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γ^* -OPERATION IN THE PRODUCT OF TOPOLOGICAL SPACES

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Abstract: In this paper, we define the concept of γ^* -operation in the topological product of topological spaces $(X_i, \mathcal{T}_i), i \in I$, through γ_i -operations of spaces $(X_i, \mathcal{T}_i), i \in I$. In addition, we examine whether the product of γ_i -open sets from $(X_i, \mathcal{T}_i), i \in I$ spaces, is a γ^* -open set in the topological product of these spaces. Further, we examine the γ^* -continuity of projections $p_i : \prod_{i \in I} X_i \to X_i$. Also we prove, whether the product of $\gamma_i - T_0$ spaces, $\gamma_i - T_1$ spaces and γ_i -Hausdorff's spaces, is $\gamma^* - T_0$ space, $\gamma^* - T_1$ space or γ^* -Hausdorff space, respectively.

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1. Introduction

The notion of γ -operation in a topological space and the notion of the γ -open set were introduced by the Japanese mathematician H. Ogata in 1991. Further, through the notion of the γ -open set, Ogata has defined the $\gamma - T_i (i = 0, \frac{1}{2}, 1, 2)$ spaces.

In 1992, Rehman and Ahmad defined the γ -interior, γ -exterior, γ -closure and γ -boundary of a subset of a topological space (see [6]).

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In 2003 Ahmad and Hussain studied many properties of a γ -operation in a topological space, they defined the meaning of γ -neighbourhood as well as γ -neighbourhood of the base (see [1]).

In 2009 Basu, Uzzal Afsan and Chash defined γ -continuity of function (see [3]).

In 2012 Ibrahim defined another type of continuous functions, the almost γ -continuous functions (see [4]).

2. Preliminaries

Definition 1. ([5]) Let (X, \mathcal{T}) be the topological space. The function $\gamma : \mathcal{T} \to \mathcal{P}(X)(\mathcal{P}(X))$, is the partition of X), such that for each $G \in \mathcal{T}, G \subseteq \gamma(G)$, is called γ -operation in the topological space (X, \mathcal{T}) .

Examples of γ -operation in a topological space (X, \mathcal{T}) , are the functions: $\gamma : \mathcal{T} \to \mathcal{P}(X)$, given by: $\gamma(G) = G, \gamma(G) = clG, \gamma(G) = int(clG)$, etc.

Definition 2. ([5]) Let (X, \mathcal{T}) be the topological space and $\gamma : \mathcal{T} \to \mathcal{P}(X)$ a γ -operation in this space. The set $G \subseteq X$ is called γ -open, if for every $x \in G$, there exists the set $U \in \mathcal{T}$, such that $x \in U$ and $\gamma(U) \subseteq G$.

The family of all γ -open sets of topological space (X, \mathcal{T}) is written by: \mathcal{T}_{γ} . This means:

$$\mathcal{T}_{\gamma} = \{ G \subseteq X : \forall x \in G, \exists U \in \mathcal{T} : x \in U \subseteq \gamma(U) \subseteq G \}.$$

We note that any γ -open set is open, i.e. $\mathcal{T}_{\gamma} \subseteq \mathcal{T}$. It is true, because from $G \in \mathcal{T}_{\gamma}$ it follows that for every $x \in G$, there exists $U_x \in \mathcal{T}$ such that $x \in U_x \subseteq \gamma(U_x) \subseteq G$, so the union of sets $U_x \in \mathcal{T}$ is $\cup \{U_x : x \in G\} = G \in \mathcal{T}$.

Definition 3. ([2]) The set F, is called γ -closed, if its complement $F^C = X \setminus F$ is γ -open set in X.

Theorem 4. ([2]) Let (X, \mathcal{T}) be the topological space and $\gamma : \mathcal{T} \to \mathcal{P}(X)$ a γ -operation in this space, then:

- 1. If for every $i \in I, G_i \in \mathcal{T}_{\gamma}$ and then $\bigcup_{i \in I} G_i \in \mathcal{T}_{\gamma}$.
- 2. If $G_1, G_2 \in \mathcal{T}_{\gamma}$, then $G_1 \cap G_2$, is not always the γ -open set.

- Proof. 1. Let $x \in \bigcup_{i \in I} G_i(G_i \in \mathcal{T}_{\gamma})$. Then, there exists $i_0 \in I$, such that $x \in G_{i_0}$. Since $G_{i_0} \in \mathcal{T}_{\gamma}$, there exists $G \in \mathcal{T}$ such as $x \in G$ and $\gamma(G) \subseteq G_{i_0} \subseteq \bigcup_{i \in I} G_i$. It means $\bigcup_{i \in I} G_i \in \mathcal{T}_{\gamma}$.
- 2. That the intersection of the two γ -open sets is not always a γ -open set, shows the following example.

