac{k+b}{1+b} \right)^s \right| \times \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\beta)}{j-1} \right)$$

$$\leq |f'(z)| \leq$$

$$1 + 2(1-\beta)r \sum_{k=2}^{\infty} \frac{k}{(k-1)(1-\alpha(1-\lambda)(1-k))^n} \left| \left(\frac{k+b}{1+b} \right)^s \right| \times \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\beta)}{j-1} \right).$$

Proof. Let $f(z) \in A$ in the form of (1). Then by Theorem 2, we obtain $|f(z)| \leq |z| + \sum_{k=2}^{\infty} |a_k||z^k|$

$$\leq r + r^2 \sum_{k=2}^{\infty} \frac{2(1-\beta)}{(k-1)(1-\alpha(1-\lambda)(1-k))^n} \left| \left(\frac{k+b}{1+b} \right) \right| \times \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\beta)}{j-1} \right)$$

$$= r + 2r^2(1-\beta)\sum_{k=2}^{\infty}\frac{1}{(k-1)(1-\alpha(1-\lambda)(1-k))^n}\left|\binom{k+b}{1+b}\right| \times \prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)}{j-1}\right)$$

and

$$|f(z)| \geq |z| - \sum_{k=2}^{\infty}|a_k||z^k|$$

$$\geq r - r^2\sum_{k=2}^{\infty}\frac{2(1-\beta)}{(k-1)(1-\alpha(1-\lambda)(1-k))^n}\left|\binom{k+b}{1+b}\right| \times \prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)}{j-1}\right)$$

$$= r - 2r^2(1-\beta)\sum_{k=2}^{\infty}\frac{1}{(k-1)(1-\alpha(1-\lambda)(1-k))^n}\left|\binom{k+b}{1+b}\right| \times \prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)}{j-1}\right)$$

for $|z| = r < 1$.

From (1), upon differentiating

$$|f'(z)| \leq 1 + \sum_{k=2}^{\infty}k|a_k||z^{k-1}|$$

$$\leq 1 + r\sum_{k=2}^{\infty}\frac{2(1-\beta)}{(k-1)(1-\alpha(1-\lambda)(1-k))^n}\left|\binom{k+b}{1+b}\right| \times \prod_{j=2}^{k-1}\left(1+\frac{2(1-\beta)}{j-1}\right)$$

$$= 1 + 2(1-\beta)r \sum_{k=2}^{\infty} \frac{1}{(k-1)(1-\alpha(1-\lambda)(1-k))^n} \left| \binom{k+b}{1+b} \right| \times \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\beta)}{j-1}\right)$$

and

$$|f'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k||z^{k-1}|$$

$$\geq 1 - r \sum_{k=2}^{\infty} \frac{2(1-\beta)}{(k-1)(1-\alpha(1-\lambda)(1-k))^n} \left| \binom{k+b}{1+b} \right| \times \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\beta)}{j-1}\right)$$

$$= 1 - 2(1-\beta)r \sum_{k=2}^{\infty} \frac{1}{(k-1)(1-\alpha(1-\lambda)(1-k))^n} \left| \binom{k+b}{1+b} \right| \times \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\beta)}{j-1}\right).$$

$\square$

4. Extreme points

Let $\tilde{S}^{n,s}_{\alpha,\lambda}(\beta)$ be subclass of $S^{n,s}_{\alpha,\lambda}(\beta)$ that consists of all functions $f(z) \in \mathcal{A}$ which satisfy the inequality (5). Then the extreme points of $\tilde{S}^{n,s}_{\alpha,\lambda}(\beta)$ are given as follows:

Theorem 4. *Let*

$$f_1(z) = z$$

and

$$f_k := z + \frac{1-\beta}{(k-\beta)(1-\alpha(1-\lambda)(1-k))^n} \left| \binom{k+b}{1+b}^s \right|.$$

Then

$$f \in \tilde{S}^{n,s}_{\alpha,\lambda}(\beta)$$

if and only if it can be expressed in the following form:

$$f(z) = \sum_{k=2}^{\infty} t_k f_k(z) \quad \left(t_k > 0; \quad \sum_{k=1}^{\infty} t_k = 1 \right).$$

