International Journal of Applied Mathematics

Volume 30 No. 3 2017, 267-272
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v30i3.6

ON THE NONEXISTENCE OF UNITS IN ALGEBRAIC SYMBOLS AND THE CONSEQUENT IDENTIFICATION OF (dy/dx) WITH (dy/b)/(dx/a), AS ILLUSTRATED BY ECONOMIC COMPARATIVE STATICS

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Abstract: This short communication seeks to draw researchers' attention to the simple fact that algebraic symbols, while capable of *carrying* units externally, do not *contain* these units internally. As a result, a derivative such as $dy/dx \in \mathbb{R}$, when evaluated at a point p in a domain of definition, is a scalar in a 1−dimensional vector space (x) bounded at p that multiplies the tensor $(dx) \equiv (1)$ and then applies (dy/dx) to any vector (x) to result in a scalar. Any vector space contains exactly two kinds of objects: vectors and scalars, with scalars closed as an algebraic field; hence the scalars cannot contain units or else they do not form a (closed) field. If dy/dx is a unit-free pure number to begin with, then $x = 1$ and $y = 1$ are subject to arbitrary underlying unit specifications. As such, one can identify (dy/dx) with $(dy/b)/(dx/a)$, with $a, b > 0$ conveniently chosen.

AMS Subject Classification: 15A69, 26B10, 26A24
Key Words: quantitative perturbation analyzes, quantitative comparative statics, sign analyzes reconsidered, implicit function theorem strengthened

1. Introduction

Previously we introduced the idea of "relative derivative,"

Received: April 30, 2017

$$(dy/b)/(dx/a),$$

(cf. [3], and its citation in [2]). This short note advances our prior results to a statement that all (standard) derivatives are in fact relative derivatives. Here immediately below we will present a simple example for a quick demonstration. In Section 2 we will show that the implicit function theorem as cast in relative derivatives has the same form as that cast in the standard derivatives, and we will present three examples from economic comparative statics to show that the hitherto qualitative analyzes of evaluating the involve signs can readily be elevated to quantitative predictions. In Section 3 we will close with a summary remark.

Consider $y =$ quantity supplied of apples, $x =$ their price, and $y = x^2 + 1$, so that $dy = 2xdx$, with $dy = 10dx$ at $x = 5$. Then in detailed tensorial expressions, we have tensors $dy = T_y = (1) = T_x = dx$ and

$$\begin{aligned} (1)\,(10 \text{ apples}) &= \left(10 \cdot \frac{apples}{\$}\right) \cdot (1) \cdot (\$1) \equiv \alpha T_x w \\ &= \left(10 \cdot \frac{apples}{\$}\right) \cdot \left(\frac{\$1.25}{Euro\ 1}\right) \cdot (Euro\ 0.8) \\ &= \alpha\,(A^* T_x)\,v, \end{aligned} \tag{1}$$

where

$$\begin{aligned} A &: \quad v = Euro\ 0.8 \mapsto w = \$1, \\ T_x\,(Av) &= T_x w = \$1 = (A^* T_x)\,v \\ &= \left(\frac{\$1.25}{Euro\ 1}\right) \cdot (Euro\ 0.8)\,, \end{aligned}$$

where A^*, the transpose of A, is the pull-back operator.

Continuing with Equation (1), we have

$$\begin{aligned} 10\ apples &= \alpha\,(A^* T_x)\,v \\ &= (A^* T_x)\,(\alpha v) \\ &= \left(\frac{\$1.25}{Euro\ 1}\right)\left(10 \cdot \frac{apples}{\$}\right)(1)\,(Euro\ 0.8) \\ &= A^* \alpha T_x v; \\ i.e., \alpha\,(A^* T_x) &= A^*\,(\alpha T_x)\,. \end{aligned}$$

A^* is an algebraic homomorphism, where $\alpha \in \mathbb{R}$, a field F closed under $(+, \cdot)$. Thus,

$$\alpha \neq \left(10 \cdot \frac{apples}{\$}\right)$$

since α^2 would equal $100 \cdot \left(\frac{apples^2}{\$^2}\right)$, breaking out of F. I.e., $\alpha = \frac{dy}{dx}$ must not carry any units.