Example 1. Let $X = \{a, b, c\}$ with the topology

$$\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}.$$

In X, we define γ -operation in this way:

$$\gamma(G) = \begin{cases} G, b \in G \\ clG, b \notin G \end{cases}.$$

We define the \mathcal{T}_{γ} family, by proving which of the sets from the \mathcal{T} family, complete the terms of Definition 2.

We consider the set $\{a\}$. Sets $\{a\}, \{a, b\}$ and $\{a, c\}$, are open sets that contain the single element a, of the set $\{a\}$. On the other hand, we have:

$$\gamma(\{a\}) = cl\{a\} = \{a, c\} \not\subseteq \{a\}, \gamma(\{a, b\}) = \{a, b\} \not\subseteq \{a\}, \gamma(\{a, c\}) = \{a, c\} \not\subseteq \{a\}. \text{ It means } \{a\} \not\in \mathcal{T}_{\gamma}.$$

Similarly proving to all open sets of given space, we find that:

$$\mathcal{T}_{\gamma} = \{\emptyset, \{b\}, \{a, b\}, \{a, c\}, X\}.$$

We notice that $\{a,b\} \in \mathcal{T}_{\gamma}$ and $\{a,c\} \in \mathcal{T}_{\gamma}$, but $\{a,b\} \cap \{a,c\} = \{a\} \notin \mathcal{T}_{\gamma}$.

Remark 1. The statement 2. of Theorem 4 shows that \mathcal{T}_{γ} , is not always the topology in X.

Remark 2. It is easily shown by examples that the union of sets that are not γ -open is not necessarily γ -open set.

Based on Theorem 4 and De-Morgan laws, it follows:

Theorem 5. Let (X, \mathcal{T}) be the topological space and $\gamma : \mathcal{T} \to \mathcal{P}(X)$ a γ -operation in this space, then:

- 1. If for any $i \in I$, F_i is the γ -closed set in X, then $\cap_{i \in I} F_i$ is the γ -closed set in X.
- 2. If F_1, F_2 , are γ -closed sets in X, then $F_1 \cup F_2$, it is not always the γ -closed set.

The statement 2. of Theorem 5 is proved by Example 1, according to which $\{c\}^C = \{a,b\} \in \mathcal{T}_{\gamma}$ and $\{b\}^C = \{a,c\} \in \mathcal{T}_{\gamma}$. So, the sets $\{c\}$ and $\{b\}$, are γ -closed, but their union $\{b,c\}$, is not γ -closed because $\{b,c\}^C = \{a\} \notin \mathcal{T}_{\gamma}$.

The following example shows that the difference of the γ -open (γ -closed) sets is not always γ -open (γ -closed):

Example 2. Let $X = \{a, b, c, d\}$ with the topology

$$\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, X\}$$

and γ -operation in X, given with:

$$\gamma(G) = \begin{cases} int(clG) : G = \{a\} \\ clG : G \neq \{a\} \end{cases}.$$

It is easy to prove that:

$$\mathcal{T}_{\gamma} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b, d\}, X\}.$$

The γ -closed sets are: \emptyset , $\{b, c, d\}$, $\{a, b, d\}$, $\{c\}$ and X. We notice that $\{a, b, d\} \in \mathcal{T}_{\gamma}$ and $\{a, c\} \in \mathcal{T}_{\gamma}$, but $\{a, b, d\} \setminus \{a, c\} = \{b, d\} \notin \mathcal{T}_{\gamma}$.

Also, we note that $\{a, b, d\}$ and $\{b, d\}$, are a γ -closed sets, but their difference is the set $\{a\}$, which is not a γ -closed set.

Definition 6. ([3]) Let (X, \mathcal{T}) and (Y, \mathcal{T}') be two topological space and $\gamma : \mathcal{T} \to \mathcal{P}(X)$ a γ -operation in (X, \mathcal{T}) . The function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is called the γ -continuous if the inverse image of any V-open set in Y is the γ -open set in X.

Definition 7. ([5]) Let (X, \mathcal{T}) be the topological space and $\gamma : \mathcal{T} \to \mathcal{P}(X)$ a γ -operation in (X, \mathcal{T}) . The space (X, \mathcal{T}) is called $\gamma - T_0$, if for any two different points $x, y \in X$, there exists an open set U such that $x \in U$ and $y \notin \gamma(U)$.

