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SOME SUFFICIENT CONDITIONS FOR THE CLASS $\mathcal{U}$

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Abstract: Let $\mathcal{A}$ be the class of functions f that are analytic in the unit disk and normalized by $f(0) = f'(0) - 1 = 0$. The classes $\widetilde{\mathcal{U}}(\mu)$ and $\mathcal{U}(\lambda, \mu)$ ($\mu > 0$ and $0 < \lambda \leq 1$) consist of functions f from $\mathcal{A}$ that satisfy

$$\operatorname{Re} U(f, \mu; z) > 0 \quad (z \in \mathbb{D}),$$

and respectively

$$|U(f, \mu; z) - 1| < \lambda \quad (z \in \mathbb{D}),$$

where

$$U(f, \mu; z) = \left[\frac{z}{f(z)}\right]^{1+\mu} \cdot f'(z).$$

In this paper, using methods from the theory of first order differential subordinations, sufficient conditions (some of them sharp) are obtained in terms of the analytical representations of starlikeness and convexity that embed a function f from $\mathcal{A}$ in the class $\mathcal{U}(\lambda, \mu)$ or $\widetilde{\mathcal{U}}(\mu)$.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f that are analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$, i.e., such that $f(z) = z + a_2z^2 + \cdots$.

A function $f \in \mathcal{A}$ is said to be *starlike*, if and only if

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0, \quad z \in \mathbb{D}.$$

We denote by $\mathcal{S}$ the class of all such functions. Also, we denote by $\mathcal{K}$ the class of *convex functions*, i.e., the class of functions $f(z) \in \mathcal{A}$ for which

$$\operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0, \quad z \in \mathbb{D}.$$

Both these classes are subclasses of the class of univalent functions in $\mathbb{D}$ and even more, $\mathcal{K} \subset \mathcal{S}^*$. For details, see [2]. We will just recall their geometrical characterizations. A function f is convex if, and only if, it maps the unit disk onto a convex region, i.e.,

$$w_1, w_2 \in f(\mathbb{D}) \quad \Rightarrow \quad tw_1 + (1-t)w_2 \in f(\mathbb{D}) \qquad (0 \le t \le 1).$$

Fixing $w_2 = 0$ gives the geometrical characterizations of starlike functions, that is: f is starlike if, and only if, $t\omega \in f(\mathbb{D})$ for all $\omega \in f(\mathbb{D})$ and all $t \in [0, 1]$, i.e., for all $z \in \mathbb{D}$, $f(z)$ is visible from the origin.

Further, for $f \in \mathcal{A}$ and μ being a real number, let us define the operator

$$U(f, \mu; z) = \left[\frac{z}{f(z)}\right]^{1+\mu} \cdot f'(z)$$

which is closely related to the expression involved in the analytical definition of starlikeness. Using this operator, the following class is defined

$$\widetilde{\mathcal{U}}(\mu) = \left\{f \in \mathcal{A} : \frac{z}{f(z)} \neq 0 \text{ and } \operatorname{Re} U(f, \mu; z) > 0,\ z \in \mathbb{D}\right\},$$

and further $\widetilde{\mathcal{U}} \equiv \widetilde{\mathcal{U}}(1)$.

For $\mu = 0$ we obviously have a subclass of the class of starlike functions, and for $\mu < 0$ we receive a subclass of the class of Bazilevič functions, which is also widely studied. On the other hand, the case when $\mu > 0$ still attracts attention and offers interesting open problems. For example, the subclass of $\widetilde{\mathcal{U}}(\mu)$, defined by

$$\mathcal{U}(\lambda,\mu)=\left\{f\in\mathcal{A}:\frac{z}{f(z)}\neq 0 \text{ and } |U(f,\mu;z)-1|<\lambda,\ z\in\mathbb{D}\right\},$$

where $0<\lambda\leq 1$, is widely studied in the past decades, especially the cases $\mathcal{U}(\lambda)\equiv\mathcal{U}(\lambda,1)$ and $\mathcal{U}\equiv\mathcal{U}(1)=\mathcal{U}(1,1)$ (see [1], [3], [7]-[15]).

