

COMMON FIXED POINT THEOREM IN CONE METRIC SPACE UNDER CONTRACTIVE MAPPINGS

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Abstract: In this paper we prove a common fixed point theorem for two Banach pairs of mappings which satisfy the contraction conditions in cone metric spaces without the assumption of normality condition.

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1. Introduction

Recently, Huang and Zhang [1] introduced the notion of cone metric spaces. They replaced real number system by ordered Banach space. They also gave

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the condition in the setting of cone metric spaces. The authors also described the convergence of sequences in the cone metric spaces and introduce the corresponding notion of completeness. Subsequently, many authors have generalized the results of Haung and Zang and have studied fixed point theorems for normal and non-normal cones, see for instance [4], [8], [10], [11], [13], [14], etc.

In 2009, Beiranvand [2] et al introduced new classes of contractive functions T -contraction and T -contractive mappings and then they established and extended the Banach contraction principle. Morales and Rojas [5], [6] obtained sufficient conditions for the existence of a unique fixed point of T -Kannan contractive, T -Zamfirescu, T -contractive mappings on complete cone metric spaces.

In [8], authors have proved some common fixed point theorems for a Banach pair of mappings satisfying T -Hardy Rogers type contraction condition in cone metric spaces. In sequel, Ozturk and Basarir [3] proved some common fixed point theorems for f -contraction mappings in cone metric spaces without the assumption of normality condition of the cone. Subrahmanyam [9] introduced Banach operator of type k . Recently, Chen and Li [7] extended the concept of Banach operator of type k to Banach operator pair and proved various best approximation results using common fixed point theorems for f -nonexpansive mappings.

The aim of this paper is to prove common fixed point theorems for two Banach pairs of mappings which satisfy contraction conditions in cone metric spaces without the assumption of normality condition of the cone.

2. Preliminaries

We recall some standard definitions and other results that will be needed in the sequel.

Definition 2.1. A self mapping T of a metric space (X, d) is said to be contraction mapping, if there exists a real number $0 \leq k < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y).$$

Definition 2.2. Let T and f be two self mappings of a metric space (X, d) . The self mapping f of X is said to be T -contraction, if there exists a real number $0 \leq k < 1$ such that

$$d(Tfx, Tfy) \leq kd(Tx, Ty) \text{ for all } x, y \in X.$$

If $T = I$, the identity mapping, Definition 2.1 reduces to Banach contraction mapping.

Definition 2.3. Let T be a self mapping of a metric space (X, d) , then:

- (i) A mapping T is said to be sequentially convergent, if the sequence $\{y_n\}$ in X is convergent whenever $\{Ty_n\}$ is convergent.
- (ii) A mapping T is said to be subsequentially convergent, if $\{y_n\}$ has a convergent subsequence whenever $\{Ty_n\}$ is convergent.

Definition 2.4. Let T be a self mapping of a normed space X . Then T is called a Banach operator of type k , if

$$\|T^2x - Tx\| \leq k \|Tx - x\|,$$

for some $k \geq 0$ and for all $x \in X$.

Definition 2.5. Let T and f be two self mappings of a non-empty subset M of a normed linear space X . Then (T, f) is a Banach operator pair, if any one of the following conditions is satisfied:

- (i) $T[F(f)] \subseteq F(f)$, i.e. $F(f)$ is T -invariant;
- (ii) $fTx = Tx$ for each $x \in F(f)$;
- (iii) $fTx = Tf x$ for each $x \in F(f)$;
- (iv) $\|Tfx - fx\| \leq k \|fx - x\|$ for some $k \geq 0$.

Definition 2.6. Let E be a real Banach space and P a subset of E . P is called a cone if and only if:

- (i) P is closed, non-empty and $P \neq \{0\}$;
- (ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b ;
- (iii) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, a partial ordering is defined as $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. It is denoted as $x \ll y$ will stand for $y - x \in \text{int}P$ denotes the interior of P . The cone P is called normal if there is number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K \|y\|.$$

The least positive number K satisfying (2.3) is called normal constant of P .

