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SUFFICIENT CONDITIONS FOR CONVEX FUNCTIONS OF ORDER γ

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Abstract: Sufficient conditions for functions to be starlike of order γ have been studied by authors Ramesha et al. and Nunokawa et al. Sufficient conditions for convexity involving higher order derivatives were investigated by Silverman. Taking ideas from the works of many authors we obtain interesting sufficient conditions for functions involving higher order derivatives to be univalently convex of order γ ($0 \le \gamma < 1$), using a well known best result in function theory and obtain as a special case, conditions of sufficiency for convexity if they belong to \mathcal{A}_m and $\mathcal{A} := \mathcal{A}_1$.

AMS Subject Classification: 30C45

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1. Introduction

Let \mathcal{A}_m denote the set of all normalized regular functions of the form

$$g(z) = z + \sum_{k=m}^{\infty} b_k z^k$$

which are regular and univalent in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$.

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Define

$$\mathcal{A}_1 := \mathcal{A} = \left\{ g : g(z) = z + \sum_{k=2}^{\infty} b_k z^k \right\}, \quad z \in \mathbb{E}.$$
 (1)

A function g(z) belonging to \mathcal{A} is said to be starlike of order γ in \mathbb{E} , and denoted by \mathcal{S}^* if it satisfies

$$\Re\left(\frac{zg'(z)}{g(z)}\right) > \gamma \qquad (0 \le \gamma < 1) \quad (z \in \mathbb{E}). \tag{2}$$

If $g(z) \in \mathcal{A}$ satisfies

$$\Re\left(\frac{(zg'(z))'}{g'(z)}\right) > \gamma \qquad (0 \le \gamma < 1) \quad (z \in \mathbb{E}),\tag{3}$$

then we say that g(z) is convex of order γ in \mathbb{E} , and we denote by \mathcal{C} the class of all such functions.

Similarly, if by $\mathcal{S}^*(\gamma)$, $\mathcal{C}(\gamma)$, respectively we denote the classes of all starlike and convex functions of order γ , $(0 \le \gamma < 1)$, then $\mathcal{S}_1^*(0) = \mathcal{S}^*$ and $\mathcal{C}_1(0) = \mathcal{C}$, respectively the starlike functions with respect to origin and convex functions.

2. Preliminaries

In order to derive our main results, we have to recall here the following lemma.

Lemma 1. ([6]) Let Ω be a region in the complex plane \mathbb{C} . Suppose that φ is a mapping from $\mathbb{C}^2 \times \mathbb{E} \longrightarrow \mathbb{C}$ which satisfies $\varphi(xi, y, z) \notin F$ for $z \in \mathbb{E}$, and for all real x, y such that

$$y \le -\frac{n(1+x^2)}{2}.$$

If the functions $p(z) = 1 + c_n z^n + ...$ are regular in \mathbb{E} with p(0) = 1 and $\varphi(p(z), zp'(z), z) \in \mathcal{F}$ for all $z \in \mathbb{E}$, then $\Re(p(z)) > 0$, $(z \in \mathbb{E})$.

3. Main Results

Theorem 2. If $g(z) \in \mathcal{A}_m$ satisfies

$$\Re\left\{ \left[\left(1 + \frac{zg''(z)}{g'(z)} \right) \left(\frac{z^2g'''(z)}{z(g'(z))'} + 1 \right) \right] \right\} > \frac{1}{2} + (1 - \gamma)[(1 - \gamma) - m]$$
 (4)

for some γ (0 $\leq \gamma < 1$), then $g(z) \in C(\gamma)$.

Proof. Define p(z) by

$$(1 - \gamma)p(z) + \gamma = \frac{(zg'(z))'}{g'(z)}, \quad (z \in \mathbb{E}).$$
 (5)

Then $p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$ is regular in \mathbb{E} .

From (5), we have

$$(1 - \gamma)p(z) + \gamma = \frac{zg''(z) + g'(z)}{g'(z)} = \frac{zg''(z)}{g'(z)} + 1.$$
 (6)

Differentiating (5), we get

$$(zg'(z))'' = [(1 - \gamma)p(z) + \gamma]g''(z) + g'(z)(1 - \gamma)p'(z).$$

Now

$$\frac{(zg'(z))''}{(zg'(z))'} = \frac{[(1-\gamma)p(z)+\gamma]g''(z)+g'(z)(1-\gamma)p'(z)]}{[(1-\gamma)p(z)+\gamma]g'(z)}$$
$$= \frac{g''(z)}{g'(z)} + \frac{(1-\gamma)p'(z)}{[(1-\gamma)p(z)+\gamma]}$$

which gives

$$\frac{zg'''(z) + 2g''(z)}{(zg'(z))'} = \frac{g''(z)}{g'(z)} + \frac{(1 - \gamma)p'(z)}{[(1 - \gamma)p(z) + \gamma]},$$

or

$$\frac{zg'''(z)}{(zg'(z))'} + \frac{2g''(z)}{(zg''(z) + g'(z))} = \frac{g''(z)}{g'(z)} + \frac{(1 - \gamma)p'(z)}{[(1 - \gamma)p(z) + \gamma]} \ .$$