It is known [1],[15] that functions in $\mathcal{U}(\lambda)$ are univalent if $0<\lambda\leq 1$, but not necessarily univalent if $\lambda>1$. Further, Fournier and Ponnusamy [3] proved that assuming $\operatorname{Re}\mu<1$ the following equivalency holds:

$$\mathcal{U}(\lambda,\mu)\subset\mathcal{S}^* \quad\Leftrightarrow\quad 0<\lambda\leq\frac{|1-\mu|}{\sqrt{(1-\mu)^2+\mu^2}},$$

i.e., in general case, $\mathcal{U}(\lambda,\mu)$ is not a subset of $\mathcal{S}^*$. In particular,

$$\mathcal{U}(1,\mu)\subset\mathcal{S}^* \quad\Leftrightarrow\quad \mu=0,$$

i.e., $\mathcal{U}\not\subseteq\mathcal{S}^*$, which can be also verified by the function

$$f(z)=\frac{z}{1+\frac{1}{2}z+\frac{1}{2}z^3}\in\mathcal{U}\setminus\mathcal{S}^*.$$

As already stated, the classes $\mathcal{U}$ defined above are relatively new and attract interest among authors working in the area. One direction of their study is finding sufficient conditions for a function to be in $\mathcal{U}(\lambda,\mu)$ and/or $\widetilde{\mathcal{U}}(\mu)$. In this paper we obtain sharp sufficient conditions in terms of the analytical representations of starlikeness and convexity, that is in terms of

$$1+\frac{zf''(z)}{f'(z)} \quad\text{and}\quad \frac{zf'(z)}{f(z)}.$$

For that purpose we will make use of results from the theory of first order differential subordination. Here are some basic definitions and results from the theory.

Let $f(z)$ and $g(z)$ be analytic in the unit disk. We say that $f(z)$ is subordinate to $g(z)$, and we write $f(z)\prec g(z)$, if $g(z)$ is univalent in $\mathbb{D}$, $f(0)=g(0)$ and $f(\mathbb{D})\subseteq g(\mathbb{D})$. Further, we use the method of differential subordination introduced by Miler and Mocanu [4, 5]. In fact, if $\phi:\mathbb{C}^2\to\mathbb{C}$ ($\mathbb{C}$ is the complex plane) is analytic in domain $D\subset\mathbb{C}$, if $h(z)$ is univalent in $\mathbb{D}$, and $p(z)$

is analytic in $\mathbb{D}$ with $(p(z), zp'(z)) \in D$, when $z \in \mathbb{D}$, then we say that $p(z)$ satisfies a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \tag{1}$$

The univalent function $q(z)$ is called a dominant of the differential subordination (1) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1). If $\tilde{q}(z)$ is a dominant of (1) and $\widetilde{q}(z) \prec q(z)$ for all dominants of (1), then we say that $\tilde{q}(z)$ is the best dominant of the differential subordination (1).

A remarkable result from the theory of first order differential subordination is the following one due to Miller and Mocanu [4, 6].

Lemma 1. *Let q be univalent in the unit disk $\mathbb{D}$, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(\mathbb{D})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that:*

(i) Q is starlike in the unit disk $\mathbb{D}$,

(ii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re} \left[\dfrac{\theta'(q(z))}{\phi(q(z))} + \dfrac{zQ'(z)}{Q(z)}\right] > 0, \; z \in \mathbb{D}.$

If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{2}$$

then $p(z) \prec q(z)$, and q is the best dominant of (2).

2. Main results and consequences

First we prove a useful lemma that describes the behaviour of the operator U and will be applied for obtaining results over the class $\mathcal{U}$.