Definition 2.7. Let X be a non-empty set. Suppose E is a real Banach space, P is a cone with $\text{int}P \neq \emptyset$ and $\leq$ is a partial ordering with respect to P . if the mapping $d : X \times X \rightarrow E$ satisfies:

- (i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.8. Let (X, d) be a cone metric space and $\{x_n\}$ be a sequence in X . Then:

- (i) $\{x_n\}$ converges to $x \in X$, if for every $c \in E$ with $0 \ll c$, there is $n_0 \in N$, the set of all natural numbers such that for all $n \geq n_0, d(x_n, x) \ll c$. It is denoted by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x, (n \rightarrow \infty)$;
 - (ii) If for any $c \in E$, there is a number $n_0 \in N$ such that for all $m, n \geq n_0, d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X ; (X, d) is a complete cone metric space, if every Cauchy sequence in X is convergent;
- A self mapping $T : X \rightarrow X$ is said to be continuous at a point $x \in X$, if $\lim_{n \rightarrow \infty} x_n = x$ implies that $\lim_{n \rightarrow \infty} Tx_n = Tx$ for every $\{x_n\}$ in X .

Definition 2.9. Let (X, d) be a cone metric space and $T, S : X \rightarrow X$ two functions:

A mappings S is said to be T -Reich contraction, if there is $\alpha + \beta + \gamma + \delta < 1$ such that

$$d(TSx, TSy) \leq \alpha d(Tx, TSx) + \beta d(Ty, TSy) + \gamma d(Tx, Ty) + \delta [d(Tx, TSy) + d(Ty, TSx)]. \quad (1)$$

3. Main Result

Theorem. Let T, f and g be three continuous self mappings of a complete cone metric space (X, d) . Assume that T is an injective mapping. If the mapping T, f and g satisfy

$$d(Tfx, Tgy) \leq \alpha d(Tx, Tfx) + \beta d(Ty, Tgy) + \gamma d(Tx, Ty) + \delta [d(Tx, Tgy) + d(Ty, Tfx)]$$

for all $x, y \in X$ where $\alpha, \beta, \gamma, \delta$ are all non-negative constants such that $\alpha + \beta + \gamma + \delta < 1$, then f and g have a unique common fixed point in X . Moreover, if (T, f) and (T, g) are Banach pairs, then T, f and g have a unique common fixed point in X .

Proof. Let $x_0 \in X$ as an arbitrary element and define the sequence $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for each $n = 0, 1, 2, \dots, \infty$. Then by using equation (1) and triangle inequality

$$\begin{aligned}
 d(Tx_{2n+1}, Tx_{2n}) &= d(Tfx_{2n}, Tgx_{2n-1}) \\
 &\leq \alpha d(Tx_{2n}, Tfx_{2n}) + \beta d(Tx_{2n-1}, Tgx_{2n-1}) + \gamma d(Tx_{2n}, Tx_{2n-1}) \\
 &\quad + \delta [d(Tx_{2n}, Tgx_{2n-1}) + d(Tx_{2n-1}, Tfx_{2n})] \\
 &= \alpha d(Tx_{2n}, Tx_{2n+1}) + \beta d(Tx_{2n-1}, Tx_{2n}) + \gamma d(Tx_{2n}, Tx_{2n-1}) \\
 &\quad + \delta [d(Tx_{2n}, Tx_{2n}) + d(Tx_{2n-1}, Tx_{2n+1})] \\
 &= \alpha d(Tx_{2n}, Tx_{2n+1}) + \beta d(Tx_{2n-1}, Tx_{2n}) + \gamma d(Tx_{2n}, Tx_{2n-1}) \\
 &\quad + (1 - \alpha - \beta) d(Tx_{2n-1}, Tx_{2n+1}) \\
 &\leq (\gamma + \delta) d(Tx_{2n}, Tx_{2n-1}) d(Tx_{2n-1}, Tx_{2n+1}) \\
 &\leq \frac{\gamma + \delta}{1 - \alpha - \beta} d(Tx_{2n}, Tx_{2n-1}).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 d(Tx_{2n+3}, Tx_{2n+2}) &= d(Tfx_{2n+2}, Tgx_{2n+1}) \\
 &\leq \alpha d(Tx_{2n+2}, Tfx_{2n+2}) + \beta d(Tx_{2n+1}, Tgx_{2n+1}) + \gamma d(Tx_{2n+2}, Tx_{2n+1}) \\
 &\quad + \delta [d(Tx_{2n+2}, Tgx_{2n+1}) + d(Tx_{2n+1}, Tfx_{2n+2})] \\
 &= \alpha d(Tx_{2n+2}, Tx_{2n+3}) + \beta d(Tx_{2n+1}, Tx_{2n+2}) + \gamma d(Tx_{2n+2}, Tx_{2n+1}) \\
 &\quad + \delta [d(Tx_{2n+2}, Tx_{2n+2}) + d(Tx_{2n+1}, Tx_{2n+3})] \\
 &= \alpha d(Tx_{2n+2}, Tx_{2n+3}) + \beta d(Tx_{2n+1}, Tx_{2n+2}) + \gamma d(Tx_{2n+2}, Tx_{2n+1}) \\
 &\quad + \delta d(Tx_{2n+1}, Tx_{2n+3}) \\
 &= \frac{\gamma + \delta}{1 - \alpha - \beta} d(Tx_{2n+2}, Tx_{2n+1}).
 \end{aligned}$$