If $\frac{dy}{dx}\mid_{x=5} = 10 \in \mathbb{R}$, with units suppressed in the background, then we might just as well choose discretionarily

$$\begin{aligned} b\,(apples) &> 0 \text{ and } a\,(\$) > 0, \text{ and consider} \\ \frac{dy/b}{dx/a} &= \frac{\%\text{ changes in y}}{\%\text{ changes in x}}, \end{aligned}$$

where a and b can be taken to be the existing (or current) levels. The values of a and b vary with the analysts, who may simulate by professional insights

$$\frac{dy/b}{dx/a} = 0.5, 1, 2, etc..$$

Thus we are not limiting ourselves to the concept of elasticity in economics,

$$\frac{dy/26}{dx/5} = 10 \cdot \frac{5}{26}$$

(which actually is not a fixed value due to the different functional forms $\{f\}$ of $f(x) = y$).

Nor are we referring to logarithmic differentiation

$$\frac{d\ln y}{d\ln x} = \frac{dy/y}{dx/x},$$

where x and y are variables. In fact, the perspective of logarithmic differentiation misses the point that any derivative is evaluated at a fixed point; letting $y = x^2 + 1$ vary with x would mean taking derivatives of the whole function, resulting in $\frac{dy}{dx} = 2x$.

2. The Implicit Function Theorem and Economic Comparative Statics

Let $F \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, $F(\mathbf{x}_0, \mathbf{c}_0) = \mathbf{y}_0$, where

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ & \vdots & \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} = - \begin{bmatrix} \frac{\partial y_1}{\partial c_1} & \cdots & \frac{\partial y_1}{\partial c_m} \\ & \vdots & \\ \frac{\partial y_n}{\partial c_1} & \cdots & \frac{\partial y_n}{\partial c_m} \end{bmatrix} \begin{bmatrix} dc_1 \\ \vdots \\ dc_m \end{bmatrix}$$

is true if and only if its left-hand-side as re-expressed

$$\begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{bmatrix}_{diag} \begin{bmatrix} \frac{\partial y_1}{\partial x_1}\frac{a_1}{b_1} & \cdots & \frac{\partial y_1}{\partial x_n}\frac{a_n}{b_1} \\ & \vdots & \\ \frac{\partial y_n}{\partial x_1}\frac{a_1}{b_n} & \cdots & \frac{\partial y_n}{\partial x_n}\frac{a_n}{b_n} \end{bmatrix} \begin{bmatrix} \frac{1}{a_1} & & \\ & \ddots & \\ & & \frac{1}{a_n} \end{bmatrix}$$

$$\circ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \begin{bmatrix} \frac{dx_1}{a_1} \\ \vdots \\ \frac{dx_1}{a_1} \end{bmatrix} \text{ is equal to the following:}$$

$$-\begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{bmatrix}_{diag} \begin{bmatrix} \frac{\partial y_1}{\partial x_1}\frac{c_1}{b_1} & \cdots & \frac{\partial y_1}{\partial x_n}\frac{c_m}{b_1} \\ & \vdots & \\ \frac{\partial y_n}{\partial x_1}\frac{c_1}{b_n} & \cdots & \frac{\partial y_n}{\partial x_n}\frac{c_m}{b_n} \end{bmatrix} \begin{bmatrix} \frac{1}{c_1} & & \\ & \ddots & \\ & & \frac{1}{c_m} \end{bmatrix}$$

$$\circ \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_m \end{bmatrix} \begin{bmatrix} \frac{dc_1}{c_1} \\ \vdots \\ \frac{dc_m}{c_m} \end{bmatrix},$$

or

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1}\frac{a_1}{b_1} & \cdots & \frac{\partial y_1}{\partial x_n}\frac{a_n}{b_1} \\ & \vdots & \\ \frac{\partial y_n}{\partial x_1}\frac{a_1}{b_n} & \cdots & \frac{\partial y_n}{\partial x_n}\frac{a_n}{b_n} \end{bmatrix} \begin{bmatrix} \frac{dx_1}{a_1} \\ \vdots \\ \frac{dx_1}{a_1} \end{bmatrix} = -\begin{bmatrix} \frac{\partial y_1}{\partial x_1}\frac{c_1}{b_1} & \cdots & \frac{\partial y_1}{\partial x_n}\frac{c_m}{b_1} \\ & \vdots & \\ \frac{\partial y_n}{\partial x_1}\frac{c_1}{b_n} & \cdots & \frac{\partial y_n}{\partial x_n}\frac{c_m}{b_n} \end{bmatrix} \begin{bmatrix} \frac{dc_1}{c_1} \\ \vdots \\ \frac{dc_m}{c_m} \end{bmatrix},$$

thus the same form. To demonstrate the advantages of our relative-derivative interpretation of the standard derivatives, we will now present three examples from economics comparative statics in the following.