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} t_k f_k(z) \\ &= z + \sum_{k=2}^{\infty} t_k \frac{1-\beta}{(k-\beta)(1-\alpha(1-\lambda)(1-k))^n} \left| \left(\frac{k+b}{1+b} \right)^s \right| z^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty} (k-\beta)(1-\alpha(1-\lambda)(1-k))^n \left| \left(\frac{1+b}{k+b} \right)^n \right| |a_k| \\ &= \sum_{k=2}^{\infty} (k-\beta)(1-\alpha(1-\lambda)(1-k))^n \left| \left(\frac{1+b}{k+b} \right)^n \right| \\ &\times t_k \frac{1-\beta}{(k-\beta)(1-\alpha(1-\lambda)(1-k))^n} \left| \left(\frac{k+b}{1+b} \right)^n \right| \\ &= (1-\beta) \sum_{k=2}^{\infty} t_k = (1-\beta)(1-t_1) \le 1-\beta. \end{aligned}$$

Therefore, by the definition of class $\tilde{S}^{n,s}_{\alpha,\lambda}(\beta)$, we get

$$f \in \tilde{S}^{n,s}_{\alpha,\lambda}(\beta) \quad (0 \le \beta < 1).$$

Conversely, suppose that

$$f \in \tilde{S}^{n,s}_{\alpha,\lambda}(\beta) \quad (0 \le \beta < 1).$$

Then, by using equation (5), we may set

$$t_k = (k-\beta)(1-\alpha(1-\lambda)(1-k))^n \left| \left(\frac{1+b}{k+b} \right)^n \right| \quad (k \in \mathbb{N}/\{1\})$$

$$t_1 = 1 - \sum_{k=2}^{\infty} t_k.$$

We note that $f(z) = \sum_{k=1}^{\infty} t_k f_k(z)$ and the proof of Theorem 4 is thus completed. □

5. Fekete-Szegö problem

The aim of this section is to obtain the Fekete-Szegö inequality for functions in the class $S_{\alpha,\lambda}^{n,s}$ provided

$$s > 0, \quad b > 0, \quad 0 \le \beta < 1.$$

To derive the results, we recall the lemma from [8].

Lemma 5. *If $p(z) = 1 + c_1 z + c_2 z^2 + ...$ is an analytic function in $\mathbb{U}$ such that $Re\{p(z)\} > 0$ for $z \in \mathbb{U}$, then*

$$|c_2 - \nu c_1^2| \le \begin{cases} -4\nu + 2, & \text{if } \nu \le 0 \\ 2, & \text{if } 0 \le \nu \le 1 \\ 4\nu - 2, & \text{if } \nu \ge 1. \end{cases}$$

When $\nu < 0$ or $\nu > 1$ the equality holds true if and only if

$$p(z) = \frac{1+z}{1-z},$$

or one of its rotations. If $0 < \nu < 1$, then the equality holds true if and only if

$$p(z) = \frac{1+z^2}{1-z^2},$$

or one of its rotations. If $\nu = 0$, the equality holds true, if and only if

$$p(z) = \left(\frac{1+\omega}{2}\right)\left(\frac{1+z}{1-z}\right) + \left(\frac{1-\omega}{2}\right)\left(\frac{1-z}{1+z}\right) \quad (0 \le \omega \le 1)$$

or one of its rotations. If $\nu = 1$, then the equality holds true if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds true in the case $\nu = 0$.

Theorem 6. *Let $s > 0$, $b > 0$, and $0 \le \beta < 1$. If $f(z) \in S^{n,s}_{\alpha,\lambda}$, then*

$$|a_3 - \mu a_2^2| \le \begin{cases} (1-\beta)^2 \Big\{ \dfrac{2}{D_2^n} \left(\dfrac{3+b}{1+b}\right)^s - \dfrac{4\mu}{D_1^{2n}} \left(\dfrac{2+b}{1+b}\right)^{2n} \\ \qquad + \dfrac{1}{(1-\beta)D_2^n} \left(\dfrac{3+b}{1+b}\right)^s \Big\}, \quad \text{if} \quad \mu \le \sigma_1, \\ \dfrac{(1-\beta)}{D_2^n} \left(\dfrac{3+b}{1+b}\right)^s, \quad \text{if} \quad \sigma_1 \le \mu \le \sigma_2, \\ (1-\beta)^2 \Big\{ \dfrac{4\mu}{D_1^{2n}} \left(\dfrac{2+b}{1+b}\right)^{2s} - \dfrac{2}{D_2^n} \left(\dfrac{3+b}{1+b}\right)^s \\ \qquad - \dfrac{1}{(1-\beta)D_2^n} \left(\dfrac{3+b}{1+b}\right)^s \Big\}, \quad \text{if} \quad \mu \ge \sigma_2, \end{cases}$$

where

$$D_1 = 1 + \alpha(1-\lambda) \quad \text{and} \quad D_2 = 1 + 2\alpha(1-\lambda),$$
$$\sigma_1 = \frac{1}{2}\left(\frac{D_1^2}{D_2}\right)^n \left(\frac{1+b}{2+b}\right)^s \left(\frac{3+b}{2+b}\right)^s$$

and

$$\sigma_2 = \frac{2-\beta}{2(1-\beta)}\left(\frac{D_1^2}{D_2}\right)^n \left(\frac{1+b}{2+b}\right)^s \left(\frac{3+b}{2+b}\right)^s.$$

The result is sharp.