Lemma 2. *Let $f \in \mathcal{A}$ with $f'(z) \neq 0$ for all $z \in \mathbb{D}$ and q be univalent unction that does not vanish on the unit disk with $q(0) = 1$ and*

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right] > 0 \qquad (z \in \mathbb{D}). \tag{3}$$

If for some real μ,

$$1 + \frac{zf''(z)}{f'(z)} - (1+\mu)\left[\frac{zf'(z)}{f(z)}\right] + \mu \prec \frac{zq'(z)}{q(z)} \equiv h(z), \tag{4}$$

then

$$U\left(f,\mu;z\right)\prec q(z), \tag{5}$$

and q is the best dominant of (4).

Proof. Let $p(z)=U\left(f,\mu;z\right)$, $\theta(\omega)=0$ and $\phi(\omega)=1/\omega$. Then $\theta(\omega)$ and $\phi(\omega)$ are analytic with domain $D=\mathbb{C}\setminus\{0\}$ which contains $q(\mathbb{D})$ and $\phi(\omega)\neq 0$ when $\omega\in q(\mathbb{D})$. On the other hand,

$$Q(z)=zq'(z)\phi\left(q(z)\right)=\frac{zq'(z)}{q(z)}.$$

and

$$\frac{zQ'(z)}{Q(z)}=\frac{zh'(z)}{Q(z)}=1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)},$$

i.e.,

$$\mathrm{Re}\left[\frac{zQ'(z)}{Q(z)}\right]=\mathrm{Re}\left[1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right]>0 \qquad (z\in\mathbb{D}).$$

So, conditions (i) and (ii) from Lemma 1 are satisfied.

Further, p is analytic in $\mathbb{D}$ and $p(0)=q(0)=1$. Also, $p(z)\neq 0$ for all $z\in\mathbb{D}$, i.e. $p(\mathbb{D})\subseteq D$, since $f'(z)\neq 0$ for all $z\in\mathbb{D}$ (condition of the theorem); $z/f(z)=1\neq 0$ for $z=0$ (because $f\in\mathcal{A}$) and $f(z)$ has no poles on $\mathbb{D}$. Hence from Lemma 1 and the fact that

$$1+\frac{zf''(z)}{f'(z)}-(1+\mu)\left[\frac{zf'(z)}{f(z)}\right]+\mu$$

$$=\frac{zp'(z)}{p(z)}\prec\frac{zq'(z)}{q(z)}=\theta\left(q(z)\right)+zq'(z)\phi\left(q(z)\right),$$

we obtain $p(z)\prec q(z)$, i.e. relation (5), and we also receive that $q(z)$ is the best dominant of (4). $\square$

Using Lemma 2 and the definition of subordination we receive the main result of the paper.

Theorem 1. *Let $f\in\mathcal{A}$ with $f'(z)\neq 0$ and $z/f(z)\neq 0$ for all $z\in\mathbb{D}$. Also, let $\mu>0$.*

(i) If $0 < a \le 1/2$ *and*

$$\left|1+\frac{zf''(z)}{f'(z)}-(1+\mu)\left[\frac{zf'(z)}{f(z)}\right]+\mu\right|<a \quad (z\in\mathbb{D}), \tag{6}$$

then $f \in \mathcal{U}(\lambda(a),\mu)$, *where* $\lambda(a) \equiv a/(1-a)$. *This result is sharp, in the sense that for fixed* $\mu > 0$ *and* $0 < a \le 1/2$, $\lambda(a)$ *is the smallest number such that the implication holds.*

(ii) If

$$1+\frac{zf''(z)}{f'(z)}-(1+\mu)\left[\frac{zf'(z)}{f(z)}\right]+\mu \in \mathbb{C}\setminus\{ix : |x|\ge 1\} \quad (z\in\mathbb{D}), \tag{7}$$

then $f \in \widetilde{\mathcal{U}}(\mu)$.