Definition 8. ([5]) Let (X, \mathcal{T}) be the topological space and $\gamma : \mathcal{T} \to \mathcal{P}(X)$ a γ -operation in (X, \mathcal{T}) . The space (X, \mathcal{T}) is called $\gamma - T_1$, if for any two different points $x, y \in X$, there exists an open set U and V such that $x \in U, y \in V$ and $x \notin \gamma(V), y \notin \gamma(U)$.

Definition 9. ([5]) Let (X, \mathcal{T}) be the topological space and $\gamma : \mathcal{T} \to \mathcal{P}(X)$ a γ -operation in (X, \mathcal{T}) . The space (X, \mathcal{T}) is called γ -Hausdorff, if for every two different points $x, y \in X$, there exists an open set U and V such that $x \in U, y \in V$ and $\gamma(U) \cap \gamma(V) = \emptyset$.

3. Main Results

Naturally, the following questions are raised:

- 1. Let (X, \mathcal{T}) be the topological space and $\gamma : \mathcal{T} \to \mathcal{P}(X)$ a γ -operation in (X, \mathcal{T}) . Let $X^n = X \times X \times ... \times X$, be the topological product of n-samples of space X. Can a γ^* -operation be induced in (X^n, \mathcal{T}_{X^n}) , through γ -operation of space (X, \mathcal{T}) , and is the product $\prod_{i=1}^n G_i, G_i \in \mathcal{T}_{\gamma}, \gamma^*$ -open set in the space (X^n, \mathcal{T}_{X^n}) ?
- 2. Let (X, \mathcal{T}) be the topological space and $\gamma_1, \gamma_2, ..., \gamma_n, \gamma_i$ -operations, i = 1, 2, 3, ..., n, in this space. Can it be induced any γ^* -operation in the space (X^n, \mathcal{T}_{X^n}) , through γ_i -operations, in the space (X, \mathcal{T}) and is the product $\prod_{i=1}^n G_i, G_i \in \mathcal{T}_{\gamma_i}, \gamma^*$ -open set in the space (X^n, \mathcal{T}_{X^n}) ?
- 3. Let (X_i, \mathcal{T}_i) , i = 1, 2, 3, ..., n and $\gamma_i : \mathcal{T}_i \to \mathcal{P}(X_i)$, γ_i —operations in (X_i, \mathcal{T}_i) , i = 1, 2, 3, ..., n. Can it be induced any γ^* -operation in the space $(\prod_{i=1}^n X_i, \mathcal{T}_{\prod_{i=1}^n X_i})$, through γ_i -operations, in the spaces (X_i, \mathcal{T}_i) , i = 1, 2, 3, ..., n and is the product $\prod_{i=1}^n G_i, G_i \in \mathcal{T}_{\gamma_i}, \gamma^*$ —open set in the space $(\prod_{i=1}^n X_i, \mathcal{T}_{\prod_{i=1}^n X_i})$?
- 4. What if, in each of the above cases, n is an infinite number? In other words, can the product of an infinite number of γ_i -open sets be γ^* -open set?

We note that the problems raised in question (1) and (2) are particular cases of the problem raised in question (3).

If (X_i, \mathcal{T}_i) , i = 1, 2, 3, ..., n, are the topological spaces, with (X, \mathcal{T}) we denote the topological product of spaces (X_i, \mathcal{T}_i) , i = 1, 2, 3, ..., n, where $X = \prod_{i=1}^n X_i$ and \mathcal{T} is the topology of product (Tihonov) in X. We recall that the base of topology of product \mathcal{T} is formed by all sets of the form $B = G_1 \times G_2 \times \cdots \times G_n$, where $G_i \in \mathcal{T}_i$, i = 1, 2, ..., n, whereas the open sets of product \mathcal{T} are of form $U = \bigcup \{B_i : i \in I\}$, where $B_i = G_{i_1} \times G_{i_2} \times ... \times G_{i_{n_i}}$, $G_{i_j} \in \mathcal{T}_{i_j}$, $j = 1, 2, ..., n_i$, $i \in I$, so $U = \bigcup \{G_{i_1} \times G_{i_2} \times ... \times G_{i_{n_i}} : i \in I\}$.

The answer to question (3) is given by the following theorem.