Proof. Suppose $f(z) \in S^{n,s}_{\alpha,\lambda}$, let

$$p(z) = \frac{1}{1-\beta}\left\{ \frac{z(\vartheta^{n,s}_{\alpha,\lambda} f(z))'}{\vartheta^{n,s}_{\alpha,\lambda} f(z)} - \beta \right\} = 1 + c_1 z + c_2 z^2 + \dots.$$

Then, by virtue of equation (3) and with the help of (9), we have

$$a_2 = \frac{(1-\beta)c_1}{D_1^n}\left(\frac{2+b}{1+b}\right)^s$$

$$a_3 = \frac{1-\beta}{2D_2^n}\left(\frac{3+b}{1+b}\right)^s (c_2 + (1-\beta)c_1^2).$$

We obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{(1-\beta)}{2D_2^n}\left(\frac{3+b}{1+b}\right)^s (c_2 + (1-\beta)c_1^2) - \mu\frac{(1-\beta)^2}{D_1^{2n}}\left(\frac{2+b}{1+b}\right)^{2s} c_1^2 \\ &= \frac{(1-\beta)}{2D_2^n}\left(\frac{3+b}{1+b}\right)^s (c_2 - \nu c_1^2), \end{aligned}$$

where

$$\nu = (1-\beta)\left(2\mu\frac{D_2^n}{D_1^{2n}}\left(\frac{2+b}{1+b}\right)^s\left(\frac{2+b}{3+b}\right)^s - 1\right).$$

By inserting ν in Lemma 5, we have

$$|c_2 - \nu c_1^2| \le \begin{cases} -4(1-\beta)\left(2\mu\left(\frac{D_2}{D_1^2}\right)^n\left(\frac{2+b}{1+b}\right)^s\left(\frac{2+b}{3+b}\right)^s - 1\right) + 2, & \text{if } \mu \le \sigma_1, \\ 2, & \text{if } \sigma_1 \le \mu \le \sigma_2, \\ 4(1-\beta)\left(\left(2\mu\frac{D_2}{D_1^2}\right)^n\left(\frac{2+b}{1+b}\right)^s\left(\frac{2+b}{3+b}\right)^s - 1\right) - 2, & \text{if } \mu \ge \sigma_2. \end{cases}$$

Applying the lemma, the result asserted by Theorem 6 follows.

In addition, if $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds true if and only if

$$\vartheta_{\alpha\lambda}^{n,s} f(z) = \frac{z}{(1-e^{i\theta}z)^{2(1-\beta)}} \quad (\theta \in \mathbb{R}).$$

If $\sigma_1 < \mu < \sigma_2$, the equality holds true if and only if

$$\vartheta_{\alpha\lambda}^{n,s} f(z) = \frac{z}{(1-e^{i\theta}z^2)^{1-\beta}} \quad (\theta \in \mathbb{R}).$$

If $\mu = \sigma_1$, then the equality holds true if and only if

$$\begin{aligned} \vartheta_{\alpha\lambda}^{n,s} f(z) &= \left(\frac{z}{(1-e^{i\theta}z)^{2(1-\beta)}}\right)^{(1+\omega)/2}\left(\frac{z}{(1+e^{i\theta}z)^{2(1-\beta)}}\right)^{(1-\omega)/2} \\ &= \frac{z}{[(1-e^{i\theta}z)^{1+\omega}(1+e^{i\theta}z)^{1-\omega}]^{1-\beta}}. \end{aligned}$$

If $\mu = \sigma_2$, then the equality holds true if and only if $\vartheta_{\alpha\lambda}^{n,s} f(z)$ satisfies the condition below:

$$\frac{z(\vartheta_{\alpha\lambda}^{n,s} f(z))'}{\vartheta_{\alpha\lambda}^{n,s} f(z)} = (1-\beta)p(z) + \beta,$$

where

$$\frac{1}{p(z)} = \left(\frac{1+\omega}{2}\right)\left(\frac{1+z}{1-z}\right) + \left(\frac{1-\omega}{2}\right)\left(\frac{1-z}{1+z}\right) \quad (0 \le \omega \le 1).$$

□

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