Proof. (i) In the beginning let us note that $\lambda(a)$ increases from 0 to 1, as a goes from 0 to 1/2. Next, for $q(z) = 1 + \lambda(a)z$,

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1}{1+\lambda(a)z}$$

and the condition (3) holds. Next, the function $h(z) = zq'(z)/q(z) = \lambda(a)z/(1+\lambda(a)z)$ defined by the expression (4) is univalent in the unit disk such that

$$\min\{|h(z)| : |z| = 1\} = \frac{\lambda(a)}{1+\lambda(a)} = a.$$

So, the disk $\{w : |w| < a\}$ is contained in $h(\mathbb{D})$ which, having in mind the definition of subordination, means that inequality (6) implies subordination (4). Further, from Lemma 2 follows subordination (5), which is equivalent to

$$|U(f,\mu;z)-1| < \lambda(a) \quad (z\in\mathbb{D}).$$

Even more, Lemma 2 says that $q(z) = 1 + \lambda(a)z$ is the best dominant of (4).

In order to prove the sharpness of the result let us assume the opposite, i.e., there exists λ_*, $0 < \lambda_* < \lambda(a)$, such that inequality (6) implies

$$|U(f,\mu;z)-1| < \lambda_* \quad (z\in\mathbb{D}),$$

i.e.

$$U(f,\mu;z) \prec 1+\lambda_* z \equiv q_*(z).$$

On the other hand, inequality (6) implies subordination (4) with best dominant $q(z)$, meaning that $q(z) \prec q_*(z)$. This is a contradiction to the assumption $\lambda_* < \lambda(a)$ which proves the sharpness of the result.

(ii) Similarly as in (i) for $q(z) = (1+z)/(1-z)$, we receive

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1+z^2}{1-z^2},$$

i.e., (3) holds and further, $h(z) = zq'(z)/q(z) = 2z/(1-z^2)$ is univalent in the unit disk and maps it onto $\mathbb{C} \setminus \{ix : |x| \geq 1\}$. So, (7) is equivalent to (4), implying (5), i.e., $f \in \widetilde{\mathcal{U}}(\mu)$. □

Specifying values for μ and/or a gives the following corollary.

Corollary 1. *Let $f \in \mathcal{A}$ with $f'(z) \neq 0$ and $z/f(z) \neq 0$ for all $z \in \mathbb{D}$.*

(i) If for some $0 < a \leq 1/2$,

$$\left|1 + \frac{zf''(z)}{f'(z)} - 2\left[\frac{zf'(z)}{f(z)}\right] + 1\right| < a \quad (z \in \mathbb{D}),$$

then $f \in \mathcal{U}(\lambda(a))$, where $\lambda(a) = a/(1-a)$ (obtained for $\mu = 1$ in Theorem 1(i)). Especially, fixing $a = 1/2$ brings that

$$\left|1 + \frac{zf''(z)}{f'(z)} - 2\left[\frac{zf'(z)}{f(z)}\right] + 1\right| < \frac{1}{2} \quad (z \in \mathbb{D}),$$

implies $f \in \mathcal{U}$.

(ii) If for some real and positive μ,

$$\left|\mathrm{Im}\left\{1 + \frac{zf''(z)}{f'(z)} - (1+\mu)\left[\frac{zf'(z)}{f(z)}\right] + \mu\right\}\right| < 1 \quad (z \in \mathbb{D}),$$

then $f \in \widetilde{\mathcal{U}}(\mu)$ (obtained from Theorem 1(ii)).

(iii) If for some real and positive μ,

$$\left|1 + \frac{zf''(z)}{f'(z)} - (1+\mu)\left[\frac{zf'(z)}{f(z)}\right] + \mu\right| < 1 \quad (z \in \mathbb{D}),$$

then $f \in \widetilde{\mathcal{U}}(\mu)$ (obtained from (ii)).

All results are sharp.

Now we will apply the results obtained in this paper on specific functions f from $\mathcal{A}$, and obtain some interesting conclusions.

Example 1.