Theorem 10. Let (X, \mathcal{T}) be the topological product of spaces (X_i, \mathcal{T}_i) , i = 1, 2, 3, ..., n and $\gamma_i : \mathcal{T}_i \to \mathcal{P}(X_i)$ let be γ_i -operations in (X_i, \mathcal{T}_i) , i = 1, 2, 3, ..., n. The function $\gamma^* : \mathcal{T} \to \mathcal{P}(X)$ defined by $\gamma^*(U) = \bigcup \{\gamma_{i_1}(G_{i_1}) \times \gamma_{i_2}(G_{i_2}) \times ... \times \gamma_{i_{n_i}}(G_{i_{n_i}}), i \in I\}$ is a γ^* -operation in (X, \mathcal{T}) . Moreover, if $A_i \in \mathcal{T}_{\gamma_i}$, i = 1, 2, ..., n, then $A_1 \times A_2 \times ... \times A_n \in \mathcal{T}_{\gamma^*}$.

Proof. First we will show that $\gamma^* : \mathcal{T} \to \mathcal{P}(X)$ is $\gamma * - \text{operation in } (X, \mathcal{T})$. Let be $U \in \mathcal{T}$. Then

$$U = \bigcup \{G_{i_1} \times G_{i_2} \times ... \times G_{i_{n_i}} : i \in I\} \subseteq \bigcup \{\gamma_{i_1}(G_{i_1}) \times \gamma_{i_2}(G_{i_2}) \times ... \times \gamma_{i_{n_i}}(G_{i_{n_i}}), i \in I\} = \gamma^*(U).$$

By the definition of γ^* – operation in (X, \mathcal{T}) , we notice that for the open sets of the base, we have

$$\gamma^*(G_1 \times G_2 \times ... \times G_n) = \gamma_1(G_1) \times \gamma_2(G_2) \times ... \times \gamma_n(G_n), G_i \in \mathcal{T}_i,$$

$$i = 1, 2, ..., n.$$

Let $A_i \in \mathcal{T}_{\gamma_i}, i = 1, 2, ..., n$ and $(x_1, x_2, ..., x_n) \in A_1 \times A_2 \times ... \times A_n$ be any point. Then, $x_i \in A_i, i = 1, 2, ..., n$ and since $A_i \in \mathcal{T}_{\gamma_i}$, there exists $G_i \in \mathcal{T}_i, i = 1, 2, ..., n$, such that $x_i \in G_i \subseteq \gamma_i(G_i) \subseteq A_i$. Therefore, $(x_1, x_2, ..., x_n) \in G_1 \times G_2 \times ... G_n \subseteq \gamma_1(G_1) \times \gamma_2(G_2) \times ... \times \gamma_n(G_n) = \gamma^*(G_1 \times G_2 \times ... \times G_n) \subseteq A_1 \times A_2 \times ... \times A_n$. Since, $G_1 \times G_2 \times ... \times G_n$ is open (base) set in (X, \mathcal{T}) , it turns out that $A_1 \times A_2 \times ... \times A_n$ is a γ^* - open set in (X, \mathcal{T}) .

Theorem 11. Let (X^n, \mathcal{T}_{X^n}) be the *n*-topological power of space $(X, \mathcal{T}), \gamma_i : \mathcal{T} \to \mathcal{P}(X), i = 1, 2, ..., n$ and let be the n, γ -operations in (X, \mathcal{T}) . The function $\gamma^* : \mathcal{T}_{X^n} \to \mathcal{P}(X^n)$ defined with $\gamma^*(U) = \bigcup \{\gamma_{i_1}(G_{i_1}) \times \gamma_{i_2}(G_{i_2}) \times ... \times \gamma_{i_{n_i}}(G_{i_{n_i}}), i \in I\}$ is a γ^* -operation in (X^n, \mathcal{T}_{X^n}) . In addition, if $A_i \in \mathcal{T}_{\gamma_i} : i = 1, 2, ..., n$, then $A_1 \times A_2 \times ... \times A_n$, is γ^* -open in (X^n, \mathcal{T}_{X^n}) .

Proof. It follows directly from Theorem 10, for $X_1 = X_2 = \dots = X_n = X$.

Theorem 12. Let (X^n, \mathcal{T}_{X^n}) be the n- topological power of space (X, \mathcal{T}) and $\gamma: \mathcal{T} \to \mathcal{P}(X)$ let be γ -operations in (X, \mathcal{T}) . The function $\gamma^*: \mathcal{T}_{X^n} \to \mathcal{P}(X^n)$ defined with $\gamma^*(U) = \bigcup \{\gamma_{i_1}(G_{i_1}) \times \gamma_{i_2}(G_{i_2}) \times ... \times \gamma_{i_{n_i}}(G_{i_{n_i}}), i \in I\}$ is a γ^* -operation in (X^n, \mathcal{T}_{X^n}) . In addition, if $A_i \in \mathcal{T}_{\gamma_i}: i = 1, 2, ..., n$, then $A_1 \times A_2 \times ... \times A_n$, is γ^* -open in (X^n, \mathcal{T}_{X^n}) .