Multiplying throughout by z and using (6) we find that

$$\frac{zg'''(z)}{(zg'(z))'} + \frac{\frac{2g''(z)}{g'(z)}}{\frac{(zg''(z)+g'(z))}{g'(z)}} = (1-\gamma)p(z) + \gamma - 1 + \frac{(1-\gamma)p'(z)}{[(1-\gamma)p(z)+\gamma]},$$

$$\frac{zg'''(z)}{(zg'(z))'} + 1 = \frac{(1-\gamma)p'(z)}{[(1-\gamma)p(z)+\gamma]} - \frac{2[(1-\gamma)p(z)+\gamma-1]}{[(1-\gamma)p(z)+\gamma]} + (1-\gamma)p(z) + \gamma.$$
(7)

Using (6) and (7) we have that

$$\left(1 + \frac{zg''(z)}{g'(z)}\right) \left(\frac{z^2g'''(z)}{z(g'(z))'} + 1\right)
= (1 - \gamma)^2 p^2(z) + 2[\gamma(1 - \gamma) - (1 - \gamma)]p(z)
+ (1 - \gamma)zp'(z) + (\gamma)^2 - 2\gamma + 2$$

$$= (1 - \gamma)^2 p^2(z) - 2(1 - \gamma)^2 p(z) + (1 - \gamma)zp'(z) + (1 - \gamma)^2 + 1.$$

Let

$$\varphi(p(z), zp'(z), z) = (1 - \gamma)^2 p^2(z) - 2(1 - \gamma)^2 p(z) + (1 - \gamma)zp'(z) + (1 - \gamma)^2 + 1$$
 and

Then

$$\Re \left\{ \varphi(ix, y, z) \right\} = (1 - \gamma)y + (1 - \gamma)^2 + 1 - x^2(1 - \gamma)^2$$

$$\leq -\frac{m}{2}(1 - \gamma)(1 + x^2) + (1 - \gamma)^2 + 1 - x^2(1 - \gamma)^2$$

$$= -\frac{(1 - \gamma)}{2}[2(1 - \gamma) + m]x^2$$

$$+ \frac{(1 - \gamma)}{2}[2(1 - \gamma) + m] + 1$$

$$\leq \frac{1}{2} + (1 - \gamma)[(1 - \gamma) - m]$$

for $x, y \in \mathbb{R}$ and $y \leq -\frac{m}{2}(1+x^2)$.

By using Lemma 1, we conclude that $g \in \mathcal{C}(\gamma)$.

Letting $\gamma = 0$ and m = 1 in Theorem 2, we obtain

Corollary 3. If $g(z) \in A$ satisfies

$$\Re\left\{\left(1+\frac{zg''(z)}{g'(z)}\right)\left(\frac{z^2g'''(z)}{z(g'(z))'}+1\right)\right\} > \frac{3}{2} \quad (z \in \mathbb{E}),$$

then $g(z) \in \mathcal{C}$.

Theorem 4. If $g(z) \in A_m$ satisfies

$$\Re\left\{ \left[\left(1 + \frac{zg''(z)}{g'(z)} \right) \left(\mu \frac{z^2 g'''(z)}{z(g'(z))'} + 1 \right) \right] \right\} > \frac{\mu(1-\gamma)}{2} [2(2-\gamma) - m] + \gamma$$

for some γ (0 $\leq \gamma < 1$), $\mu \geq 0$, then $g(z) \in C(\gamma)$.

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Proof. From (5), we have

$$\frac{\mu z^2 g'''(z)}{(zg'(z))'} + 1 = \frac{\mu(1-\gamma)zq'(z)}{[(1-\gamma)p(z)+\gamma]} - \frac{2\mu[(1-\gamma)p(z)+\gamma-1]}{[(1-\gamma)p(z)+\gamma]} + \mu[(1-\gamma)p(z)+\gamma-1] + 1.$$
(8)

Now

$$\left(1 + \frac{zg''(z)}{g'(z)}\right) \left(\mu \frac{z^2 g'''(z)}{z(g'(z))'} + 1\right)$$

$$= \mu(1 - \gamma)zq'(z) - 2\mu[(1 - \gamma)p(z) + \gamma - 1] + \mu[(1 - \gamma)p(z) + \gamma - 1][(1 - \gamma)p(z) + \gamma] + (1 - \gamma)p(z) + \gamma$$

$$= \mu(1 - \gamma)zp'(z) - 2\mu[(1 - \gamma)(p(z) - 1)] + \mu(1 - \gamma)(p(z) - 1)[(1 - \gamma)p(z) + \gamma] + (1 - \gamma)p(z) + \gamma$$

$$= \mu(1 - \gamma)zp'(z) + (1 - \gamma)(-3\mu + 2\gamma\mu + 1)p(z) + \mu(1 - \gamma)^2p^2(z) + \mu(1 - \gamma)(2 - \gamma) + \gamma.$$