Thus $d(Tx_{n+1}, Tx_n) \leq \lambda d(Tx_n, Tx_{n-1}) \leq \dots \leq \lambda^n d(Tx_1, Tx_0)$, for all $n \geq 0$, where $\lambda = \frac{\gamma + \delta}{1 - \alpha - \beta} < 1$.

Now for $n > m$ we have

$$\begin{aligned}
 d(Tx_n, Tx_m) &\leq d(Tx_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_{n-2}) + \dots + d(Tx_{m+1}, Tx_m) \\
 &\leq (\lambda^{n-1} + \lambda^{n-2} + \dots + \lambda^m) d(Tx_1, Tx_0) \\
 &\leq \frac{\lambda^m}{1 - \lambda} d(Tx_1, Tx_0).
 \end{aligned}$$

Let $0 \ll c$ be given. Choose $\rho > 0$ such that $c + N_\rho(0) \subseteq P$, where

$$N_\rho(0) = \{y \in E : \|y\| < \rho\}.$$

Also, choose a natural number N_1 such that $\frac{\lambda^m}{1-\lambda}d(Tx_1, Tx_0) \in N_\rho(0)$, for all $m \geq N_1$.

Then

$$\frac{\lambda^m}{1-\lambda}d(Tx_1, Tx_0) \ll c, \text{ for all } m \geq N_1.$$

Thus

$$d(Tx_n, Tx_m) \leq \frac{\lambda^m}{1-\lambda}d(Tx_1, Tx_0)$$

and

$$\frac{\lambda^m}{1-\lambda}d(Tx_1, Tx_0) \ll c,$$

for all $m > n$. Then we get $d(Tx_n, Tx_m) \ll c$ for all $n < m$. Therefore $\{Tx_n\}$ is a Cauchy sequence in (X, d) . As X is complete, there exists $z \in X$ such that $\lim_{n \rightarrow \infty} Tx_n = z$.

Since T is a sub-sequentially convergent, $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $\lim_{m \rightarrow \infty} x_m = u$. As T is continuous $\lim_{m \rightarrow \infty} Tx_m = Tu$.

By the uniqueness of the limit, $z = Tu$. Since f and g are continuous, $\lim_{m \rightarrow \infty} gx_m = gu$ and $\lim_{m \rightarrow \infty} fx_m = fu$. Again since T is continuous, $\lim_{m \rightarrow \infty} Tgx_m = Tgu$ and $\lim_{m \rightarrow \infty} Tfx_m = Tfu$.