Proof. It follows directly from Theorem 10, for $X_1 = X_2 = ... = X_n = X$ and $\gamma_1 = \gamma_2 = ... = \gamma_n = \gamma$.

The answer to question 4. (outlined above) is negative, because the product of an infinite number of open sets is not necessarily the open set of topological product and as such can not be a γ^* -open set, because $\mathcal{T}_{\gamma^*} \subseteq \mathcal{T}$.

Remark 3. The product $\prod_{i\in I}G_i$ of an infinite number of γ_i -opened sets from the spaces (X_i, \mathcal{T}_i) is γ^* -open set of product, if $G_i = X_i$, for almost all $i \in I$. True, from $X_i = \gamma_i(X_i), \forall i \in I$, we have $G_{i_1} \times G_{i_2} \times \ldots \times G_{i_n} \times \prod_{i \in I \setminus \{i_1, i_2, \ldots, i_n\}} X_i = G_{i_1} \times G_{i_2} \times \ldots \times G_{i_n} \times \prod_{i \in I \setminus \{i_1, i_2, \ldots, i_n\}} \gamma_i(X_i) = \gamma^*(G_{i_1} \times G_{i_2} \times \ldots \times G_{i_n} \times \prod_{i \in I \setminus \{i_1, i_2, \ldots, i_n\}} X_i)$.

So for $x \in G_{i_1} \times G_{i_2} \times ... \times G_{i_n} \times \prod_{i \in I \setminus \{i_1, i_2, ..., i_n\}} X_i$, the set $G_{i_1} \times G_{i_2} \times ... \times G_{i_n} \times \prod_{i \in I \setminus \{i_1, i_2, ..., i_n\}} X_i$ is the open set that contains x and $\gamma^*(G_{i_1} \times G_{i_2} \times ... \times G_{i_n} \times \prod_{i \in I \setminus \{i_1, i_2, ..., i_n\}} X_i) \subseteq G_{i_1} \times G_{i_2} \times ... \times G_{i_n} \times \prod_{i \in I \setminus \{i_1, i_2, ..., i_n\}} X_i$.

Example 3. Let $X = \{a, b\}$ and $Y = \{c, d\}$ be spaces provided with topologies $\mathcal{T}_1 = \{\emptyset, \{a\}, X\}, \mathcal{T}_2 = \{\emptyset, \{c\}, Y\}$, respectively. We assign $\mathcal{T}_{X \times Y}$. First, we define the sub base of topology of product,

$$\begin{split} p_1^{-1}(\emptyset) &= \emptyset \times Y = \emptyset, \\ p_1^{-1}(X) &= X \times Y, \\ p_1^{-1}(\{a\}) &= \{a\} \times Y = \{(a,c),(a,d)\}, \\ p_2^{-1}(\emptyset) &= X \times \emptyset = \emptyset, \\ p_2^{-1}(Y) &= X \times Y, \\ p_2^{-1}(\{c\}) &= X \times \{c\} = \{(a,c),(b,c)\}. \end{split}$$

This means

$$S = \{\emptyset, X \times Y, \{(a, c), (a, d)\}, \{(a, c), (b, c)\}\}.$$

Since all finite intersection of members of S represent the topology base of product, it turns out that:

$$B = \{\emptyset, X \times Y, \{(a,c), (a,d)\}, \{(a,c), (b,c)\}, \{(a,c)\}\}.$$

Consequently,

$$\mathcal{T}_{X\times Y} = \{\emptyset, X \times Y, \{(a,c), (a,d)\}, \{(a,c), (b,c)\}, \{(a,c)\}, \{(a,c), (a,d), (b,c)\}\}.$$

Let $\gamma_1: \mathcal{T}_1 \to P(X)$ and $\gamma_2: \mathcal{T}_2 \to P(Y), \gamma$ -operation in the given spaces, defined by $\gamma_1(G) = G, \forall G \in \mathcal{T}_1$ and $\gamma_2(H) = clH, \forall H \in \mathcal{T}_2$.