Let

$$\psi(p(z), zp'(z), z) = \mu(1 - \gamma)^2 p^2(z) - (1 - \gamma)(3\mu - 2\gamma\mu - 1)p(z) + \gamma(1 - \gamma)zp'(z) + \mu(1 - \gamma)(2 - \gamma) + \gamma$$

and

$$F = \left\{ \tau \in \mathbb{C} : \Re \ \tau > \frac{\mu(1-\gamma)}{2} [2(1-\gamma) - m] + \gamma \right\}.$$

Then

$$\begin{split} \Re\left\{\psi(ix,y,z)\right\} &= -\mu(1-\gamma)^2 x^2 + (1-\gamma)y + \mu(1-\gamma)(2-\gamma) + \gamma \\ &\leq -\mu(1-\gamma)^2 x^2 - \mu(1-\gamma)(1+x^2) \left(\frac{m}{2}\right) \\ &\quad + \mu(1-\gamma)(2-\gamma) + \gamma \\ &= -\left[\mu(1-\gamma)^2 + \frac{\mu m}{2}(1-\gamma)\right] x^2 \\ &\quad + \mu(1-\gamma)(2-\gamma) - \frac{\mu m}{2}(1-\gamma) + \gamma \\ &\leq \mu(1-\gamma)(2-\gamma) - \frac{\mu m}{2}(1-\gamma) + \gamma \\ &= \frac{\mu}{2}(1-\gamma)[2(1-\gamma) - m] + \gamma \end{split}$$

for real x and y and for $y \le -\frac{n}{2}(1+x^2)$.

By application of Lemma 1, we find that $g \in \mathcal{C}(\gamma)$.

Choosing $\gamma = 0$ and m = 1 in Theorem 4, we obtain

Corollary 5. If $g(z) \in A$ satisfies

$$\Re\left\{ \left(1 + \frac{zg''(z)}{g'(z)} \right) \left(\mu \frac{z^2 g'''(z)}{z(g'(z))'} + 1 \right) \right\} > \frac{3\mu}{2}$$

which by Lemma 1 implies that

$$\Re\left(1 + \frac{zg''(z)}{g'(z)}\right) > 0 \Rightarrow g(z) \in \mathcal{C},$$

for some $\mu \geq 0$.

Theorem 6. If $g(z) \in A_m$ satisfies

$$\Re\left\{ \left[\left(1 + \frac{zg''(z)}{g'(z)} \right) \left(\mu \frac{z^2 g'''(z)}{z(g'(z))'} + 1 \right) \right] \right\} > \frac{\gamma \mu}{2} [2(\gamma - 3) + m] + \frac{\mu}{2} (4 - m) + \gamma$$

for some γ $(0 \le \gamma < 1)$, $\mu \ge 0$ then $g(z) \in \mathcal{C}(\gamma)$.

Proof. From (8), we have

$$\left(1 + \frac{zg''(z)}{g'(z)}\right) \left(\mu \frac{z^2 g'''(z)}{z(g'(z))'} + 1\right)
= \mu(1 - \gamma)^2 p^2(z) + (1 - \gamma)(1 + 2\gamma\mu - 3\mu)p(z)
+ \mu(1 - \gamma)zp'(z) + \gamma\mu(\gamma - 3) + \gamma + 2\mu.$$

Let

$$\eta(p(z), zp'(z), z) = \mu(1 - \gamma)^2 p^2(z) + (1 - \gamma)(1 + 2\gamma\mu - 3\mu)p(z) + \mu(1 - \gamma)zp'(z) + \gamma\mu(\gamma - 3) + (\gamma + 2\mu)$$

and

$$F = \left\{ \tau \in \mathbb{C} : \Re \tau > \frac{\gamma \mu}{2} [2(\gamma - 3) + m] + \frac{\mu}{2} (4 - m) + \gamma \right\}.$$