Therefore, if m is odd, then

$$\lim_{n \rightarrow \infty} Tgx_{2n+1} = Tgu.$$

Choose a natural number N_2 such that

$$d(Tx_{2n+1}, Tu) \ll \left[\frac{c}{2} \left(\frac{\gamma + \delta}{1 - \alpha - \beta} \right) \right] \text{ for all } n \geq N_2.$$

Now consider

$$\begin{aligned} d(Tgu, Tu) &\leq d(Tgu, Tx_{2n+1}) + d(Tx_{2n+1}, Tu) \\ &\leq \alpha d(Tgu, Tfgu) + \beta d(Tx_{2n+1}, Tgx_{2n+1}) + \gamma d(Tgu, Tx_{2n+1}) \\ &\quad + \delta [d(Tgu, Tgx_{2n+1}) + d(Tx_{2n+1}, Tfgu)] + d(Tx_{2n+1}, Tu) \\ &= \alpha d(Tu, Tgu) + \beta d(Tx_{2n+1}, Tx_{2n+2}) + \gamma d(Tu, Tx_{2n+1}) \\ &\quad + \delta [d(Tu, Tx_{2n+2}) + d(Tx_{2n+1}, Tgu)] + d(Tx_{2n+1}, Tu). \end{aligned}$$

So

$$d(Tu, Tgu) \leq \left(\frac{\gamma + \delta}{1 - \alpha - \beta} \right) d(Tx_{2n}, Tu)$$

$$+\left(\frac{1+\gamma}{1-\alpha-\beta}\right)d(Tu, Tx_{2n+1}) \ll c,$$

for all $n \geq N_2$. Therefore, $d(Tu, Tgu) \ll \frac{c}{i}$ for all $i \geq 1$. Hence, $\frac{c}{i} - d(Tu, Tgu) \in P$ for all $i \geq 1$. since P is closed, $-d(Tu, Tgu) \in P$ and so $d(Tu, Tgu) = 0$. Hence $Tu = Tgu$. As T is injective, $u = gu$. Thus u is the fixed point of and if m is even, then we have

$$\lim_{n \rightarrow \infty} Tfx_{2n} = Tfu.$$

Now, by using triangle inequality, we have

$$\begin{aligned} d(Tu, Tfu) &\leq \left(\frac{\gamma+\delta}{1-\alpha-\beta}\right)d(Tx_{2n+1}, Tu) \\ &+ \left(\frac{1+\gamma}{1-\alpha-\beta}\right)d(Tu, Tx_{2n+2}) \ll c, \end{aligned}$$

for all $n \geq N_2$. therefore, $d(tu, Tfu) \ll \frac{c}{i}$ for all $i \geq 1$. Hence, $\frac{c}{i} - d(Tu, Tfu) \in P$ for all $i \geq 1$. since P is closed, $-d(Tu, Tfu) \in P$ and so $d(Tu, Tfu) = 0$. Hence $Tu = Tfu$. As T is injective, $u = fu$. Thus u is also fixed point of f .

Uniqueness: Suppose that u^* is another common fixed point of f and g ,

$$\begin{aligned} d(Tu, Tu^*) &= d(Tfu, Tgu^*) \\ &\leq \alpha d(Tu, Tfu) + \beta d(Tu^*, Tgu^*) + \gamma d(Tu, Tu^*) \\ &\quad + \delta [d(Tu, Tgu^*) + d(Tu^*, Tfu)] \\ d(Tu, Tu^*) &\leq (\alpha + \beta + \gamma + \delta) d(Tu, Tu^*). \end{aligned}$$

Since $\alpha + \beta + \gamma + \delta < 1$, $d(Tu, Tu^*) = 0$ which implies that $Tu = Tu^*$. We know that T is injective, $u = u^*$ is the unique common fixed point of f and g . Since we have assumed that $\{T, f\}$ and $\{T, g\}$ are Banach pairs; $\{T, f\}$ and $\{T, g\}$ commutes at the fixed point of f and, respectively. This implies that $Tfu = fTu$ for $u \in F(f)$. So $Tu = fTu$ which gives that Tu is another fixed point f . It is also true for g . By the uniqueness of fixed point of f , $Tu = u$. Hence $u = Tu = fu = gu$, u is unique common fixed point of T, f and g in X . $\square$

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