(i) *For the function* $f(z) = ze^{\alpha z}$, $0 < \alpha \le 1/2$, *it is easy to check that* $f \in \mathcal{A}$, *and that* $f'(z) \neq 0$ *and* $z/f(z) \neq 0$ *for all* $z \in \mathbb{D}$. *Further,*

$$\left|1 + \frac{zf''(z)}{f'(z)} - 2\left[\frac{zf'(z)}{f(z)}\right] + 1\right| = \left|-\frac{\alpha^2 z^2}{1+\alpha z}\right| < \frac{\alpha^2}{1-\alpha} \equiv a,$$

$0 < a \le 1/2$ *and*

$$\lambda(a) = \frac{a}{1-a} = \frac{\alpha^2}{1-\alpha-\alpha^2}.$$

So, Corollary 1(i) brings $ze^{\alpha z} \in \mathcal{U}(\lambda(\alpha))$ *and* $ze^{z/2} \in \mathcal{U}$.

(ii) *For the function*

$$f(z) = \frac{z}{1 + z/2 + z^2/4}$$

we have

$$1 + \frac{zf''(z)}{f'(z)} - 2\left[\frac{zf'(z)}{f(z)}\right] + 1 = \frac{2z^2}{z^2-4},$$

which, after using Corollary 1(iii) (with $\mu = 1$*) implies that* $f \in \widetilde{\mathcal{U}}$.

References

[1] L.A. Aksentév, Sufficient conditions for univalence of regular functions (In Russian), *Izv. Vysš. Učebn. Zaved. Matematika*, **4**, No 3 (1958), 3–7.

[2] P.L. Duren, *Univalent Functions*, Springer-Verlag, New York (1983).

[3] R. Fournier, S. Ponnusamy, A class of locally univalent functions defined by a differential inequality, *Complex Var. Elliptic Equ.*, **52**, No 1 (2007), 1–8.

[4] S.S. Miller, P.T. Mocanu, *Differential Subordinations, Theory and Applications*, Marcel Dekker, New York-Basel (2000).

[5] S.S. Miller, P.T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.*, **28**, No 2 (1981), 157–172.

[6] S.S. Miller, P.T. Mocanu, On some classes of first-order differential subordinations, *Michigan Math. J.*, **32**, No 2 (1985), 185–195.

[7] M. Nunokawa, M. Obradovic, S. Owa, One criterion for univalency, *Proc. Amer. Math. Soc.*, **106** (1989), 1035–1037.

[8] M. Obradovic, S. Ponnusamy, Product of univalent functions, *Math. Comput. Modelling*, **57**, No 3-4 (2013), 793–799.

[9] M. Obradovic, S. Ponnusamy, Criteria for univalent functions in the unit disk, *Arch. Math. (Basel)*, **100**, No 2 (2013), 149–157.

[10] M. Obradovic, S. Ponnusamy, On harmonic combination of univalent functions, *Bull. Belg. Math. Soc. Simon Stevin*, **19**, No 3 (2012), 461–472.

[11] M. Obradovic, S. Ponnusamy, New criteria and distortion theorems for univalent functions, *Complex Variables Theory Appl.*, **44** (2001), 173–191.

[12] M. Obradovic, S. Ponnusamy, V. Singh, P. Vasundhra, Univalency, starlikeness and convexity applied to certain classes of rational functions, *Analysis (Munich)*, **22**, No 3 (2002), 225–242.

[13] M. Obradovic, S. Ponnusamy, Univalence and starlikeness of certain integral transforms defined by convolution of analytic functions, *J. Math. Anal. Appl.*, **336** (2007), 758–767.

[14] M. Obradovic, S. Ponnusamy, Coefficient characterization for certain classes of univalent functions, *Bull. Belg. Math. Soc. (Simon Stevin)*, **16** (2009), 251–263.

[15] S. Ozaki, M. Nunokawa, The Schwarzian derivative and univalent functions, *Proc. Amer. Math. Soc.*, **33** (1972), 392–394.