Then

$$\Re \left\{ \eta(ix, y, z) \right\} = \mu(1 - \gamma)^2 (-x^2) + \mu(1 - \gamma)y + \gamma \mu(\gamma - 3) + (\gamma + 2\mu)$$

$$\leq -\mu(1 - \gamma)^2 x^2 - \frac{\mu m}{2} (1 - \gamma)(1 + x^2)$$

$$+ \gamma \mu(\gamma - 3) + (\gamma + 2\mu)$$

$$= -\mu(1 - \gamma)[2(1 - \gamma) + m]x^2 - \frac{\mu m}{2} (1 - \gamma)$$

$$+\gamma\mu(\gamma-3) + (\gamma+2\mu)$$

$$\leq \frac{\gamma\mu}{2}[2(\gamma-3)+m] + \frac{\mu}{2}(4-m) + \gamma$$

for real x and y for $y \leq -\frac{n}{2}(1+x^2)$, which shows that $\Re[\eta(ix,y,z)] \notin F$. Therefore by Lemma 1, we have $\Re(p(z)) > 0$, that is,

$$\Re\left\{\left(1 + \frac{zg''(z)}{g'(z)}\right)\right\} > \gamma$$

which implies that $g \in \mathcal{C}(\gamma)$.

Letting $\gamma = 0$ and m = 1 in Theorem 6, we obtain

Corollary 7. If $g(z) \in A$ satisfies

$$\Re\left\{\left(1+\frac{zg''(z)}{g'(z)}\right)\left(\mu\frac{z^2g'''(z)}{z(g'(z))'}+1\right)\right\}>\frac{3\mu}{2}$$

for some $\mu \geq 0$ which by Lemma 1 implies that $\Re(p(z)) > 0$, that is,

$$\Re\left(1 + \frac{zg''(z)}{g'(z)}\right) > 0 \Rightarrow g(z) \in \mathcal{C}.$$

Theorem 8. If $g(z) \in A_m$ satisfies

$$\Re\left\{ \left[\left(1 + \frac{zg''(z)}{g'(z)} \right) \left(\mu \frac{z^2 g'''(z)}{z(g'(z))'} \right) \right] \right\} \ge \frac{-\mu(1-\gamma)}{2} [2(\gamma-2) + m]$$

for some γ (0 $\leq \gamma < 1$), $\mu \geq 0$, then $g(z) \in C(\gamma)$.

Proof. From (6) and (8) we have

$$\left(1 + \frac{zg''(z)}{g'(z)}\right) \left(\mu \frac{z^2 g'''(z)}{z(g'(z))'}\right)
= [(1 - \gamma)p(z) + \gamma]
\left[\frac{\mu(1 - p\gamma)zp'(z)}{(1 - \gamma)p(z) + \gamma} - \frac{2\mu[(1 - \gamma)p(z) + \gamma - 1]}{(1 - \gamma)p(z) + \gamma} + \mu(1 - \gamma)p(z) + \gamma - 1\right]
= \mu(1 - \gamma)^2 p^2(z) + \mu(1 - \gamma)zp'(z)
+ \mu(1 - \gamma)(2\gamma - 3)p(z)\mu(\gamma - 1)(\gamma - 2).$$

Let

$$\delta(p(z), zp'(z), z) = \mu(1 - \gamma)^2 p^2(z) + \mu(1 - \gamma)zp'(z) + \mu(1 - \gamma)(2\gamma - 3)p(z) + \mu(\gamma - 1)(\gamma - 2)$$

and

$$\digamma = \left\{ \tau \in \mathbb{C} : \Re \ \tau > \frac{-\mu(1-\gamma)}{2} [2(\gamma-2) + m] \right\}.$$

Then

$$\Re \left\{ \delta(ix, y, z) \right\} = -\mu (1 - \gamma)^2 (x^2) + \mu (1 - \gamma) y + \mu (\gamma - 1) (\gamma - 2)$$

$$\leq -\mu (1 - \gamma)^2 x^2 - \frac{\mu m}{2} (1 - \gamma) (1 + x^2)$$

$$+ \mu (\gamma - 1) (\gamma - 2)$$

$$\leq -\frac{\mu m}{2} (1 - \gamma) + \mu (\gamma - 1) (\gamma - 2)$$

$$= -\frac{\mu (1 - \gamma)}{2} [2(\gamma - 2) + m]$$

for all real x and y for $y \le -\frac{m}{2}(1+x^2)$.

Which implies that $\Re[\delta(ix, y, z)] \notin F$.

Hence by Lemma 1, we have $\Re(p(z)) > 0$, that is,

$$\Re\left\{\left(1+\frac{zg''(z)}{g'(z)}\right)\right\} > \gamma \Rightarrow g(z) \in \mathcal{C}(\gamma).$$

Choosing $\gamma = 0$ and m = 1 in Theorem 8, we obtain

Corollary 9. If $g(z) \in A$ satisfies

$$\Re\left\{\left(1 + \frac{zg''(z)}{g'(z)}\right) \left(\mu \frac{z^2 g'''(z)}{z(g'(z))'}\right)\right\} \ge \frac{3\mu}{2}$$

for some $\beta \geq 0$, then $g \in \mathcal{C}$.

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