We have: $\gamma_1(\emptyset) = \emptyset, \gamma_1(X) = X, \gamma_1(\{a\}) = \{a\} \text{ and } \gamma_2(\emptyset) = cl\emptyset = \emptyset, \gamma_2(Y) = clY = Y \text{ and } \gamma_2(\{c\}) = cl\{c\} = Y.$

 $\mathcal{T}_{X\times Y}$ will be presented in the form:

$$\mathcal{T}_{X\times Y} = \{\emptyset, X\times Y, \{a\}\times Y, X\times \{c\}, \{a\}\times \{c\}, (\{a\}\times Y)\cup (X\times \{c\})\}.$$

From the definition of function γ^* , we have:

$$\gamma^{*}(\emptyset) = \gamma^{*}(\emptyset \times \emptyset) = \gamma_{1}(\emptyset) \times \gamma_{2}(\emptyset) = \emptyset \times cl\emptyset = \emptyset,$$

$$\gamma^{*}(X \times Y) = \gamma_{1}(X) \times \gamma_{2}(Y) = X \times Y$$

$$= \{(a, c), (a, d), (b, c), (b, d)\},$$

$$\gamma^{*}(\{(a, c), (a, d)\}) = \gamma^{*}(\{a\} \times Y) = \gamma_{1}(\{a\}) \times \gamma_{2}(Y) = \{a\} \times Y$$

$$= \{(a, c), (a, d)\},$$

$$\gamma^{*}(\{(a, c), (b, c)\}) = \gamma^{*}(X \times \{c\}) = X \times Y$$

$$= \{(a, c), (a, d), (b, c), (b, d)\},$$

$$\gamma^{*}(\{(a, c)\}) = \gamma^{*}(\{a\} \times \{c\}) = \{a\} \times Y = \{(a, c), (a, d)\},$$

$$\gamma^{*}(\{(a, c), (a, d), (b, c)\}) = \gamma^{*}((\{a\} \times Y) \cup (X \times \{c\}))$$

$$= (\gamma_{1}\{a\} \times \gamma_{2}(Y)) \cup (\gamma_{1}(X) \times \gamma_{2}\{c\}) = (\{a\} \times Y) \cup (X \times Y)$$

$$= X \times Y.$$

Now we examine which of the open sets of $\mathcal{T}_{X\times Y}$ completes the terms of Definition 2.

Clearly, empty sets and the set $X \times Y$ complete the terms of Definition 2. Consider the set $\{(a,c),(a,d)\}.$

For element (a,c), the set $\{(a,c)\}$ is open set that contains the element (a,c) and $\gamma^*(\{(a,c)\}) = \{(a,c),(a,d)\} \subseteq \{(a,c),(a,d)\}.$

For element (a, d), the set $\{(a, c), (a, d)\}$ is open set that contains that element and $\gamma^*(\{(a, c), (a, d)\}) = \{(a, c), (a, d)\} \subseteq \{(a, c), (a, d)\}$.

It means that $\{(a,c),(a,d)\}$ is γ^* -open set.

Similarly, we notice that the sets $\{(a,c),(b,c)\},\{(a,c)\}$ and $\{(a,c),(a,d),(b,c)\}$ do not complete the terms of Definition 2.

So, finally we have that $\mathcal{T}_{X\times Y}^{\gamma} = \{\emptyset, \{(a,c), (a,d)\}, X\times Y\}.$

Whether the production of γ -closed sets, is the γ -closed set, shows the following theorem.

Theorem 13. Let (X, \mathcal{T}) be the topological product of spaces (X_i, \mathcal{T}_i) , i = 1, 2, 3, ..., n and $\gamma_i : \mathcal{T}_i \to \mathcal{P}(X_i)$ let be γ_i -operations in (X_i, \mathcal{T}_i) , i = 1, 2, 3, ..., n. Further, let be $F_i \subseteq X_i$, γ_i -closed sets in X_i , i = 1, 2, 3, ..., n. Then,

- 1. $\prod_{i=1}^n F_i$ is γ^* closed set in $(\prod_{i=1}^n X_i, \mathcal{T}_{\prod_{i=1}^n X_i})$.
- 2. $\prod_{i \in I} F_i$ is γ^* -closed set in $(\prod_{i \in I} X_i, \mathcal{T}_{\prod_{i \in I} X_i})$.