Example 1. From [4], Ch. 9, Section: Stability and Dynamics, p. 263, where $p \equiv$ price, $H \equiv$ speed of adjustment, $H^{'}(0) > 0$, $q_D = D(p^*, \alpha^*) \equiv$ demand, and $q_s = S(p^*) \equiv$ supply:

$$\begin{aligned} \frac{dp}{dt} &= H(q_D - q_s) = H[D(p^*, \alpha^*) - S(p^*)] \\ &\approx H^{'}[D(p^*, \alpha^*) - S(p^*)] \cdot \left(\frac{\partial D}{\partial p}\Delta p + \frac{\partial D}{\partial \alpha}\Delta\alpha - \frac{\partial S}{\partial p}\Delta p \right); \end{aligned}$$

then we can have the interpretation that

$$\begin{aligned} \frac{dp/p^*}{dt} &\approx (say)\, 5 \times (-0.5 + 1 - 2) \cdot \frac{\Delta p}{p^*} \\ &= -7.5\frac{\Delta p}{p^*}, \end{aligned}$$

where $H^{'}(0) = 5$ would mean a 1% excess demand over the equilibrium quantity to cause a 5% rise from the equilibrium price. Otherwise by the established paradigm H is simply analytically intractable.

Example 2. Ibid., Ch. 9, Section: The Stability of Multiple Markets, p. 276, which lent to the book's front cover about the comparative statics on $C(i,Y) - Y + I = -\alpha$, $F(i,Y) - I = -\beta$, $L(i,Y) = M$, where $i \equiv$ $(1+$ the ordinary usage of an interest rate$)$, $Y \equiv$ *income*, $I \equiv$ *investment*, $C \equiv$ *consumption*, $F \equiv$ marginal efficiency of capital, $L \equiv$ liquidity preference, $M \equiv$ money stock,

$$\begin{pmatrix} & i & Y & I \\ \alpha & + & + & ? \\ \beta & + & + & ?,+ \\ M & - & ?,+ & ?,+ \end{pmatrix}.$$

We now recast the various derivatives as in the previous example to simulate a 1% increase in money stock to cause a 0.25% increase in investment:

$$\begin{aligned} \frac{dI}{dM} &= \frac{1}{\Delta}\left[F_Y\left(F_i + C_i\right) + \left(1 - C_Y - F_Y\right)F_i\right], \text{ where} \\ \Delta &= L_Y\left(F_i + C_i\right) + L_i\left(1 - C_Y - F_Y\right). \text{ Suppose} \\ \Delta &= 1 \times (-0.4 - 0.1) + (-0.1)(1 - 1 - 1) = -0.4; \text{ then} \\ \frac{dI}{dM} &= \frac{1}{-0.4}\left[1 \times (-0.4 - 0.1) + (1 - 1 - 1)(-0.4)\right] \\ &= (-2.5) \times (-0.5 + 0.4) = 0.25. \end{aligned}$$

Example 3. From [1], Ch. 5, p. 87, comparative statics on the IS-LM model, where

$$i^{'} \equiv \frac{d\text{investment}}{d \text{ interest rate}},\ l^{'} \equiv \frac{d\text{money demand for financial speculation}}{d \text{ interest rate}},$$

$$c^{'} \equiv \frac{d \text{ consumption}}{d \text{ disposable income}},\ t^{'} \equiv \frac{d \text{ income tax}}{d \text{ income } y},$$

$$k^{'} \equiv \frac{d \text{ money demand for transaction}}{d \text{ income } y}$$

(note that the notations here are distinct as defined by the referred book):

$$\begin{aligned} dy &= \frac{\frac{i'}{l'}}{1 - c'\left(1 - t'\right) + \frac{i'k'}{l'}} dm \\ &= \frac{\frac{-2}{-1}}{1 - 1(1 - 1) + \frac{-2\times 1}{-1}} = \frac{2}{1+2} = \frac{2}{3}, \end{aligned}$$

meaning that an increase in money supply by 1% is to cause a $\frac{2}{3}$% increase in national income.

3. Summary Remark

This brief communication alerts all scholars that, e.g., $\frac{d\ output\ y}{d\ input\ x_i}$ can henceforth be simulated such as 0.3 to mean a 1%− *increase* in x_i to cause a 0.3% − *increase* in y, regardless of the underlying units. The fact of the matter is that algebraic symbols such as x and y represent numbers with no units; that is, one would write $\$x = \$7$, not $x = \$7$, for $x \in$ an algebraic field F, closed under addition and multiplication. As such, all expositions of comparative statics in economics or perturbative analyzes across disciplines can maintain their existing textbook/literature appearances yet all the involved derivatives can henceforth assume the meanings of proportional changes.

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