Proof. 1. From $F_i \subseteq X_i, i = 1, 2, 3, ..., n$, we have

$$(F_1 \times F_2 \times \dots \times F_n)^C = (X_1 \times X_2 \times \dots \times X_n) \setminus (F_1 \times F_2 \times \dots \times F_n)$$
$$= [(X_1 \setminus F_1) \times X_2 \times \dots \times X_n] \cup [X_1 \times (X_2 \setminus F_2) \times \dots \times X_n]$$
$$\cup \dots \cup [X_1 \times X_2 \times \dots \times (X_n \setminus F_n)].$$

Since

$$X_i \in \mathcal{T}_{\gamma_i}, i = 1, 2, 3, ..., n,$$

and

$$X_j \backslash F_j \in T_{\gamma_j}, j = 1, 2, 3, ..., n,$$

it follows that $(F_1 \times F_2 \times ... \times F_n)^C$ is a union of γ^* -open sets (the definition of the γ^* -function) and as such is a γ^* -open set, that means $F_1 \times F_2 \times ... \times F_n$ is γ^* -closed.

2. From $F_i \subseteq X_i, i \in I (\prod_{i \in I} F_i)^C = \bigcup_{j \in I} (\prod_{i \in I} H_i : H_i = X_i, \forall i \neq j, H_j = X_j \setminus F_j)$.

So, based on Remark 3, $(\prod_{i \in I} F_i)^C$ is the γ^* -open in

 $(\prod_{i\in I} X_i, \mathcal{T}_{\prod_{i\in I} X_i})$ as union of the γ^* -open sets. Consequently $\prod_{i\in I} F_i$ is the γ^* -closed in $(\prod_{i\in I} X_i, \mathcal{T}_{\prod_{i\in I} X_i})$.

Theorem 14. Let (X, \mathcal{T}) be the topological product of spaces (X_i, \mathcal{T}_i) , i = 1, 2, 3, ..., n and $\gamma_i : \mathcal{T}_i \to \mathcal{P}(X_i)$ let be γ_i -operations in (X_i, \mathcal{T}_i) , $i \in I$. Then projections $p_i : X \to X_i$, given by $p_i(x) = x(i) = x_i, \forall i \in I$, are γ^* -continuous functions.

Proof. Let $V_i \in \mathcal{T}_i$, then we have to prove that $p_i^{-1}(V_i) = V_i \times \prod_{j \in I \setminus \{i\}} X_j$ is γ^* -open set. Let $x \in V_i \times \prod_{j \in I \setminus \{i\}} X_j$. From $\gamma_j(X_j) = X_j, \forall j \in I \setminus \{i\}$, we have

$$V_i \times \prod_{j \in I \setminus \{i\}} X_j = V_i \times \prod_{j \in I \setminus \{i\}} \gamma_j(X_j) = \gamma^*(V_i \times \prod_{j \in I \setminus \{i\}} X_j).$$

So the set $V_i \times \prod_{j \in I \setminus \{i\}} X_j$ is open in (X, \mathcal{T}) which contains x and $\gamma^*(V_i \times \prod_{j \in I \setminus \{i\}} X_j) \subseteq V_i \times \prod_{j \in I \setminus \{i\}} X_j$. It means $p_i : X \to X_i$ is γ^* -continuous functions, for $\forall i \in I$.

Theorem 15. Let (X, \mathcal{T}) be the topological product of spaces (X_i, \mathcal{T}_i) , i = 1, 2, 3, ..., n and $\gamma_i : \mathcal{T}_i \to \mathcal{P}(X_i)$ let be γ_i -operations in (X_i, \mathcal{T}_i) , $i \in I$. If (X_i, \mathcal{T}_i) , $i \in I$ are $\gamma_i - T_0$ spaces, than (X, \mathcal{T}) is $\gamma^* - T_0$ space.

Proof. Let $x, y \in X$ and $x \neq y$. Then, there exists k, such as the k-coordinate of x and y be different. It means, $x_k \neq y_k$. Since, $x_k, y_k \in X_k$ and X_k is $\gamma_k - T_0$, there exists $U_k, V_k \in \mathcal{T}_k$, such that $x_k \in U_k$ and $y_k \notin \gamma_k(U_k)$. Now $p_k^{-1}(U_k)$ is the open set in X that contains x and $y \notin p_k^{-1}(\gamma_k(U_k)) \Rightarrow$

$$y \notin \gamma_k(U_k) \times \prod_{i \in I \setminus \{k\}} X_i = \gamma_k(U_k) \times \prod_{i \in I \setminus \{k\}} \gamma_i(X_i) = \gamma^*(U_k \times \prod_{i \in I \setminus \{k\}} X_i).$$

It means that $y \notin \gamma^*(U_k \times \prod_{i \in I \setminus \{k\}} X_i) = \gamma^*(p_k^{-1}(U_k))$, consequently (X, \mathcal{T}) is $\gamma^* - T_0$.

Theorem 16. Let (X, \mathcal{T}) be the topological product of spaces (X_i, \mathcal{T}_i) , i = 1, 2, 3, ..., n and $\gamma_i : \mathcal{T}_i \to \mathcal{P}(X_i)$ let be γ_i -operations in (X_i, \mathcal{T}_i) , $i \in I$. If (X_i, \mathcal{T}_i) , $i \in I$ are $\gamma_i - T_1$ spaces, then (X, \mathcal{T}) is $\gamma^* - T_1$ space.

Proof. Let $x,y\in X$ and $x\neq y$. Then, there exists k such as the k-coordinate of x and y be different. It means, $x_k\neq y_k$. Since, $x_k,y_k\in X_k$ and X_k is γ_k-T_1 , there exists $U_k,V_k\in \mathcal{T}_k$, such that $x_k\in U_k,y_k\in V_k,x_k\notin \gamma_k(V_k),y_k\notin \gamma_k(U_k)$. Now $p_k^{-1}(U_k)$ is the open set in X that contains x and $p_k^{-1}(V_k)$ is the open set in X that contains y and $x\notin p_k^{-1}(\gamma_k(V_k))$ and $y\notin p_k^{-1}(\gamma_k(U_k))\Rightarrow$

$$x \notin \gamma_k(V_k) \times \prod_{i \in I \setminus \{k\}} X_i = \gamma_k(V_k) \times \prod_{i \in I \setminus \{k\}} \gamma_i(X_i)$$

and

$$y \notin \gamma_k(U_k) \times \prod_{i \in I \setminus \{k\}} X_i = \gamma_k(U_k) \times \prod_{i \in I \setminus \{k\}} \gamma_i(X_i) \Rightarrow$$
$$x \notin \gamma^*(V_k \times \prod_{i \in I \setminus \{k\}} X_i)$$

and

$$y \notin \gamma^*(U_k \times \prod_{i \in I \setminus \{k\}} X_i) \Rightarrow x \notin \gamma^*(p_k^{-1}(V_k)), y \notin \gamma^*(p_k^{-1}(U_k)).$$

That means that (X, \mathcal{T}) is $\gamma^* - T_1$.

Theorem 17. Let (X, \mathcal{T}) be the topological product of spaces (X_i, \mathcal{T}_i) , i = 1, 2, 3, ..., n and $\gamma_i : \mathcal{T}_i \to \mathcal{P}(X_i)$ let be γ_i -operations in (X_i, \mathcal{T}_i) , $i \in I$. If (X_i, \mathcal{T}_i) , $i \in I$ are γ_i -Hausdorf spaces, then (X, \mathcal{T}) is γ^* -Hausdorf space.

Proof. Let $x, y \in X$ and $x \neq y$. Then, there exists k such as the k-coordinate of x and y be different. It means, $x_k \neq y_k$. Since $x_k, y_k \in X_k$ and X_k is γ_k -of Hausdorf, there exists $U_k, V_k \in \mathcal{T}_k$, such that $x_k \in U_k, y_k \in V_k$ and $\gamma_k(U_k) \cap \gamma_k(V_k) = \emptyset \Rightarrow$,

$$\begin{split} p_k^{-1}(\gamma_k(U_k)) &\cap p_k^{-1}(\gamma_k(V_k)) = p_k^{-1}(\gamma_k(U_k) \cap \gamma_k(V_k)) = \emptyset \Rightarrow \\ &(\gamma_k(U_k) \times \prod_{i \in I \setminus \{k\}} X_i) \cap (\gamma_k(V_k) \times \prod_{i \in I \setminus \{k\}} X_i) = \emptyset \Rightarrow \\ &(\gamma_k(U_k) \times \prod_{i \in I \setminus \{k\}} \gamma_i(X_i)) \cap (\gamma_k(V_k) \times \prod_{i \in I \setminus \{k\}} \gamma_i(X_i)) = \emptyset \Rightarrow \\ &\gamma^*(U_k \times \prod_{i \in I \setminus \{k\}} X_i) \cap \gamma^*(V_k \times \prod_{i \in I \setminus \{k\}} X_i) = \emptyset \Rightarrow \\ &\gamma^*(p_k^{-1}(U_k)) \cap \gamma^*(p_k^{-1}(V_k)) = \emptyset. \end{split}$$

It means that (X, \mathcal{T}) is γ^* -Hausdorf space.

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